

Linear Programming

MAC175 Operational Research — Part 2: additional worked examples

Three worked examples

1. **A workshop with three products.** Three variables, so no picture: the tableau on its own. Two iterations; one product is never made.

Three worked examples

1. **A workshop with three products.** Three variables, so no picture: the tableau on its own. Two iterations; one product is never made.
2. **A diet problem.** Minimisation with $\geq$ constraints: artificial variables, the method of penalties and the two-phase method, side by side with the picture.

Three worked examples

1. **A workshop with three products.** Three variables, so no picture: the tableau on its own. Two iterations; one product is never made.
2. **A diet problem.** Minimisation with $\geq$ constraints: artificial variables, the method of penalties and the two-phase method, side by side with the picture.
3. **A bakery.** Maximisation with three resources; then read the dual prices from the final tableau and do a sensitivity analysis.

Example 1: a workshop with three products

A workshop makes three products from timber, labour and machine time.

	Product 1	Product 2	Product 3	Available per week
Timber (units)	1	1	2	5
Labour (hours)	1	2	1	10
Machine time (hours)	2	1	2	7

Example 1: a workshop with three products

A workshop makes three products from timber, labour and machine time.

	Product 1	Product 2	Product 3	Available per week
Timber (units)	1	1	2	5
Labour (hours)	1	2	1	10
Machine time (hours)	2	1	2	7
Profit (£100 per unit)	6	5	4	

Example 1: a workshop with three products

A workshop makes three products from timber, labour and machine time.

	Product 1	Product 2	Product 3	Available per week
Timber (units)	1	1	2	5
Labour (hours)	1	2	1	10
Machine time (hours)	2	1	2	7
Profit (£100 per unit)	6	5	4	

With x_1, x_2, x_3 the weekly production of each product:

$$\begin{aligned} & \text{maximise } z = 6x_1 + 5x_2 + 4x_3 \quad \text{subject to} \quad \begin{aligned} & x_1 + x_2 + 2x_3 \leq 5 \\ & x_1 + 2x_2 + x_3 \leq 10 \\ & 2x_1 + x_2 + 2x_3 \leq 7 \end{aligned} \quad x_1, x_2, x_3 \geq 0. \end{aligned}$$

Example 1: a workshop with three products

A workshop makes three products from timber, labour and machine time.

	Product 1	Product 2	Product 3	Available per week
Timber (units)	1	1	2	5
Labour (hours)	1	2	1	10
Machine time (hours)	2	1	2	7
Profit (£100 per unit)	6	5	4	

With x_1, x_2, x_3 the weekly production of each product:

$$\begin{aligned} & \text{maximise } z = 6x_1 + 5x_2 + 4x_3 \quad \text{subject to} \quad \begin{aligned} & x_1 + x_2 + 2x_3 \leq 5 \\ & x_1 + 2x_2 + x_3 \leq 10 \\ & 2x_1 + x_2 + 2x_3 \leq 7 \end{aligned} \quad x_1, x_2, x_3 \geq 0. \end{aligned}$$

Three variables: no picture to help us. The tableau must do all the work.

Example 1: starting tableau

Add slack variables s_1, s_2, s_3 (unused timber, labour, machine time):

Example 1: starting tableau

Add slack variables s_1, s_2, s_3 (unused timber, labour, machine time):

$$x_1 + x_2 + 2x_3 + s_1 = 5$$

$$x_1 + 2x_2 + x_3 + s_2 = 10$$

$$2x_1 + x_2 + 2x_3 + s_3 = 7$$

Example 1: starting tableau

Add slack variables s_1, s_2, s_3 (unused timber, labour, machine time):

$$x_1 + x_2 + 2x_3 + s_1 = 5$$

$$x_1 + 2x_2 + x_3 + s_2 = 10$$

$$2x_1 + x_2 + 2x_3 + s_3 = 7$$

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Solution
z	-6	-5	-4	0	0	0	0
s_1	1	1	2	1	0	0	5
s_2	1	2	1	0	1	0	10
s_3	2	1	2	0	0	1	7

Example 1: starting tableau

Add slack variables s_1, s_2, s_3 (unused timber, labour, machine time):

$$x_1 + x_2 + 2x_3 + s_1 = 5$$

$$x_1 + 2x_2 + x_3 + s_2 = 10$$

$$2x_1 + x_2 + 2x_3 + s_3 = 7$$

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Solution
z	-6	-5	-4	0	0	0	0
s_1	1	1	2	1	0	0	5
s_2	1	2	1	0	1	0	10
s_3	2	1	2	0	0	1	7

Starting BFS: $x_1 = x_2 = x_3 = 0$, $s_1 = 5$, $s_2 = 10$, $s_3 = 7$, $z = 0$ (make nothing).

Example 1: iteration 1

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Solution	
z	-6	-5	-4	0	0	0	0	
s_1	1	1	2	1	0	0	5	
s_2	1	2	1	0	1	0	10	
s_3	2	1	2	0	0	1	7	

Example 1: iteration 1

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Solution	
z	-6	-5	-4	0	0	0	0	
s_1	1	1	2	1	0	0	5	
s_2	1	2	1	0	1	0	10	
s_3	2	1	2	0	0	1	7	

- Most negative entry in the z -row: -6 , so x_1 enters (the most profitable product).

Example 1: iteration 1

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Solution	Ratio
z	-6	-5	-4	0	0	0	0	
s_1	1	1	2	1	0	0	5	5
s_2	1	2	1	0	1	0	10	10
s_3	2	1	2	0	0	1	7	$\frac{7}{2}$

- Most negative entry in the z -row: -6 , so x_1 enters (the most profitable product).
- Ratios: $5 \div 1 = 5$, $10 \div 1 = 10$, $7 \div 2 = \frac{7}{2}$.

Example 1: iteration 1

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Solution	Ratio
z	-6	-5	-4	0	0	0	0	
s_1	1	1	2	1	0	0	5	5
s_2	1	2	1	0	1	0	10	10
s_3	2	1	2	0	0	1	7	$\frac{7}{2}$

- Most negative entry in the z -row: -6 , so x_1 enters (the most profitable product).
- Ratios: $5 \div 1 = 5$, $10 \div 1 = 10$, $7 \div 2 = \frac{7}{2}$.
- Smallest is $\frac{7}{2}$: s_3 leaves. Machine time is the first resource to run out.

Example 1: iteration 1

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Solution	Ratio
z	-6	-5	-4	0	0	0	0	
s_1	1	1	2	1	0	0	5	5
s_2	1	2	1	0	1	0	10	10
s_3	2	1	2	0	0	1	7	$\frac{7}{2}$

- Most negative entry in the z -row: -6 , so x_1 enters (the most profitable product).
- Ratios: $5 \div 1 = 5$, $10 \div 1 = 10$, $7 \div 2 = \frac{7}{2}$.
- Smallest is $\frac{7}{2}$: s_3 leaves. Machine time is the first resource to run out.
- Pivot on the 2: divide the s_3 row by 2; subtract $1 \times$ it from the s_1 and s_2 rows; add $6 \times$ it to the z -row.

Example 1: iteration 2

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Solution	
z	0	-2	2	0	0	3	21	
s_1	0	$\frac{1}{2}$	1	1	0	$-\frac{1}{2}$	$\frac{3}{2}$	
s_2	0	$\frac{3}{2}$	0	0	1	$-\frac{1}{2}$	$\frac{13}{2}$	
x_1	1	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	$\frac{7}{2}$	

- Now $x_1 = \frac{7}{2}$, $z = 21$. The z -row says $z = 21 + 2x_2 - 2x_3 - 3s_3$.

Example 1: iteration 2

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Solution	
z	0	-2	2	0	0	3	21	
s_1	0	$\frac{1}{2}$	1	1	0	$-\frac{1}{2}$	$\frac{3}{2}$	
s_2	0	$\frac{3}{2}$	0	0	1	$-\frac{1}{2}$	$\frac{13}{2}$	
x_1	1	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	$\frac{7}{2}$	

- Now $x_1 = \frac{7}{2}$, $z = 21$. The z -row says $z = 21 + 2x_2 - 2x_3 - 3s_3$.
- Only x_2 has a negative entry: x_2 enters. (Note that x_3 now has +2: making product 3 would *lower* the profit.)

Example 1: iteration 2

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Solution	Ratio
z	0	-2	2	0	0	3	21	
s_1	0	$\frac{1}{2}$	1	1	0	$-\frac{1}{2}$	$\frac{3}{2}$	3
s_2	0	$\frac{3}{2}$	0	0	1	$-\frac{1}{2}$	$\frac{13}{2}$	$\frac{13}{3}$
x_1	1	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	$\frac{7}{2}$	7

- Now $x_1 = \frac{7}{2}$, $z = 21$. The z -row says $z = 21 + 2x_2 - 2x_3 - 3s_3$.
- Only x_2 has a negative entry: x_2 enters. (Note that x_3 now has +2: making product 3 would *lower* the profit.)
- Ratios: $\frac{3}{2} \div \frac{1}{2} = 3$, $\frac{13}{2} \div \frac{3}{2} = \frac{13}{3}$, $\frac{7}{2} \div \frac{1}{2} = 7$.

Example 1: iteration 2

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Solution	Ratio
z	0	-2	2	0	0	3	21	
s_1	0	$\frac{1}{2}$	1	1	0	$-\frac{1}{2}$	$\frac{3}{2}$	3
s_2	0	$\frac{3}{2}$	0	0	1	$-\frac{1}{2}$	$\frac{13}{2}$	$\frac{13}{3}$
x_1	1	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	$\frac{7}{2}$	7

- Now $x_1 = \frac{7}{2}$, $z = 21$. The z -row says $z = 21 + 2x_2 - 2x_3 - 3s_3$.
- Only x_2 has a negative entry: x_2 enters. (Note that x_3 now has +2: making product 3 would *lower* the profit.)
- Ratios: $\frac{3}{2} \div \frac{1}{2} = 3$, $\frac{13}{2} \div \frac{3}{2} = \frac{13}{3}$, $\frac{7}{2} \div \frac{1}{2} = 7$.
- Smallest is 3: s_1 leaves. Timber runs out.

Example 1: the final tableau

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Solution
z	0	0	6	4	0	1	27
x_2	0	1	2	2	0	-1	3
s_2	0	0	-3	-3	1	1	2
x_1	1	0	0	-1	0	1	2

Example 1: the final tableau

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Solution
z	0	0	6	4	0	1	27
x_2	0	1	2	2	0	-1	3
s_2	0	0	-3	-3	1	1	2
x_1	1	0	0	-1	0	1	2

- No negative entries in the z -row: **optimal**.

Example 1: the final tableau

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Solution
z	0	0	6	4	0	1	27
x_2	0	1	2	2	0	-1	3
s_2	0	0	-3	-3	1	1	2
x_1	1	0	0	-1	0	1	2

- No negative entries in the z -row: **optimal**.
- $x_1 = 2$, $x_2 = 3$, $x_3 = 0$ (non-basic), $s_2 = 2$, $z = 27$: profit £2700 per week.

Example 1: the final tableau

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Solution
z	0	0	6	4	0	1	27
x_2	0	1	2	2	0	-1	3
s_2	0	0	-3	-3	1	1	2
x_1	1	0	0	-1	0	1	2

- No negative entries in the z -row: **optimal**.
- $x_1 = 2$, $x_2 = 3$, $x_3 = 0$ (non-basic), $s_2 = 2$, $z = 27$: profit £2700 per week.
- Check: timber $2 + 3 = 5$ ✓ (all used); labour $2 + 6 = 8 \leq 10$ (2 hours spare = s_2); machine $4 + 3 = 7$ ✓ (all used).

Example 1: the final tableau

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Solution
z	0	0	6	4	0	1	27
x_2	0	1	2	2	0	-1	3
s_2	0	0	-3	-3	1	1	2
x_1	1	0	0	-1	0	1	2

- No negative entries in the z -row: **optimal**.
- $x_1 = 2$, $x_2 = 3$, $x_3 = 0$ (non-basic), $s_2 = 2$, $z = 27$: profit £2700 per week.
- Check: timber $2 + 3 = 5$ ✓ (all used); labour $2 + 6 = 8 \leq 10$ (2 hours spare = s_2); machine $4 + 3 = 7$ ✓ (all used).
- Product 3 is never made: its z -row entry is $6 > 0$, so each unit of it would cost £600 of profit.

Example 1: what the z -row tells us

$$z = 27 - 6x_3 - 4s_1 - 0 \cdot s_2 - 1 \cdot s_3.$$

Example 1: what the z -row tells us

$$z = 27 - 6x_3 - 4s_1 - 0 \cdot s_2 - 1 \cdot s_3.$$

- Under the slack variables we read the **dual prices** (worth of the resources):

$$y_1 = 4, \quad y_2 = 0, \quad y_3 = 1.$$

A unit of timber is worth £400, an hour of machine time £100.

Example 1: what the z -row tells us

$$z = 27 - 6x_3 - 4s_1 - 0 \cdot s_2 - 1 \cdot s_3.$$

- Under the slack variables we read the **dual prices** (worth of the resources):

$$y_1 = 4, \quad y_2 = 0, \quad y_3 = 1.$$

A unit of timber is worth £400, an hour of machine time £100.

- Labour is worth nothing: 2 hours are already spare ($s_2 = 2$), so an extra hour cannot help.

Example 1: what the z -row tells us

$$z = 27 - 6x_3 - 4s_1 - 0 \cdot s_2 - 1 \cdot s_3.$$

- Under the slack variables we read the **dual prices** (worth of the resources):

$$y_1 = 4, \quad y_2 = 0, \quad y_3 = 1.$$

A unit of timber is worth £400, an hour of machine time £100.

- Labour is worth nothing: 2 hours are already spare ($s_2 = 2$), so an extra hour cannot help.
- Check with the dual: $w = 5 \cdot 4 + 10 \cdot 0 + 7 \cdot 1 = 27 = z$. ✓

Example 1: what the z -row tells us

$$z = 27 - 6x_3 - 4s_1 - 0 \cdot s_2 - 1 \cdot s_3.$$

- Under the slack variables we read the **dual prices** (worth of the resources):

$$y_1 = 4, \quad y_2 = 0, \quad y_3 = 1.$$

A unit of timber is worth £400, an hour of machine time £100.

- Labour is worth nothing: 2 hours are already spare ($s_2 = 2$), so an extra hour cannot help.
- Check with the dual: $w = 5 \cdot 4 + 10 \cdot 0 + 7 \cdot 1 = 27 = z$. ✓
- Why not product 3? Its resources are worth $2 \cdot 4 + 1 \cdot 0 + 2 \cdot 1 = 10$, more than its profit 4. The difference, $10 - 4 = 6$, is exactly its z -row entry.

Example 1: what the z -row tells us

$$z = 27 - 6x_3 - 4s_1 - 0 \cdot s_2 - 1 \cdot s_3.$$

- Under the slack variables we read the **dual prices** (worth of the resources):

$$y_1 = 4, \quad y_2 = 0, \quad y_3 = 1.$$

A unit of timber is worth £400, an hour of machine time £100.

- Labour is worth nothing: 2 hours are already spare ($s_2 = 2$), so an extra hour cannot help.
- Check with the dual: $w = 5 \cdot 4 + 10 \cdot 0 + 7 \cdot 1 = 27 = z$. ✓
- Why not product 3? Its resources are worth $2 \cdot 4 + 1 \cdot 0 + 2 \cdot 1 = 10$, more than its profit 4. The difference, $10 - 4 = 6$, is exactly its z -row entry.
- Product 3 becomes worth making only if its profit rises above £1000 per unit (or its resource use drops).

Example 2: a diet problem

Two foods, costing £2 and £3 per unit, must supply daily at least 4 units of protein and 6 units of vitamins.

	Food 1	Food 2	Needed
Protein	1	1	4
Vitamins	1	3	6
Cost (£)	2	3	

Example 2: a diet problem

Two foods, costing £2 and £3 per unit, must supply daily at least 4 units of protein and 6 units of vitamins.

	Food 1	Food 2	Needed
Protein	1	1	4
Vitamins	1	3	6
Cost (£)	2	3	

$$\begin{aligned} \min z = 2x_1 + 3x_2 \quad \text{s.t.} \quad & x_1 + x_2 \geq 4 \quad (1) \\ & x_1 + 3x_2 \geq 6 \quad (2) \end{aligned}$$

$$x_1, x_2 \geq 0.$$

Example 2: a diet problem

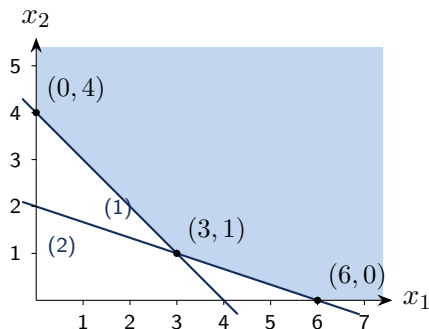
Two foods, costing £2 and £3 per unit, must supply daily at least 4 units of protein and 6 units of vitamins.

	Food 1	Food 2	Needed
Protein	1	1	4
Vitamins	1	3	6
Cost (£)	2	3	

$$\begin{aligned} \min z = 2x_1 + 3x_2 \quad \text{s.t.} \quad & x_1 + x_2 \geq 4 \quad (1) \\ & x_1 + 3x_2 \geq 6 \quad (2) \end{aligned}$$

$$x_1, x_2 \geq 0.$$

- Unbounded region; corners $(6, 0)$, $(3, 1)$, $(0, 4)$ with $z = 12, 9, 12$.



Example 2: a diet problem

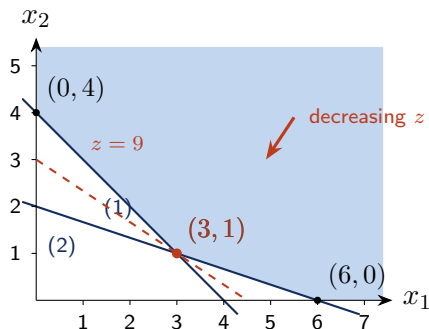
Two foods, costing £2 and £3 per unit, must supply daily at least 4 units of protein and 6 units of vitamins.

	Food 1	Food 2	Needed
Protein	1	1	4
Vitamins	1	3	6
Cost (£)	2	3	

$$\begin{aligned} \min z = 2x_1 + 3x_2 \quad \text{s.t.} \quad & x_1 + x_2 \geq 4 \quad (1) \\ & x_1 + 3x_2 \geq 6 \quad (2) \end{aligned}$$

$$x_1, x_2 \geq 0.$$

- Unbounded region; corners $(6, 0)$, $(3, 1)$, $(0, 4)$ with $z = 12, 9, 12$.
- Graphically: $(3, 1)$, $z = 9$.



Example 2: artificial variables and the penalty

- Subtract surplus variables x_3, x_4 : $x_1 + x_2 - x_3 = 4$, $x_1 + 3x_2 - x_4 = 6$.

Example 2: artificial variables and the penalty

- Subtract surplus variables x_3, x_4 : $x_1 + x_2 - x_3 = 4$, $x_1 + 3x_2 - x_4 = 6$.
- Setting $x_1 = x_2 = 0$ gives $x_3 = -4$, $x_4 = -6$: **not** a feasible start.

Example 2: artificial variables and the penalty

- Subtract surplus variables x_3, x_4 : $x_1 + x_2 - x_3 = 4$, $x_1 + 3x_2 - x_4 = 6$.
- Setting $x_1 = x_2 = 0$ gives $x_3 = -4$, $x_4 = -6$: **not** a feasible start.
- Add artificial variables x_5, x_6 and penalise them:

$$\begin{array}{ll} \text{minimise } z = 2x_1 + 3x_2 + M(x_5 + x_6) & \text{s.t.} \\ & x_1 + x_2 - x_3 + x_5 = 4 \\ & x_1 + 3x_2 - x_4 + x_6 = 6. \end{array}$$

Example 2: artificial variables and the penalty

- Subtract surplus variables x_3, x_4 : $x_1 + x_2 - x_3 = 4$, $x_1 + 3x_2 - x_4 = 6$.
- Setting $x_1 = x_2 = 0$ gives $x_3 = -4$, $x_4 = -6$: **not** a feasible start.
- Add artificial variables x_5, x_6 and penalise them:

$$\begin{array}{ll} \text{minimise } z = 2x_1 + 3x_2 + M(x_5 + x_6) & \text{s.t.} \\ x_1 + x_2 - x_3 + x_5 = 4 \\ x_1 + 3x_2 - x_4 + x_6 = 6. \end{array}$$

- Starting BFS: $x_5 = 4$, $x_6 = 6$, everything else 0.

Example 2: artificial variables and the penalty

- Subtract surplus variables x_3, x_4 : $x_1 + x_2 - x_3 = 4$, $x_1 + 3x_2 - x_4 = 6$.
- Setting $x_1 = x_2 = 0$ gives $x_3 = -4$, $x_4 = -6$: **not** a feasible start.
- Add artificial variables x_5, x_6 and penalise them:

$$\begin{array}{ll} \text{minimise } z = 2x_1 + 3x_2 + M(x_5 + x_6) & \text{s.t.} \\ & x_1 + x_2 - x_3 + x_5 = 4 \\ & x_1 + 3x_2 - x_4 + x_6 = 6. \end{array}$$

- Starting BFS: $x_5 = 4$, $x_6 = 6$, everything else 0.

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	-2	-3	0	0	$-M$	$-M$	0
x_5	1	1	-1	0	1	0	4
x_6	1	3	0	-1	0	1	6

The z -row is inconsistent (x_5, x_6 are basic but have entries $-M$).

Example 2: artificial variables and the penalty

- Subtract surplus variables x_3, x_4 : $x_1 + x_2 - x_3 = 4$, $x_1 + 3x_2 - x_4 = 6$.
- Setting $x_1 = x_2 = 0$ gives $x_3 = -4$, $x_4 = -6$: **not** a feasible start.
- Add artificial variables x_5, x_6 and penalise them:

$$\begin{array}{ll} \text{minimise } z = 2x_1 + 3x_2 + M(x_5 + x_6) & \text{s.t.} \\ & x_1 + x_2 - x_3 + x_5 = 4 \\ & x_1 + 3x_2 - x_4 + x_6 = 6. \end{array}$$

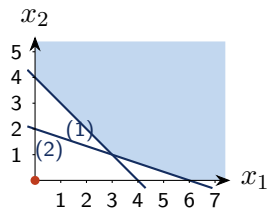
- Starting BFS: $x_5 = 4$, $x_6 = 6$, everything else 0.

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	$-2 + 2M$	$-3 + 4M$	$-M$	$-M$	0	0	$10M$
x_5	1	1	-1	0	1	0	4
x_6	1	3	0	-1	0	1	6

$z\text{-row} + M \times (x_5 \text{ row}) + M \times (x_6 \text{ row})$: now consistent, with $z = 10M$.

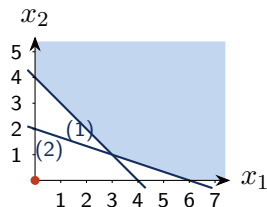
Example 2: iteration 1

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution	
z	$-2 + 2M$	$-3 + 4M$	$-M$	$-M$	0	0	$10M$	
x_5	1	1	-1	0	1	0	4	
x_6	1	3	0	-1	0	1	6	



Example 2: iteration 1

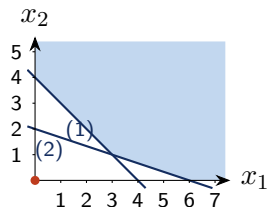
Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution	
z	$-2 + 2M$	$-3 + 4M$	$-M$	$-M$	0	0	$10M$	
x_5	1	1	-1	0	1	0	4	
x_6	1	3	0	-1	0	1	6	



- Minimisation: the *most positive* entry. M is huge, so compare the M parts: $4M > 2M$, x_2 enters.

Example 2: iteration 1

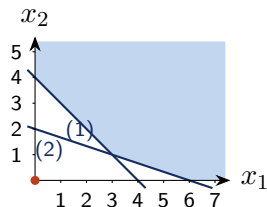
Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution	Ratio
z	$-2 + 2M$	$-3 + 4M$	$-M$	$-M$	0	0	$10M$	
x_5	1	1	-1	0	1	0	4	4
x_6	1	3	0	-1	0	1	6	2



- Minimisation: the *most positive* entry. M is huge, so compare the M parts: $4M > 2M$, x_2 enters.
- Ratios: $4 \div 1 = 4$, $6 \div 3 = 2$.

Example 2: iteration 1

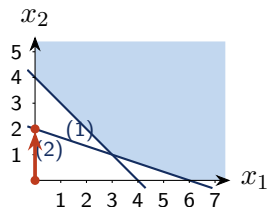
Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution	Ratio
z	$-2 + 2M$	$-3 + 4M$	$-M$	$-M$	0	0	$10M$	
x_5	1	1	-1	0	1	0	4	4
x_6	1	3	0	-1	0	1	6	2



- Minimisation: the *most positive* entry. M is huge, so compare the M parts: $4M > 2M$, x_2 enters.
- Ratios: $4 \div 1 = 4$, $6 \div 3 = 2$.
- Smallest is 2: x_6 leaves (one artificial variable gone).

Example 2: iteration 1

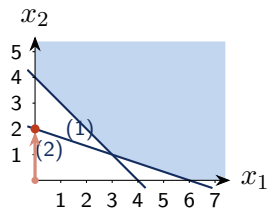
Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution	Ratio
z	$-2 + 2M$	$-3 + 4M$	$-M$	$-M$	0	0	$10M$	
x_5	1	1	-1	0	1	0	4	4
x_6	1	3	0	-1	0	1	6	2



- Minimisation: the *most positive* entry. M is huge, so compare the M parts: $4M > 2M$, x_2 enters.
- Ratios: $4 \div 1 = 4$, $6 \div 3 = 2$.
- Smallest is 2: x_6 leaves (one artificial variable gone).
- We are at $(x_1, x_2) = (0, 2)$: *outside* the feasible region; the artificial $x_5 = 2$ is covering the protein shortfall.

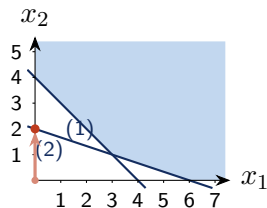
Example 2: iteration 2

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution	
z	$-1 + \frac{2}{3}M$	0	$-M$	$-1 + \frac{1}{3}M$	0	$1 - \frac{4}{3}M$	$6 + 2M$	
x_5	$\frac{2}{3}$	0	-1	$\frac{1}{3}$	1	$-\frac{1}{3}$	2	
x_2	$\frac{1}{3}$	1	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	2	



Example 2: iteration 2

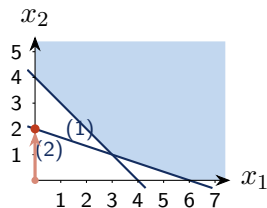
Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution	
z	$-1 + \frac{2}{3}M$	0	$-M$	$-1 + \frac{1}{3}M$	0	$1 - \frac{4}{3}M$	$6 + 2M$	
x_5	$\frac{2}{3}$	0	-1	$\frac{1}{3}$	1	$-\frac{1}{3}$	2	
x_2	$\frac{1}{3}$	1	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	2	



- The only entry with $+M$ is under x_1 : x_1 enters.

Example 2: iteration 2

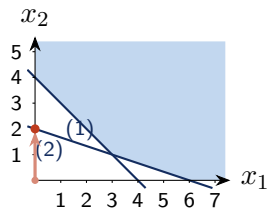
Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution	Ratio
z	$-1 + \frac{2}{3}M$	0	$-M$	$-1 + \frac{1}{3}M$	0	$1 - \frac{4}{3}M$	$6 + 2M$	
x_5	$\frac{2}{3}$	0	-1	$\frac{1}{3}$	1	$-\frac{1}{3}$	2	3
x_2	$\frac{1}{3}$	1	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	2	6



- The only entry with $+M$ is under x_1 : x_1 enters.
- Ratios: $2 \div \frac{2}{3} = 3$, $2 \div \frac{1}{3} = 6$.

Example 2: iteration 2

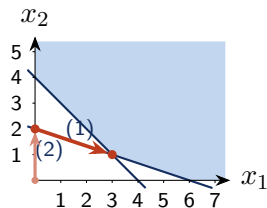
Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution	Ratio
z	$-1 + \frac{2}{3}M$	0	$-M$	$-1 + \frac{1}{3}M$	0	$1 - \frac{4}{3}M$	$6 + 2M$	
x_5	$\frac{2}{3}$	0	-1	$\frac{1}{3}$	1	$-\frac{1}{3}$	2	3
x_2	$\frac{1}{3}$	1	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	2	6



- The only entry with $+M$ is under x_1 : x_1 enters.
- Ratios: $2 \div \frac{2}{3} = 3$, $2 \div \frac{1}{3} = 6$.
- Smallest is 3: x_5 leaves (the last artificial variable).

Example 2: iteration 2

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution	Ratio
z	$-1 + \frac{2}{3}M$	0	$-M$	$-1 + \frac{1}{3}M$	0	$1 - \frac{4}{3}M$	$6 + 2M$	
x_5	$\frac{2}{3}$	0	-1	$\frac{1}{3}$	1	$-\frac{1}{3}$	2	3
x_2	$\frac{1}{3}$	1	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	2	6



- The only entry with $+M$ is under x_1 : x_1 enters.
- Ratios: $2 \div \frac{2}{3} = 3$, $2 \div \frac{1}{3} = 6$.
- Smallest is 3: x_5 leaves (the last artificial variable).
- This takes us to $(3, 1)$, into the feasible region.

Example 2: the final tableau

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	0	0	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{3}{2} - M$	$\frac{1}{2} - M$	9
x_1	1	0	$-\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	3
x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1

Example 2: the final tableau

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	0	0	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{3}{2} - M$	$\frac{1}{2} - M$	9
x_1	1	0	$-\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	3
x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1

- The entries under x_5, x_6 are $\frac{3}{2} - M, \frac{1}{2} - M$: negative. All others are ≤ 0 : **optimal** (minimisation).

Example 2: the final tableau

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	0	0	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{3}{2} - M$	$\frac{1}{2} - M$	9
x_1	1	0	$-\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	3
x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1

- The entries under x_5, x_6 are $\frac{3}{2} - M, \frac{1}{2} - M$: negative. All others are ≤ 0 : **optimal** (minimisation).
- $x_1 = 3, x_2 = 1, z = 9$: 3 units of food 1, 1 unit of food 2, £9 a day.

Example 2: the final tableau

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	0	0	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{3}{2} - M$	$\frac{1}{2} - M$	9
x_1	1	0	$-\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	3
x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1

- The entries under x_5, x_6 are $\frac{3}{2} - M, \frac{1}{2} - M$: negative. All others are ≤ 0 : **optimal** (minimisation).
- $x_1 = 3, x_2 = 1, z = 9$: 3 units of food 1, 1 unit of food 2, £9 a day.
- Both artificial variables are non-basic (zero), as they should be.

Example 2: the final tableau

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	0	0	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{3}{2} - M$	$\frac{1}{2} - M$	9
x_1	1	0	$-\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	3
x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1

- The entries under x_5, x_6 are $\frac{3}{2} - M, \frac{1}{2} - M$: negative. All others are ≤ 0 : **optimal** (minimisation).
- $x_1 = 3, x_2 = 1, z = 9$: 3 units of food 1, 1 unit of food 2, £9 a day.
- Both artificial variables are non-basic (zero), as they should be.
- The route was $(0, 0) \rightarrow (0, 2) \rightarrow (3, 1)$: the first two points are infeasible for the original problem; the penalty M drives the method into the region.

Example 2: the same by the two-phase method

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution	
old r	0	0	0	0	-1	-1	0	
new r	2	4	-1	-1	0	0	10	
x_5	1	1	-1	0	1	0	4	
x_6	1	3	0	-1	0	1	6	

- Phase I: minimise $r = x_5 + x_6$. The “new r ” row is the old one plus the x_5 and x_6 rows.

Example 2: the same by the two-phase method

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution	Ratio
old r	0	0	0	0	-1	-1	0	
new r	2	4	-1	-1	0	0	10	
x_5	1	1	-1	0	1	0	4	4
x_6	1	3	0	-1	0	1	6	2

- Phase I: minimise $r = x_5 + x_6$. The “new r ” row is the old one plus the x_5 and x_6 rows.
- x_2 enters (entry 4), ratios 4 and 2;

Example 2: the same by the two-phase method

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution	Ratio
old r	0	0	0	0	-1	-1	0	
new r	2	4	-1	-1	0	0	10	
x_5	1	1	-1	0	1	0	4	4
x_6	1	3	0	-1	0	1	6	2

- Phase I: minimise $r = x_5 + x_6$. The “new r ” row is the old one plus the x_5 and x_6 rows.
- x_2 enters (entry 4), ratios 4 and 2; x_6 leaves.

Example 2: the same by the two-phase method

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution	
r	$\frac{2}{3}$	0	-1	$\frac{1}{3}$	0	$-\frac{4}{3}$	2	
x_5	$\frac{2}{3}$	0	-1	$\frac{1}{3}$	1	$-\frac{1}{3}$	2	
x_2	$\frac{1}{3}$	1	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	2	

- After iteration 1: $r = 2$. x_1 enters (entry $\frac{2}{3}$), ratios 3 and 6;

Example 2: the same by the two-phase method

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution	Ratio
r	$\frac{2}{3}$	0	-1	$\frac{1}{3}$	0	$-\frac{4}{3}$	2	
x_5	$\frac{2}{3}$	0	-1	$\frac{1}{3}$	1	$-\frac{1}{3}$	2	3
x_2	$\frac{1}{3}$	1	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	2	6

- After iteration 1: $r = 2$. x_1 enters (entry $\frac{2}{3}$), ratios 3 and 6; x_5 leaves.

Example 2: the same by the two-phase method

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
r	0	0	0	0	-1	-1	0
x_1	1	0	$-\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	3
x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1

- After iteration 2: $r = 0$, so a feasible point has been found: $x_1 = 3$, $x_2 = 1$.

Example 2: the same by the two-phase method

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
r	0	0	0	0	-1	-1	0
x_1	1	0	$-\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	3
x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1

- After iteration 2: $r = 0$, so a feasible point has been found: $x_1 = 3$, $x_2 = 1$.
- Same pivots as the penalty method, with no M to carry around.

Example 2: Phase II

Drop x_5, x_6 and restore the real objective $z = 2x_1 + 3x_2$:

Example 2: Phase II

Drop x_5 , x_6 and restore the real objective $z = 2x_1 + 3x_2$:

Basic	x_1	x_2	x_3	x_4	Solution
z	0	0	$-\frac{3}{2}$	$-\frac{1}{2}$	9
x_1	1	0	$-\frac{3}{2}$	$\frac{1}{2}$	3
x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	1

- The z -row was made consistent by adding $2 \times (x_1 \text{ row}) + 3 \times (x_2 \text{ row})$ to $z - 2x_1 - 3x_2 = 0$.

Example 2: Phase II

Drop x_5 , x_6 and restore the real objective $z = 2x_1 + 3x_2$:

Basic	x_1	x_2	x_3	x_4	Solution
z	0	0	$-\frac{3}{2}$	$-\frac{1}{2}$	9
x_1	1	0	$-\frac{3}{2}$	$\frac{1}{2}$	3
x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	1

- The z -row was made consistent by adding $2 \times (x_1 \text{ row}) + 3 \times (x_2 \text{ row})$ to $z - 2x_1 - 3x_2 = 0$.
- All entries ≤ 0 : **already optimal**. Phase II needs no iteration here.

Example 2: Phase II

Drop x_5 , x_6 and restore the real objective $z = 2x_1 + 3x_2$:

Basic	x_1	x_2	x_3	x_4	Solution
z	0	0	$-\frac{3}{2}$	$-\frac{1}{2}$	9
x_1	1	0	$-\frac{3}{2}$	$\frac{1}{2}$	3
x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	1

- The z -row was made consistent by adding $2 \times (x_1 \text{ row}) + 3 \times (x_2 \text{ row})$ to $z - 2x_1 - 3x_2 = 0$.
- All entries ≤ 0 : **already optimal**. Phase II needs no iteration here.
- $x_1 = 3$, $x_2 = 1$, $z = 9$, as before.

Example 2: Phase II

Drop x_5 , x_6 and restore the real objective $z = 2x_1 + 3x_2$:

Basic	x_1	x_2	x_3	x_4	Solution
z	0	0	$-\frac{3}{2}$	$-\frac{1}{2}$	9
x_1	1	0	$-\frac{3}{2}$	$\frac{1}{2}$	3
x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	1

- The z -row was made consistent by adding $2 \times (x_1 \text{ row}) + 3 \times (x_2 \text{ row})$ to $z - 2x_1 - 3x_2 = 0$.
- All entries ≤ 0 : **already optimal**. Phase II needs no iteration here.
- $x_1 = 3$, $x_2 = 1$, $z = 9$, as before.
- Under the surplus variables: $-\frac{3}{2}$, $-\frac{1}{2}$. So the dual prices are $y_1 = \frac{3}{2}$, $y_2 = \frac{1}{2}$: one more unit of protein required would cost £1.50 a day, one more unit of vitamins £0.50.

Example 3: a bakery

A bakery makes two cakes. Per batch:

	Cake 1	Cake 2	Available
Flour (kg)	2	1	10
Sugar (kg)	1	1	6
Profit (£)	4	3	

At most 4 batches of cake 2 can be sold.

Example 3: a bakery

A bakery makes two cakes. Per batch:

	Cake 1	Cake 2	Available
Flour (kg)	2	1	10
Sugar (kg)	1	1	6
Profit (£)	4	3	

At most 4 batches of cake 2 can be sold.

$$2x_1 + x_2 \leq 10 \quad (1)$$

$$\max z = 4x_1 + 3x_2 \quad \text{s.t.} \quad x_1 + x_2 \leq 6 \quad (2)$$

$$x_2 \leq 4 \quad (3)$$

$$x_1, x_2 \geq 0.$$

Example 3: a bakery

A bakery makes two cakes. Per batch:

	Cake 1	Cake 2	Available
Flour (kg)	2	1	10
Sugar (kg)	1	1	6
Profit (£)	4	3	

At most 4 batches of cake 2 can be sold.

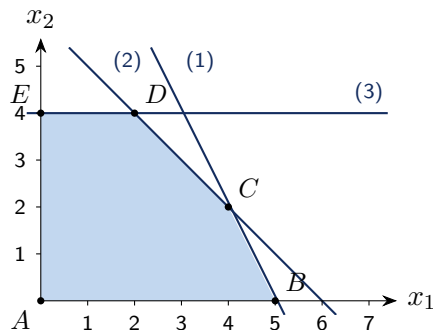
$$2x_1 + x_2 \leq 10 \quad (1)$$

$$\max z = 4x_1 + 3x_2 \quad \text{s.t.} \quad x_1 + x_2 \leq 6 \quad (2)$$

$$x_2 \leq 4 \quad (3)$$

$$x_1, x_2 \geq 0.$$

- Corners $A, \dots, E$: $z = 0, 20, 22, 20, 12$.



Example 3: a bakery

A bakery makes two cakes. Per batch:

	Cake 1	Cake 2	Available
Flour (kg)	2	1	10
Sugar (kg)	1	1	6
Profit (£)	4	3	

At most 4 batches of cake 2 can be sold.

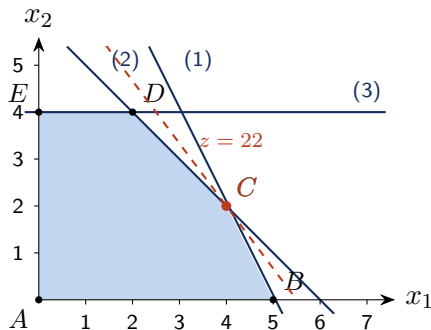
$$2x_1 + x_2 \leq 10 \quad (1)$$

$$\max z = 4x_1 + 3x_2 \quad \text{s.t.} \quad x_1 + x_2 \leq 6 \quad (2)$$

$$x_2 \leq 4 \quad (3)$$

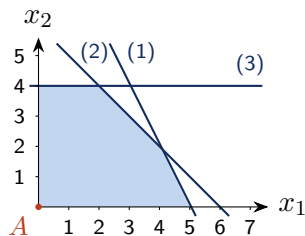
$$x_1, x_2 \geq 0.$$

- Corners $A, \dots, E$: $z = 0, 20, 22, 20, 12$.
Optimum $C = (4, 2)$, $z = 22$.



Example 3: iteration 1

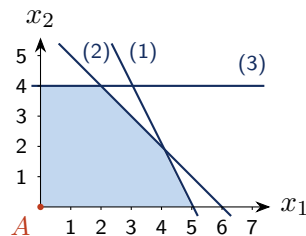
Basic	x_1	x_2	s_1	s_2	s_3	Solution	
z	-4	-3	0	0	0	0	
s_1	2	1	1	0	0	10	
s_2	1	1	0	1	0	6	
s_3	0	1	0	0	1	4	



- Slack variables: s_1 = flour left, s_2 = sugar left, s_3 = unsold demand for cake 2.

Example 3: iteration 1

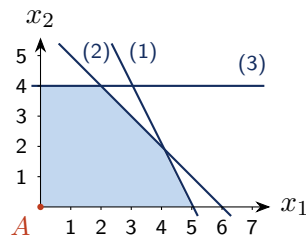
Basic	x_1	x_2	s_1	s_2	s_3	Solution	
z	-4	-3	0	0	0	0	
s_1	2	1	1	0	0	10	
s_2	1	1	0	1	0	6	
s_3	0	1	0	0	1	4	



- Slack variables: s_1 = flour left, s_2 = sugar left, s_3 = unsold demand for cake 2.
- x_1 enters (-4 is the most negative).

Example 3: iteration 1

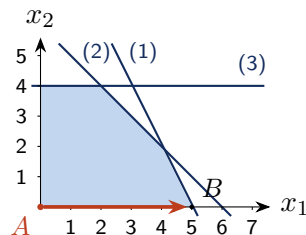
Basic	x_1	x_2	s_1	s_2	s_3	Solution	Ratio
z	-4	-3	0	0	0	0	
s_1	2	1	1	0	0	10	5
s_2	1	1	0	1	0	6	6
s_3	0	1	0	0	1	4	∞



- Slack variables: s_1 = flour left, s_2 = sugar left, s_3 = unsold demand for cake 2.
- x_1 enters (-4 is the most negative).
- Ratios: $10 \div 2 = 5$, $6 \div 1 = 6$, $4 \div 0$: no limit.

Example 3: iteration 1

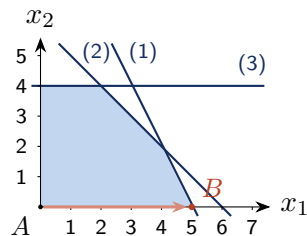
Basic	x_1	x_2	s_1	s_2	s_3	Solution	Ratio
z	-4	-3	0	0	0	0	
s_1	2	1	1	0	0	10	5
s_2	1	1	0	1	0	6	6
s_3	0	1	0	0	1	4	∞



- Slack variables: s_1 = flour left, s_2 = sugar left, s_3 = unsold demand for cake 2.
- x_1 enters (-4 is the most negative).
- Ratios: $10 \div 2 = 5$, $6 \div 1 = 6$, $4 \div 0$: no limit.
- s_1 leaves: flour runs out first, at $B = (5, 0)$.

Example 3: iteration 2

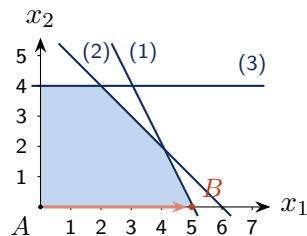
Basic	x_1	x_2	s_1	s_2	s_3	Solution	
z	0	-1	2	0	0	20	
x_1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	5	
s_2	0	$\frac{1}{2}$	$-\frac{1}{2}$	1	0	1	
s_3	0	1	0	0	1	4	



- At B : $x_1 = 5$, $z = 20$, and $z = 20 + x_2 - 2s_1$.

Example 3: iteration 2

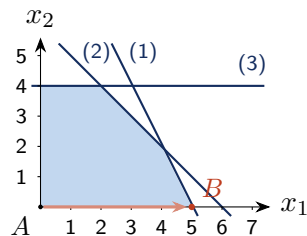
Basic	x_1	x_2	s_1	s_2	s_3	Solution	
z	0	-1	2	0	0	20	
x_1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	5	
s_2	0	$\frac{1}{2}$	$-\frac{1}{2}$	1	0	1	
s_3	0	1	0	0	1	4	



- At B : $x_1 = 5$, $z = 20$, and $z = 20 + x_2 - 2s_1$.
- x_2 enters.

Example 3: iteration 2

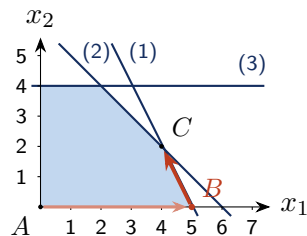
Basic	x_1	x_2	s_1	s_2	s_3	Solution	Ratio
z	0	-1	2	0	0	20	
x_1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	5	10
s_2	0	$\frac{1}{2}$	$-\frac{1}{2}$	1	0	1	2
s_3	0	1	0	0	1	4	4



- At B : $x_1 = 5$, $z = 20$, and $z = 20 + x_2 - 2s_1$.
- x_2 enters.
- Ratios: $5 \div \frac{1}{2} = 10$, $1 \div \frac{1}{2} = 2$, $4 \div 1 = 4$.

Example 3: iteration 2

Basic	x_1	x_2	s_1	s_2	s_3	Solution	Ratio
z	0	-1	2	0	0	20	
x_1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	5	10
s_2	0	$\frac{1}{2}$	$-\frac{1}{2}$	1	0	1	2
s_3	0	1	0	0	1	4	4



- At B : $x_1 = 5$, $z = 20$, and $z = 20 + x_2 - 2s_1$.
- x_2 enters.
- Ratios: $5 \div \frac{1}{2} = 10$, $1 \div \frac{1}{2} = 2$, $4 \div 1 = 4$.
- s_2 leaves: moving along the flour line (1), the sugar runs out at $C = (4, 2)$.

Example 3: the final tableau and the dual prices

Basic	x_1	x_2	s_1	s_2	s_3	Solution
z	0	0	1	2	0	22
x_1	1	0	1	-1	0	4
x_2	0	1	-1	2	0	2
s_3	0	0	1	-2	1	2

Example 3: the final tableau and the dual prices

Basic	x_1	x_2	s_1	s_2	s_3	Solution
z	0	0	1	2	0	22
x_1	1	0	1	-1	0	4
x_2	0	1	-1	2	0	2
s_3	0	0	1	-2	1	2

- Optimal: $x_1 = 4$, $x_2 = 2$, $s_3 = 2$, $z = 22$. Profit £22; two batches of cake 2 could still be sold.

Example 3: the final tableau and the dual prices

Basic	x_1	x_2	s_1	s_2	s_3	Solution
z	0	0	1	2	0	22
x_1	1	0	1	-1	0	4
x_2	0	1	-1	2	0	2
s_3	0	0	1	-2	1	2

- Optimal: $x_1 = 4$, $x_2 = 2$, $s_3 = 2$, $z = 22$. Profit £22; two batches of cake 2 could still be sold.
- z -row under s_1, s_2, s_3 : $y_1 = 1$, $y_2 = 2$, $y_3 = 0$. A kilo of flour is worth £1, a kilo of sugar £2, extra demand nothing.

Example 3: the final tableau and the dual prices

Basic	x_1	x_2	s_1	s_2	s_3	Solution
z	0	0	1	2	0	22
x_1	1	0	1	-1	0	4
x_2	0	1	-1	2	0	2
s_3	0	0	1	-2	1	2

- Optimal: $x_1 = 4$, $x_2 = 2$, $s_3 = 2$, $z = 22$. Profit £22; two batches of cake 2 could still be sold.
- z -row under s_1, s_2, s_3 : $y_1 = 1$, $y_2 = 2$, $y_3 = 0$. A kilo of flour is worth £1, a kilo of sugar £2, extra demand nothing.
- Dual problem: minimise $w = 10y_1 + 6y_2 + 4y_3$ s.t. $2y_1 + y_2 \geq 4$, $y_1 + y_2 + y_3 \geq 3$.
(1, 2, 0): $2 + 2 = 4$ ✓, $1 + 2 + 0 = 3$ ✓, $w = 10 + 12 = 22 = z$. ✓

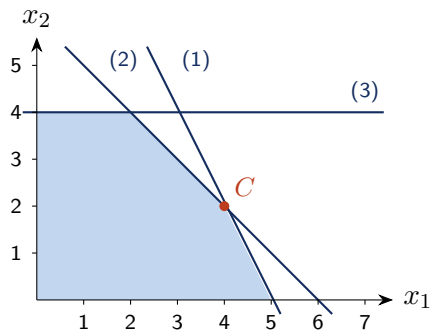
Example 3: the final tableau and the dual prices

Basic	x_1	x_2	s_1	s_2	s_3	Solution
z	0	0	1	2	0	22
x_1	1	0	1	-1	0	4
x_2	0	1	-1	2	0	2
s_3	0	0	1	-2	1	2

- Optimal: $x_1 = 4$, $x_2 = 2$, $s_3 = 2$, $z = 22$. Profit £22; two batches of cake 2 could still be sold.
- z -row under s_1, s_2, s_3 : $y_1 = 1$, $y_2 = 2$, $y_3 = 0$. A kilo of flour is worth £1, a kilo of sugar £2, extra demand nothing.
- Dual problem: minimise $w = 10y_1 + 6y_2 + 4y_3$ s.t. $2y_1 + y_2 \geq 4$, $y_1 + y_2 + y_3 \geq 3$.
(1, 2, 0): $2 + 2 = 4$ ✓, $1 + 2 + 0 = 3$ ✓, $w = 10 + 12 = 22 = z$. ✓
- A third cake needing 1 kg flour and 2 kg sugar, profit £4? Resources worth $1 \cdot 1 + 2 \cdot 2 = 5 > 4$: **not worth making**.

Example 3: sensitivity to the amount of flour

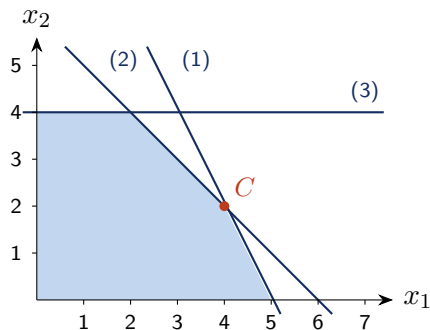
Change the flour from 10 to $10 + D$ kg.



Example 3: sensitivity to the amount of flour

Change the flour from 10 to $10 + D$ kg. The columns under s_1, s_2, s_3 in the final tableau are B^{-1} ; its first column is $(1, -1, 1)^T$, so

$$x_1 = 4 + D, \quad x_2 = 2 - D, \quad s_3 = 2 + D.$$

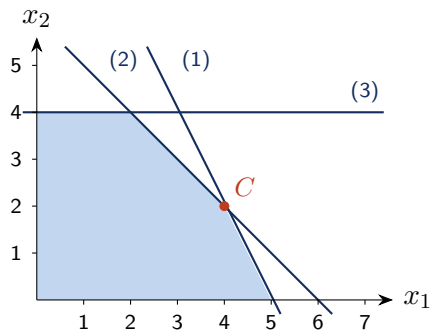


Example 3: sensitivity to the amount of flour

Change the flour from 10 to $10 + D$ kg. The columns under s_1, s_2, s_3 in the final tableau are B^{-1} ; its first column is $(1, -1, 1)^T$, so

$$x_1 = 4 + D, \quad x_2 = 2 - D, \quad s_3 = 2 + D.$$

- All ≥ 0 iff $-2 \leq D \leq 2$: $8 \leq \text{flour} \leq 12$.

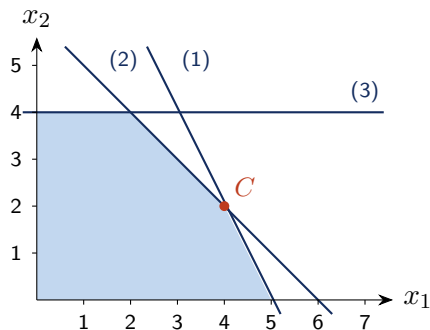


Example 3: sensitivity to the amount of flour

Change the flour from 10 to $10 + D$ kg. The columns under s_1, s_2, s_3 in the final tableau are B^{-1} ; its first column is $(1, -1, 1)^T$, so

$$x_1 = 4 + D, \quad x_2 = 2 - D, \quad s_3 = 2 + D.$$

- All ≥ 0 iff $-2 \leq D \leq 2$: $8 \leq \text{flour} \leq 12$.
- In this range the same corner (flour and sugar both used up) stays optimal and each extra kilo of flour is worth $y_1 = \pounds 1$: $z = 22 + D$.

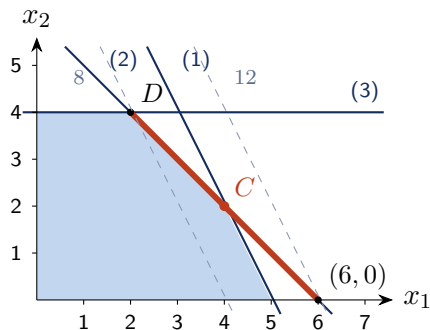


Example 3: sensitivity to the amount of flour

Change the flour from 10 to $10 + D$ kg. The columns under s_1, s_2, s_3 in the final tableau are B^{-1} ; its first column is $(1, -1, 1)^T$, so

$$x_1 = 4 + D, \quad x_2 = 2 - D, \quad s_3 = 2 + D.$$

- All ≥ 0 iff $-2 \leq D \leq 2$: $8 \leq \text{flour} \leq 12$.
- In this range the same corner (flour and sugar both used up) stays optimal and each extra kilo of flour is worth $y_1 = \pounds 1$: $z = 22 + D$.
- The corner slides along the sugar line (2): from $D = (2, 4)$ at 8 kg, where the demand limit takes over, to $(6, 0)$ at 12 kg, where cake 2 drops out.



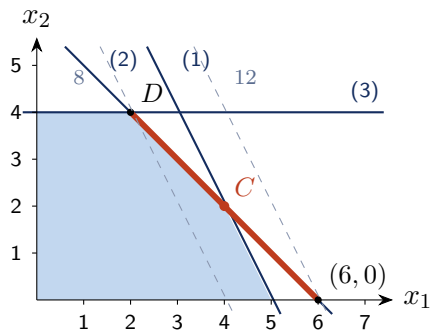
Dashed: line (1) for 8 kg and 12 kg of flour.

Example 3: sensitivity to the amount of flour

Change the flour from 10 to $10 + D$ kg. The columns under s_1, s_2, s_3 in the final tableau are B^{-1} ; its first column is $(1, -1, 1)^T$, so

$$x_1 = 4 + D, \quad x_2 = 2 - D, \quad s_3 = 2 + D.$$

- All ≥ 0 iff $-2 \leq D \leq 2$: $8 \leq \text{flour} \leq 12$.
- In this range the same corner (flour and sugar both used up) stays optimal and each extra kilo of flour is worth $y_1 = \pounds 1$: $z = 22 + D$.
- The corner slides along the sugar line (2): from $D = (2, 4)$ at 8 kg, where the demand limit takes over, to $(6, 0)$ at 12 kg, where cake 2 drops out.
- Beyond 12 kg, more flour is useless: sugar is the only limit.



Dashed: line (1) for 8 kg and 12 kg of flour.