

Linear Programming

MAC175 Examples Sheet 1 — Solutions

How to set up an LP problem

Every formulation question follows the same three steps.

1. **Variables.** Decide what you are free to choose, and give each quantity a name and a unit.
2. **Objective.** Write the quantity to be maximised or minimised (profit, cost, pleasure, ...) as a linear function of the variables.
3. **Constraints.** Translate every restriction in the question into a linear inequality or equation. Do not forget $x_j \geq 0$.

Canonical form (notes, §1.3)

minimise $\mathbf{c}^T \mathbf{x}$ subject to $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq 0$ (and we also arrange $\mathbf{b} \geq 0$).

Q1(a) The butcher's meat loaf: data and variables

	Meat	Fat	Cost (pence per kg)
Beef	80%	20%	80
Pork	68%	32%	60

Variables (as given in the question): in 1 kg of meat loaf,

$$x_1 = \text{kg of beef}, \quad x_2 = \text{kg of pork}.$$

Objective. The cost of 1 kg of meat loaf, in pence, is

$$z = 80x_1 + 60x_2 \quad (\text{to be minimised}).$$

Q1(a) The butcher's meat loaf: constraints

- **Total weight.** The loaf weighs 1 kg: $x_1 + x_2 = 1$.
- **Fat.** Beef gives $0.20x_1$ kg of fat, pork gives $0.32x_2$ kg. At most 25% of 1 kg, i.e. 0.25 kg, may be fat:

$$0.20x_1 + 0.32x_2 \leq 0.25 \iff 20x_1 + 32x_2 \leq 25.$$

- **Non-negativity.** $x_1, x_2 \geq 0$.

LP problem

$$\begin{aligned} & \text{minimise } z = 80x_1 + 60x_2 \quad \text{subject to} \quad \begin{aligned} & x_1 + x_2 = 1, \\ & 20x_1 + 32x_2 \leq 25, \\ & x_1, x_2 \geq 0. \end{aligned} \end{aligned}$$

Q1(b) The furniture maker

	Wood	Hours	Price
Model I	2	7	£120
Model II	1	8	£80
Available	6	28	

Variables:

- x_1 = number of Model I made,
- x_2 = number of Model II made.

Objective: the income (in £)

$$z = 120x_1 + 80x_2.$$

Constraints: the wood and the time used cannot exceed what is available.

LP problem

maximise $z = 120x_1 + 80x_2$ subject to

$$2x_1 + x_2 \leq 6 \quad (\text{wood})$$

$$7x_1 + 8x_2 \leq 28 \quad (\text{time})$$

$$x_1, x_2 \geq 0.$$

Strictly, x_1, x_2 should be whole numbers (an *integer* programming problem); here we ignore this.

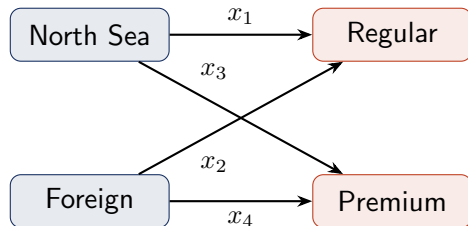
Q1(c) Aztec Oil: choosing the variables

Each petrol is a blend of the two oils, so we need to know how much of *each oil* goes into *each petrol*.

Barrels per week:

	Regular	Premium
North Sea oil	x_1	x_3
Foreign oil	x_2	x_4

- Regular made: $x_1 + x_2$; premium made: $x_3 + x_4$.
- North Sea used: $x_1 + x_3$; foreign used: $x_2 + x_4$.



Q1(c) Aztec Oil: the profit

Weekly profit = income from sales – cost of the oil.

- Income: $12(x_1 + x_2) + 14(x_3 + x_4)$.
- Cost of oil: $8(x_1 + x_3) + 15(x_2 + x_4)$.

Profit on each barrel of oil, according to where it goes:

	into Regular	into Premium
North Sea	$12 - 8 = 4$	$14 - 8 = 6$
Foreign	$12 - 15 = -3$	$14 - 15 = -1$

maximise $z = 4x_1 - 3x_2 + 6x_3 - x_4$ ($\pounds$ per week).

Q1(c) Aztec Oil: quality constraints

We assume the vapour pressure and octane rating of a blend are the *averages* of those of the oils, weighted by volume.

- **Vapour pressure of regular** at most 23:

$$\frac{25x_1 + 15x_2}{x_1 + x_2} \leq 23 \iff 25x_1 + 15x_2 \leq 23x_1 + 23x_2 \iff 2x_1 - 8x_2 \leq 0.$$

- **Vapour pressure of premium:** in the same way $2x_3 - 8x_4 \leq 0$.
- **Octane of regular** at least 88:

$$87x_1 + 98x_2 \geq 88(x_1 + x_2) \iff -x_1 + 10x_2 \geq 0 \iff x_1 - 10x_2 \leq 0.$$

- **Octane of premium** at least 93:

$$87x_3 + 98x_4 \geq 93(x_3 + x_4) \iff -6x_3 + 5x_4 \geq 0 \iff 6x_3 - 5x_4 \leq 0.$$

(Multiplying by $x_1 + x_2 > 0$ is allowed: at least 50 000 barrels of regular are delivered.)

Q1(c) Aztec Oil: quantity constraints and the full LP

- **Demand** (maximum):

$$x_1 + x_2 \leq 100\,000,$$

$$x_3 + x_4 \leq 20\,000.$$

- **Deliveries** (minimum):

$$x_1 + x_2 \geq 50\,000,$$

$$x_3 + x_4 \geq 5\,000.$$

- **Inventory** (per week):

$$x_1 + x_3 \leq 40\,000,$$

$$x_2 + x_4 \leq 60\,000.$$

- $x_1, x_2, x_3, x_4 \geq 0.$

LP problem

maximise $z = 4x_1 - 3x_2 + 6x_3 - x_4$ subject to

$$2x_1 - 8x_2 \leq 0,$$

$$2x_3 - 8x_4 \leq 0,$$

$$x_1 - 10x_2 \leq 0,$$

$$6x_3 - 5x_4 \leq 0,$$

$$x_1 + x_2 \leq 100\,000,$$

$$x_3 + x_4 \leq 20\,000,$$

$$x_1 + x_2 \geq 50\,000,$$

$$x_3 + x_4 \geq 5\,000,$$

$$x_1 + x_3 \leq 40\,000,$$

$$x_2 + x_4 \leq 60\,000,$$

$$x_1, \dots, x_4 \geq 0.$$

Q1(c) Aztec Oil: canonical form

- Maximising z = minimising $-z$.
- Add a *slack* variable to each $\leq$ constraint: $x_5, \dots, x_{10}, x_{13}, x_{14}$.
- Subtract a *surplus* variable from each $\geq$ constraint: x_{11}, x_{12} .
- All right-hand sides are already ≥ 0 .
- Result: 10 equations in 14 non-negative unknowns.

minimise $-z = -4x_1 + 3x_2 - 6x_3 + x_4$ subject to

$$2x_1 - 8x_2 + x_5 = 0,$$

$$2x_3 - 8x_4 + x_6 = 0,$$

$$x_1 - 10x_2 + x_7 = 0,$$

$$6x_3 - 5x_4 + x_8 = 0,$$

$$x_1 + x_2 + x_9 = 100\,000,$$

$$x_3 + x_4 + x_{10} = 20\,000,$$

$$x_1 + x_2 - x_{11} = 50\,000,$$

$$x_3 + x_4 - x_{12} = 5\,000,$$

$$x_1 + x_3 + x_{13} = 40\,000,$$

$$x_2 + x_4 + x_{14} = 60\,000,$$

$$x_1, \dots, x_{14} \geq 0.$$

Q2(a)

Minimise $z = x_1 + 2x_2 + 3x_3$ subject to

$$3x_1 + 4x_3 \leq 5, \quad 5x_1 + x_2 + 6x_3 \leq 7, \quad 8x_1 + 9x_3 \geq 2, \quad x_1, x_2, x_3 \geq 0.$$

- It is already a minimisation, and every right-hand side is ≥ 0 .
- Slack variables x_4, x_5 for the two $\leq$ constraints; surplus variable x_6 for the $\geq$ constraint.

Canonical form

$$\begin{aligned} & \text{minimise } z = x_1 + 2x_2 + 3x_3 \quad \text{subject to} \quad \begin{aligned} 3x_1 + 4x_3 + x_4 &= 5, \\ 5x_1 + x_2 + 6x_3 + x_5 &= 7, \\ 8x_1 + 9x_3 - x_6 &= 2, \end{aligned} \\ & x_1, \dots, x_6 \geq 0. \end{aligned}$$

Q2(a) in matrix form

The same problem as minimise $\mathbf{c}^T \mathbf{x}$ subject to $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq 0$, with

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}, \quad \mathbf{c}^T = (1, 2, 3, 0, 0, 0),$$

$$A = \begin{pmatrix} 3 & 0 & 4 & 1 & 0 & 0 \\ 5 & 1 & 6 & 0 & 1 & 0 \\ 8 & 0 & 9 & 0 & 0 & -1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 5 \\ 7 \\ 2 \end{pmatrix}.$$

The slack and surplus variables have cost 0: they do not change z .

Q2(b)

Minimise $z = 2x_1 + 3x_2 + x_3$ subject to

$$x_1 + 2x_2 + 2x_3 - x_4 + x_5 = 3, \quad 2x_1 + 3x_2 + 4x_3 + x_6 = 6, \quad x_1, \dots, x_6 \geq 0.$$

Check the requirements of the canonical form:

- a minimisation ✓ all constraints are equations ✓ all variables ≥ 0 ✓
right-hand sides ≥ 0 ✓

So the problem is **already in canonical form**; we only need to write it out:

$$\mathbf{c}^T = (2, 3, 1, 0, 0, 0), \quad A = \begin{pmatrix} 1 & 2 & 2 & -1 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}.$$

(x_4, x_5, x_6 do not appear in z , so their costs are 0.)

Q2(c) Free variables

Minimise $z = 25x_1 + 30x_2$ subject to

$$4x_1 + 7x_2 \geq 1, \quad 8x_1 + 5x_2 \geq 3, \quad 6x_1 + 9x_2 \geq -2, \quad x_1, x_2 \text{ unrestricted.}$$

Step 1: remove the free variables (notes, §1.3(iv)). Put

$$x_1 = y_1 - y_3, \quad x_2 = y_2 - y_3, \quad y_1, y_2, y_3 \geq 0.$$

- One extra variable y_3 is enough: given any x_1, x_2 , take y_3 large enough that $y_1 = x_1 + y_3 \geq 0$ and $y_2 = x_2 + y_3 \geq 0$.
- (Writing $x_1 = u_1 - v_1$, $x_2 = u_2 - v_2$ also works, with four new variables.)

Then $z = 25(y_1 - y_3) + 30(y_2 - y_3) = 25y_1 + 30y_2 - 55y_3$.

Q2(c) Constraints in the new variables

Step 2: substitute.

$$4x_1 + 7x_2 = 4y_1 + 7y_2 - 11y_3 \geq 1,$$

$$8x_1 + 5x_2 = 8y_1 + 5y_2 - 13y_3 \geq 3,$$

$$6x_1 + 9x_2 = 6y_1 + 9y_2 - 15y_3 \geq -2.$$

Step 3: make the right-hand sides non-negative. Multiply the last constraint by -1 (this reverses the inequality):

$$-6y_1 - 9y_2 + 15y_3 \leq 2.$$

Step 4: slack and surplus. Surplus y_4 , y_5 for the first two constraints, slack y_6 for the third.

Q2(c) Canonical form

Canonical form

$$\begin{aligned} \text{minimise } z = 25y_1 + 30y_2 - 55y_3 \quad \text{subject to} \quad & 4y_1 + 7y_2 - 11y_3 - y_4 = 1, \\ & 8y_1 + 5y_2 - 13y_3 - y_5 = 3, \\ & -6y_1 - 9y_2 + 15y_3 + y_6 = 2, \\ & y_1, \dots, y_6 \geq 0. \end{aligned}$$

In matrix form:

$$\mathbf{c}^T = (25, 30, -55, 0, 0, 0), \quad A = \begin{pmatrix} 4 & 7 & -11 & -1 & 0 & 0 \\ 8 & 5 & -13 & 0 & -1 & 0 \\ -6 & -9 & 15 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}.$$

Once solved, recover $x_1 = y_1 - y_3$, $x_2 = y_2 - y_3$.

Q3 Jack's day: formulation

Variables: x_1 = hours of work per day, x_2 = hours of play per day.

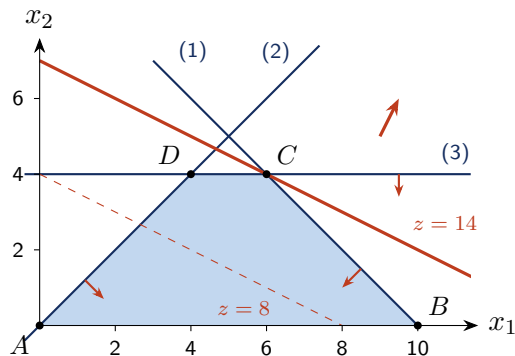
Objective. Play is twice as much fun as work. Counting one hour of work as one unit of pleasure,

$$\text{maximise } z = x_1 + 2x_2.$$

Constraints.

- 10 hours available: $x_1 + x_2 \leq 10$ (1)
- at least as much work as play: $x_1 \geq x_2$, i.e. $-x_1 + x_2 \leq 0$ (2)
- at most 4 hours of play: $x_2 \leq 4$ (3)
- $x_1, x_2 \geq 0$.

Q3 Jack's day: graphical solution



- The feasible region is the quadrilateral $ABCD$.
- Profit lines $x_1 + 2x_2 = z$ have slope $-\frac{1}{2}$; z increases in the direction $(1, 2)$.

Corner	(x_1, x_2)	$z = x_1 + 2x_2$
A	$(0, 0)$	0
B	$(10, 0)$	10
C	$(6, 4)$	14
D	$(4, 4)$	12

C is where (1) meets (3): $x_2 = 4$,
 $x_1 = 10 - 4 = 6$.

Q3 Jack's day: answer

Optimum

Jack should **work 6 hours and play 4 hours** each day: maximum pleasure $z^* = 14$ units.

- At C two constraints are *active*: all 10 hours are used, and play is at its limit of 4 hours.
- The constraint “work $\geq$ play” is *not* active: $6 > 4$.
- If the 10 hours must all be used ($x_1 + x_2 = 10$), the feasible region is just the edge BC , and the answer is the same point C .

Q4 The farmer's feed: formulation

Variables: x_1 = kg of corn per day, x_2 = kg of soya bean meal per day.

Objective: minimise the daily cost $z = 0.3x_1 + 0.9x_2$ (\$).

Constraints. The mix weighs $x_1 + x_2$ kg.

- At least 800 kg:
$$x_1 + x_2 \geq 800 \quad (1)$$

- Protein at least 30% of the mix:

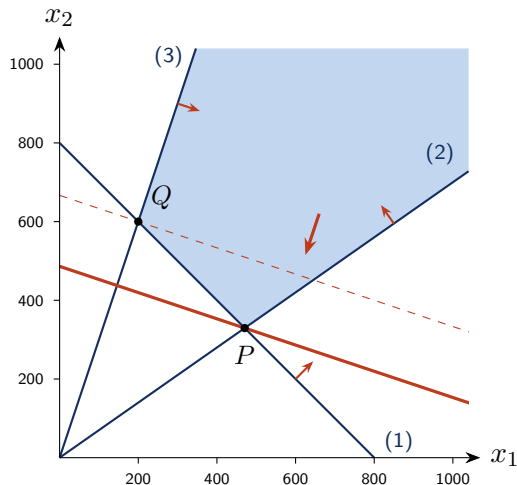
$$0.09x_1 + 0.6x_2 \geq 0.3(x_1 + x_2) \iff 0.3x_2 \geq 0.21x_1 \iff x_2 \geq 0.7x_1 \quad (2)$$

- Fibre at most 5% of the mix:

$$0.02x_1 + 0.06x_2 \leq 0.05(x_1 + x_2) \iff 0.01x_2 \leq 0.03x_1 \iff x_2 \leq 3x_1 \quad (3)$$

- $x_1, x_2 \geq 0$.

Q4 The farmer's feed: graphical solution



- (2) and (3) are lines through the origin; the feasible region is the part of the wedge between them beyond line (1). It is *unbounded*.
- Cost lines $0.3x_1 + 0.9x_2 = z$ have slope $-\frac{1}{3}$. We are *minimising*, so slide them towards the origin (thick arrow, direction $-(0.3, 0.9)$).
- The last point of the region touched is the corner P .

Corner	(x_1, x_2)	z (\$)
P	$(470.6, 329.4)$	437.65
Q	$(200, 600)$	600.00

Q4 The farmer's feed: the optimal corner

P lies on (1) and (2):

$$x_1 + x_2 = 800, \quad x_2 = 0.7x_1 \implies 1.7x_1 = 800 \implies x_1 = \frac{8000}{17} \approx 470.6,$$

$$x_2 = 0.7x_1 = \frac{5600}{17} \approx 329.4, \quad z^* = 0.3 \cdot \frac{8000}{17} + 0.9 \cdot \frac{5600}{17} = \frac{7440}{17} \approx 437.65.$$

(Q: $x_1 + x_2 = 800$, $x_2 = 3x_1$ give $x_1 = 200$, $x_2 = 600$, $z = 60 + 540 = 600$.)

Answer

Use about **470.6 kg of corn** and **329.4 kg of soya bean meal** a day: minimum cost **\$437.65** per day.

Check: protein = 240 kg = 30% of 800 kg ✓, fibre ≈ 29.2 kg $\approx 3.6\%$ ✓.

Q5 Setting up

The constraints of Q2(b) are $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq 0$, with

$$A = \begin{pmatrix} 1 & 2 & 2 & -1 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}.$$

Call the columns $\mathbf{a}_1, \dots, \mathbf{a}_6$:

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \mathbf{a}_3 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \mathbf{a}_4 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \mathbf{a}_5 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{a}_6 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Reminder (notes, §1.5)

$\mathbf{x}$ is a *basic* solution if $A\mathbf{x} = \mathbf{b}$ and the columns belonging to its non-zero components are *linearly independent*. It is a *basic feasible solution* (BFS) if, in addition, $\mathbf{x} \geq 0$.

Here $m = 2$, so a basic solution has **at most 2** non-zero components.

Q5 The vector $\mathbf{x} = (1, 0, 1, 0, 0, 0)^T$

- **It is feasible:** $\mathbf{x} \geq 0$ and

$$A\mathbf{x} = 1 \cdot \mathbf{a}_1 + 1 \cdot \mathbf{a}_3 = \begin{pmatrix} 1 + 2 \\ 2 + 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \mathbf{b}.$$

- **It is not basic:** the non-zero components are x_1 and x_3 , but

$$\mathbf{a}_3 = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2 \mathbf{a}_1,$$

so $\mathbf{a}_1, \mathbf{a}_3$ are linearly *dependent*.

- **Picture (Theorem 1.6):** it is not a vertex, because it lies between two feasible points:

$$\mathbf{x} = \frac{1}{3} (3, 0, 0, 0, 0, 0)^T + \frac{2}{3} (0, 0, \frac{3}{2}, 0, 0, 0)^T.$$

Q5 The vector $\mathbf{x} = (1, 0, 0, 0, 2, 4)^T$

- **It is feasible:** $\mathbf{x} \geq 0$ and

$$A\mathbf{x} = 1 \cdot \mathbf{a}_1 + 2 \mathbf{a}_5 + 4 \mathbf{a}_6 = \begin{pmatrix} 1 + 2 + 0 \\ 2 + 0 + 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \mathbf{b}.$$

- **It is not basic:** it has **three** non-zero components, but a basic solution has at most $m = 2$. Indeed three vectors in $\mathbb{R}^2$ are always dependent:

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \mathbf{a}_5 + 2 \mathbf{a}_6.$$

- **Picture:** again it lies between two feasible points:

$$\mathbf{x} = \frac{1}{3} (3, 0, 0, 0, 0, 0)^T + \frac{2}{3} (0, 0, 0, 0, 3, 6)^T.$$

Q5 Finding basic feasible solutions

Choose two *independent* columns, set the other four variables to zero, solve the 2×2 system, and check the signs.

- **Basic variables** x_5, x_6 : ($\mathbf{a}_5, \mathbf{a}_6$ form the identity)

$$x_5 = 3, \quad x_6 = 6 \quad \implies \quad \mathbf{x} = (0, 0, 0, 0, 3, 6)^T \text{ is a BFS.}$$

- **Basic variables** x_2, x_4 :

$$\begin{array}{l} 2x_2 - x_4 = 3 \\ 3x_2 = 6 \end{array} \implies x_2 = 2, \quad x_4 = 1 \implies \mathbf{x} = (0, 2, 0, 1, 0, 0)^T \text{ is a BFS.}$$

- Check the second one: $2 \cdot 2 - 1 = 3 \checkmark$, $3 \cdot 2 = 6 \checkmark$.

Q5 For completeness: all $\binom{6}{2} = 15$ choices

Basic	Values	Verdict
x_1, x_2	3, 0	BFS (degenerate)
x_1, x_3	—	$\mathbf{a}_3 = 2\mathbf{a}_1$: not allowed
x_1, x_4	3, 0	BFS (degenerate)
x_1, x_5	3, 0	BFS (degenerate)
x_1, x_6	3, 0	BFS (degenerate)
x_2, x_3	0, $\frac{3}{2}$	BFS (degenerate)
x_2, x_4	2, 1	BFS
x_2, x_5	2, -1	not feasible

Basic	Values	Verdict
x_2, x_6	$\frac{3}{2}, \frac{3}{2}$	BFS
x_3, x_4	0, 0	BFS (degenerate)
x_3, x_5	0, 0	BFS (degenerate)
x_3, x_6	0, 0	BFS (degenerate)
x_4, x_5	—	$\mathbf{a}_5 = -\mathbf{a}_4$: not allowed
x_4, x_6	-3, 6	not feasible
x_5, x_6	3, 6	BFS

So there are five different BFSs (vertices):

$$(0, 0, 0, 0, 3, 6)^T, \quad (0, 2, 0, 1, 0, 0)^T, \quad (0, \frac{3}{2}, 0, 0, 0, \frac{3}{2})^T, \quad (3, 0, 0, 0, 0, 0)^T, \quad (0, 0, \frac{3}{2}, 0, 0, 0)^T,$$

the last two being degenerate. Any two of them answer the question.

Q6 Setting up the equations

We want numbers $\lambda_1, \dots, \lambda_4$ with

$$\lambda_1 \begin{pmatrix} 3 \\ 6 \end{pmatrix} + \lambda_2 \begin{pmatrix} -6 \\ 9 \end{pmatrix} + \lambda_3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \lambda_4 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \end{pmatrix}, \quad \lambda_i \geq 0, \quad \sum \lambda_i = 1.$$

Written out:

$$3\lambda_1 - 6\lambda_2 + 2\lambda_3 - \lambda_4 = 0 \quad (\text{i})$$

$$6\lambda_1 + 9\lambda_2 + \lambda_3 + \lambda_4 = 7 \quad (\text{ii})$$

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 \quad (\text{iii})$$

Three equations, four unknowns, and all unknowns must be ≥ 0 . (This is exactly a feasibility question for an LP in canonical form.)

Q6 Solving

Eliminate λ_1 using (iii):

$$(ii) - 6(iii) : 3\lambda_2 - 5\lambda_3 - 5\lambda_4 = 1,$$

$$(i) - 3(iii) : -9\lambda_2 - \lambda_3 - 4\lambda_4 = -3.$$

From the first, $3\lambda_2 = 1 + 5\lambda_3 + 5\lambda_4$. Substitute into the second:

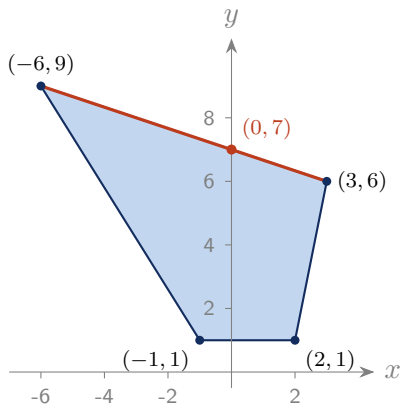
$$-3(1 + 5\lambda_3 + 5\lambda_4) - \lambda_3 - 4\lambda_4 = -3 \implies 16\lambda_3 + 19\lambda_4 = 0.$$

- Since $\lambda_3, \lambda_4 \geq 0$, this forces $\lambda_3 = \lambda_4 = 0$.
- Then $\lambda_2 = \frac{1}{3}$ and, from (iii), $\lambda_1 = \frac{2}{3}$.

Answer

Yes: $\begin{pmatrix} 0 \\ 7 \end{pmatrix} = \frac{2}{3} \begin{pmatrix} 3 \\ 6 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} -6 \\ 9 \end{pmatrix} + 0 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, and this is the only way.

Q6 Picture



- The convex combinations of the four points fill the shaded quadrilateral.
- The line through $(3, 6)$ and $(-6, 9)$ is $x + 3y = 21$.
- $(0, 7)$ is on it: $0 + 21 = 21$.
- $(2, 1)$ and $(-1, 1)$ give $x + 3y = 5$ and 2 : both < 21 , the same side.
- So $(0, 7)$ lies on the *edge* of the quadrilateral, $\frac{1}{3}$ of the way from $(3, 6)$ to $(-6, 9)$; any weight on $(2, 1)$ or $(-1, 1)$ would pull it off this edge.

Q7 The intersection of two convex sets is convex

Definition (notes, §1.4)

S is convex if $\mathbf{x}_1, \mathbf{x}_2 \in S$ and $0 < \alpha < 1$ imply $\alpha\mathbf{x}_1 + (1 - \alpha)\mathbf{x}_2 \in S$.

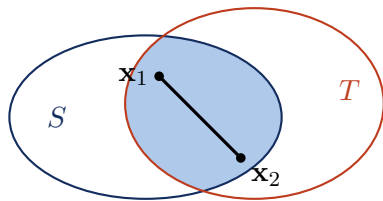
Proof. Let S and T be convex. Take any $\mathbf{x}_1, \mathbf{x}_2 \in S \cap T$ and any $0 < \alpha < 1$; put $\mathbf{y} = \alpha\mathbf{x}_1 + (1 - \alpha)\mathbf{x}_2$.

- $\mathbf{x}_1, \mathbf{x}_2 \in S$ and S is convex, so $\mathbf{y} \in S$.
- $\mathbf{x}_1, \mathbf{x}_2 \in T$ and T is convex, so $\mathbf{y} \in T$.
- Hence $\mathbf{y} \in S \cap T$.

So $S \cap T$ is convex. □

(If $S \cap T$ is empty or a single point there is nothing to check: such sets are convex.)

Q7 Picture and remarks



the segment stays in $S \cap T$

- The same argument works for any number of convex sets.
- **Use in LP:** each half-plane $\mathbf{a}^T \mathbf{x} \leq b$, each line or hyperplane $\mathbf{a}^T \mathbf{x} = b$, and each $x_j \geq 0$ is convex. The feasible region is their intersection, so it is **convex**.
- The *union* of two convex sets need not be convex: take two separate discs, and join a point of one to a point of the other.