Lecture 20
Implicit differentiation; introduction to integration

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OUTLINE

- ❑ **Implicit differentiation** (HELM 11.7)
- ❑ **Integration** (the basics; HELM 13.1, 13.4,13.5)

NOTE: The material for the first topic is identical to the *handwritten notes* posted on Moodle.

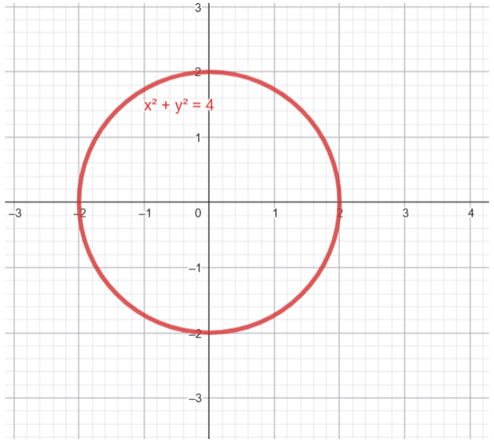
Implicit differentiation: why?

$$x^2 + y^2 = 4$$

$$F(x, y) = 0$$

where

$$F(x, y) = x^2 + y^2 - 4$$



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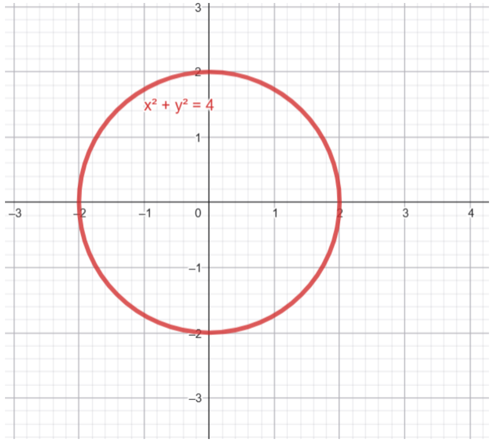

In Lecture 18 we did this....

$$x^2 + y^2 = 4 \rightarrow y^2 = 4 - x^2$$

$$y = \pm\sqrt{4 - x^2}$$

$$y_1(x) = \sqrt{4 - x^2}$$

$$y_2(x) = -\sqrt{4 - x^2}$$

Two different functions!

Implicit differentiation: why?

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↓

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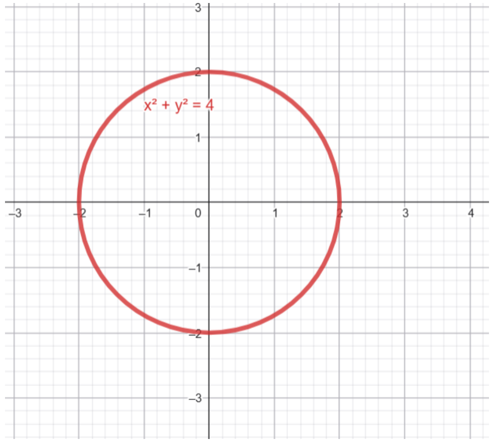


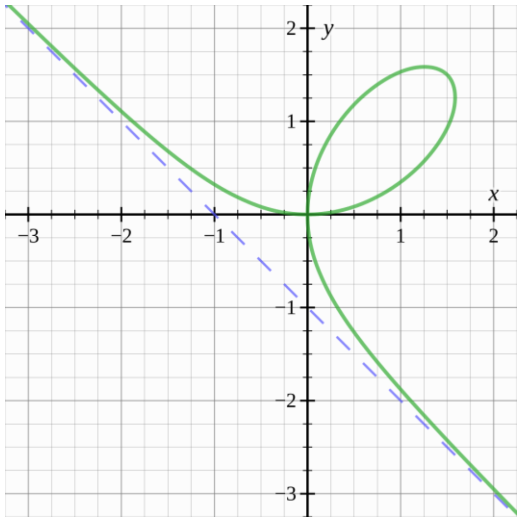
$$x^3 + y^3 - 3xy = 0$$ ('folium of Descartes')

↓

$$\boxed{G(x,y) = 0}$$

where

$$G(x,y) = x^3 + y^3 - 3xy$$



Implicit differentiation: why?

In an expression of the form

$$F(x, y) = 0$$

it is **not** always possible to solve for y in terms of x.

In other words, we can't easily find a function f such that $y = f(x)$.

Examples:

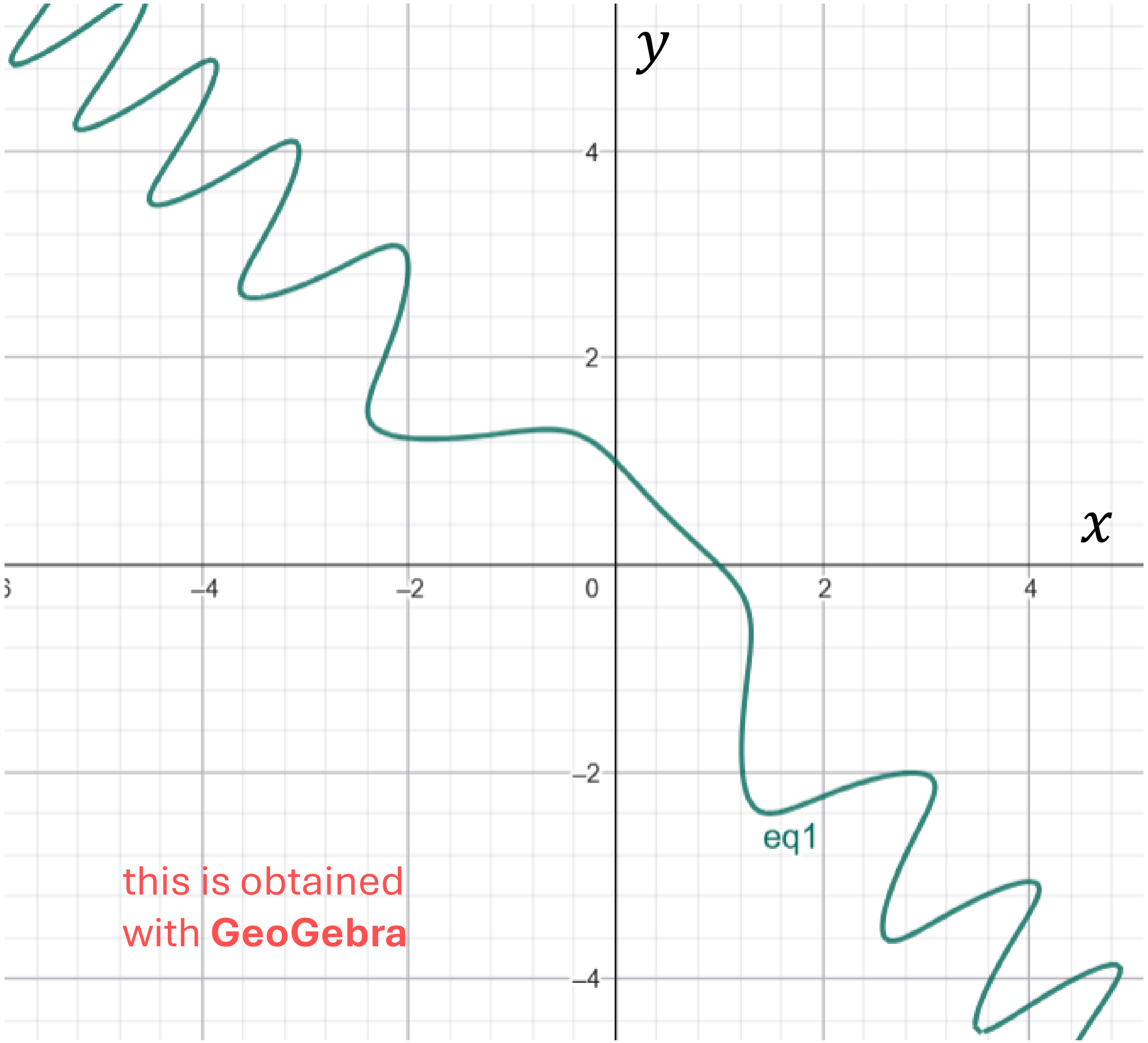
$$x + y - \cos(xy) = 0$$

$$x^2 + \sin y - 2y = 0$$

$$x^2 - xy - y^2 - 2y = 0$$

Implicit differentiation: why?

$$x + y - \cos(xy) = 0$$



this is obtained with **GeoGebra**

Implicit differentiation: why?

x_0= given value

↓

Locally, in the vicinity of this value,
y is an implicit function of x
(see graph on the right)

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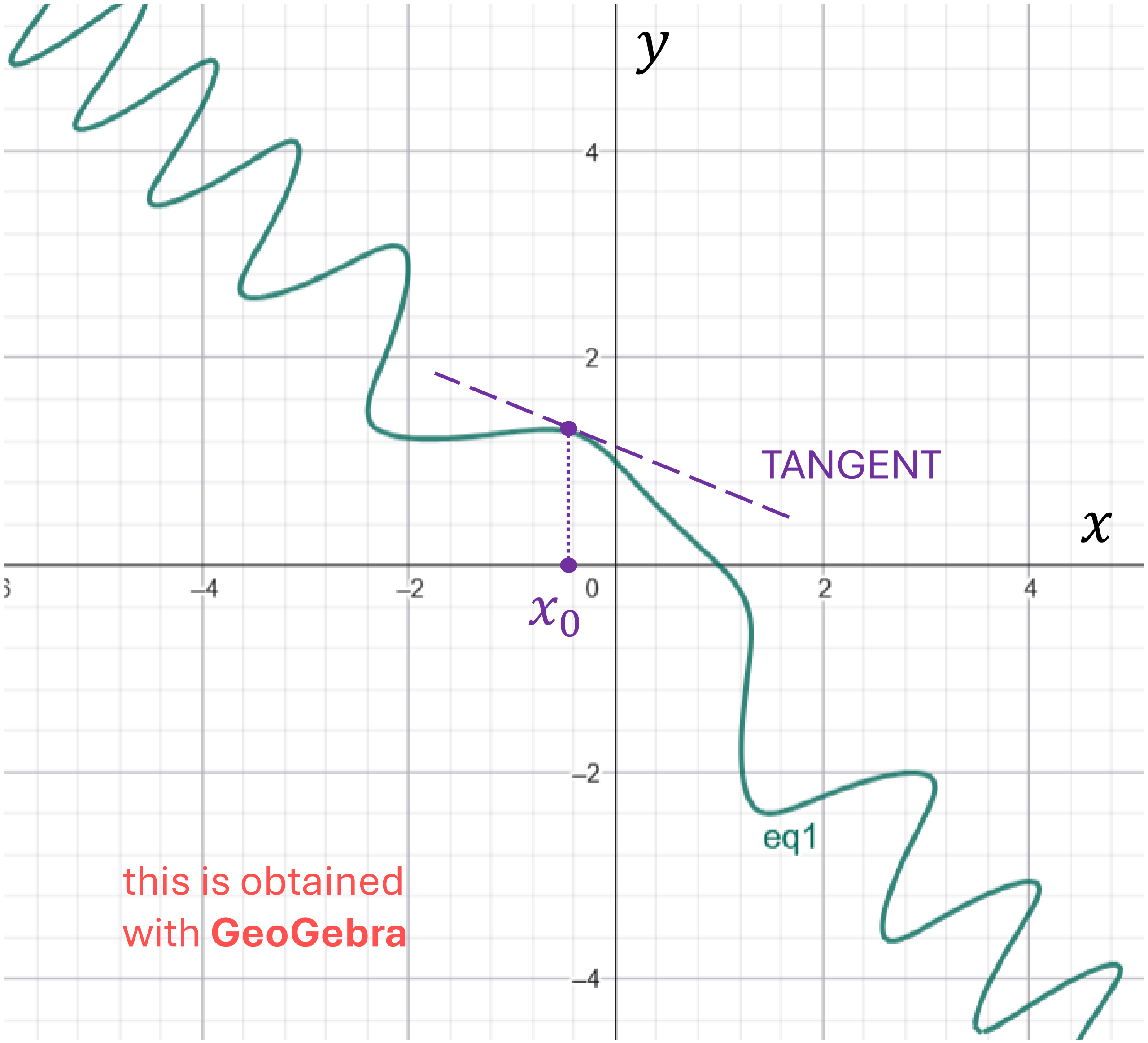

Implicit differentiation: why?

x_0 = given value

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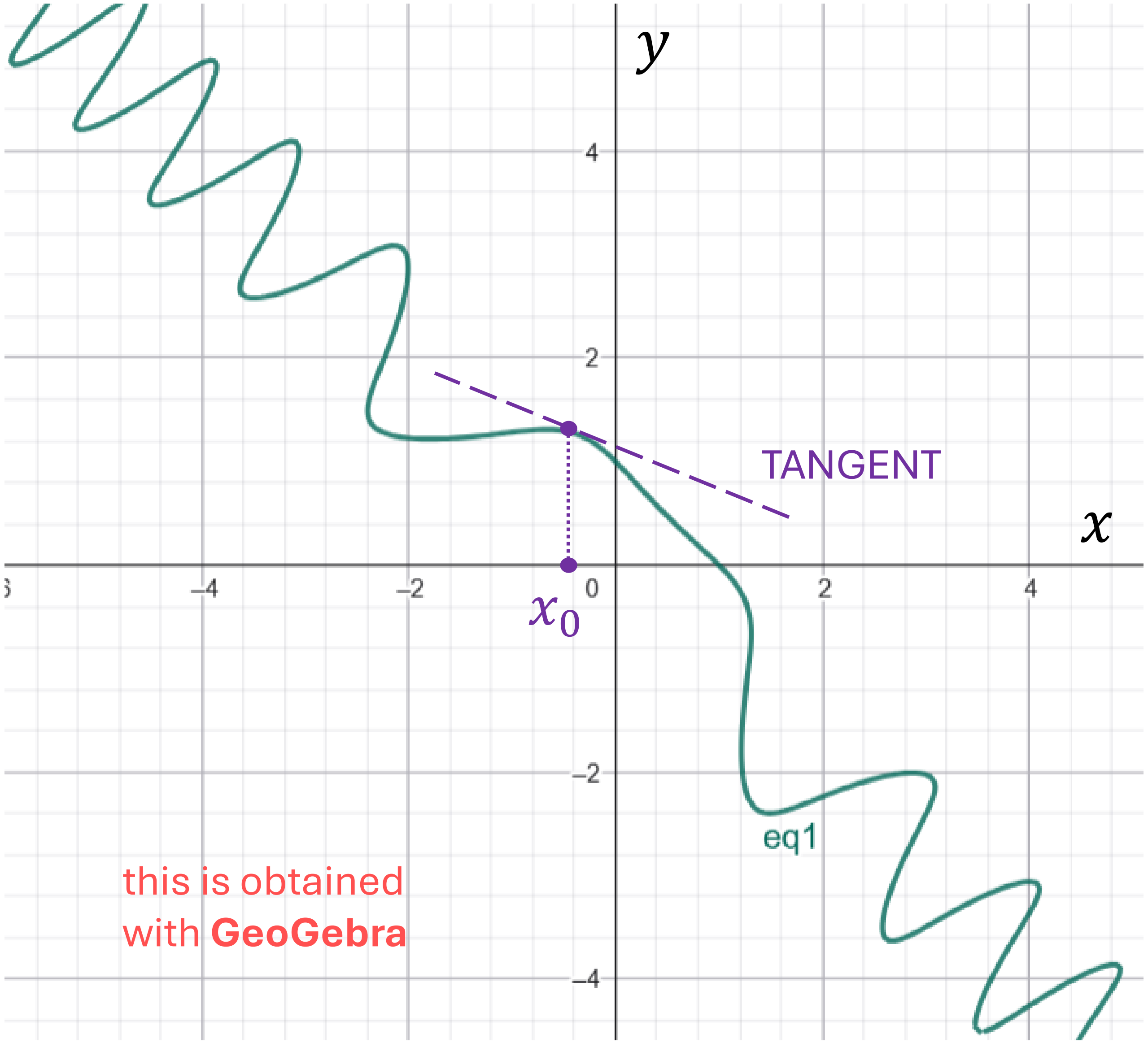
Locally, in the vicinity of this value,
y is an implicit function of x
(see graph on the right)

Implicit differentiation allows us to find

$$\frac{dy}{dx} \quad \text{at} \quad x_0$$

without needing an explicit formula for y in terms of x

$$x + y - \cos(xy) = 0$$



Implicit differentiation: how?

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Then we write:

$$x^3 + (y(x))^3 - 3x \cdot y(x) = 0$$

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$$x^3 + (y(x))^3 - 3x \cdot y(x) = 0 \longrightarrow \frac{d}{dx}\left\{x^3 + (y(x))^3 - 3x \cdot y(x)\right\} = 0$$

$$\longrightarrow \frac{d}{dx}(x^3) + \frac{d}{dx}\{(y(x))^3\} - 3\frac{d}{dx}\{x \cdot y(x)\} = 0$$

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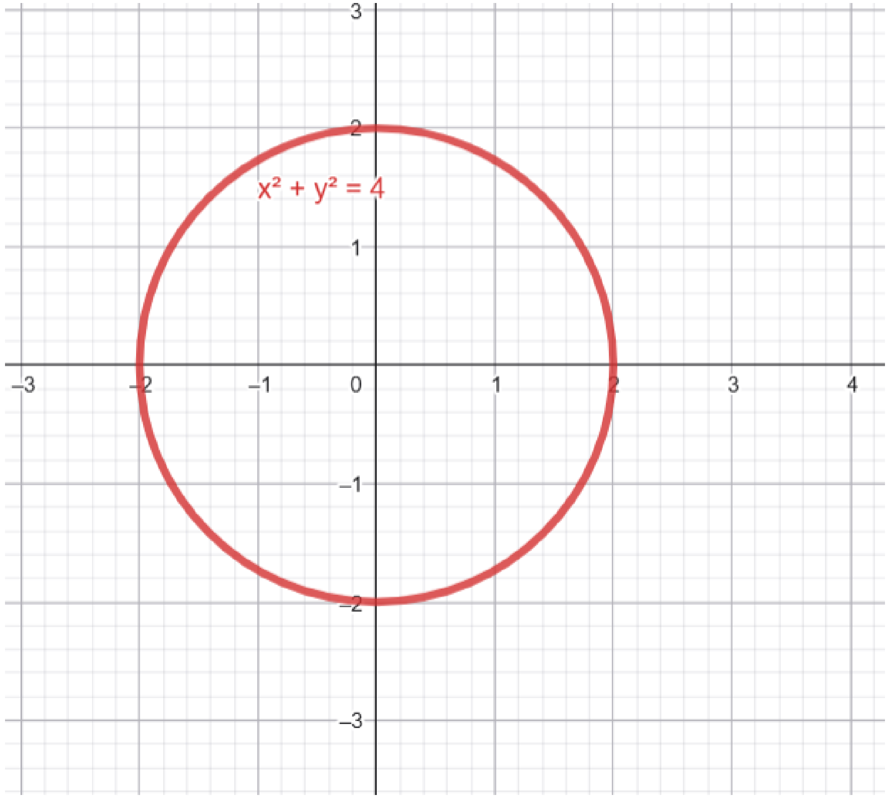
group together terms containing dy/dx: $(x^2 - y) + (y^2 - x)\dfrac{dy}{dx} = 0 \longrightarrow \boxed{\dfrac{dy}{dx} = \dfrac{y - x^2}{y^2 - x}}$ $(x \neq y^2)$

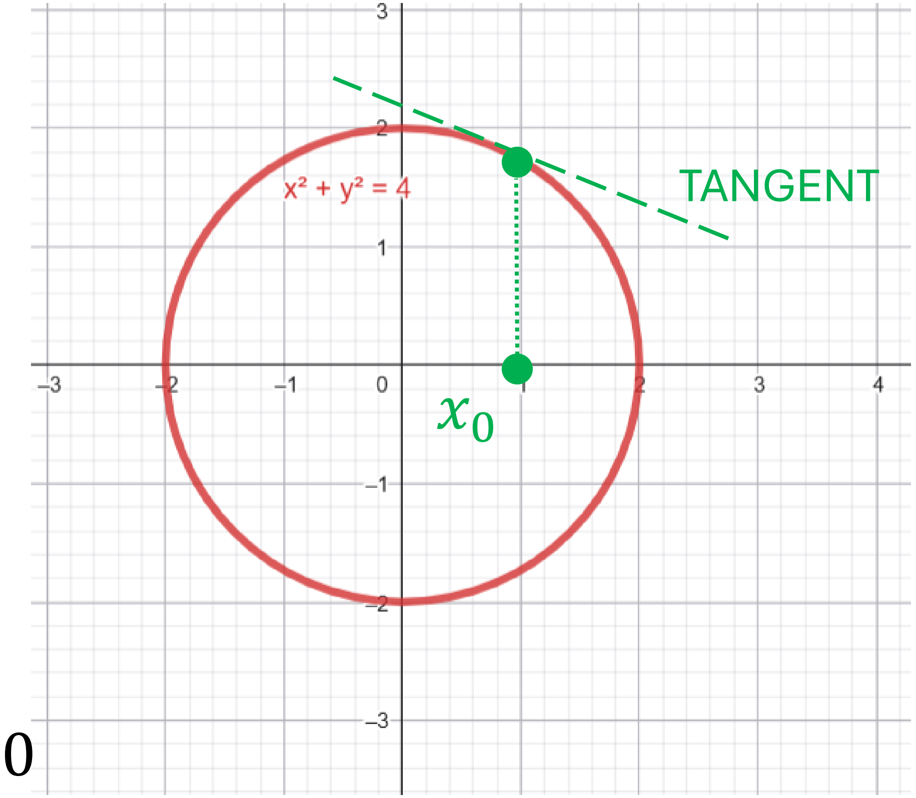
Implicit differentiation: how?

Example 2: $x^2 + y^2 = 4$



Implicit differentiation: how?

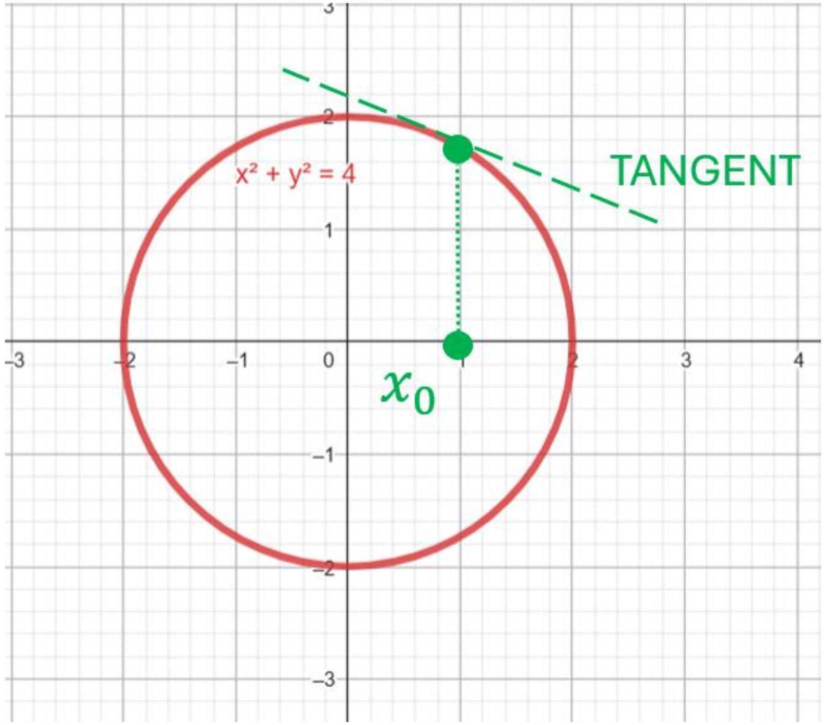
Example 2: $x^2 + y^2 = 4$



Suppose we have $y = y(x)$ in the vicinity of $x = x_0$.
Then we write:

$$x^2 + (y(x))^2 = 4 \longrightarrow \frac{d}{dx}\left\{x^2 + (y(x))^2\right\} = \frac{d}{dx}(4) = 0$$

Implicit differentiation: how?



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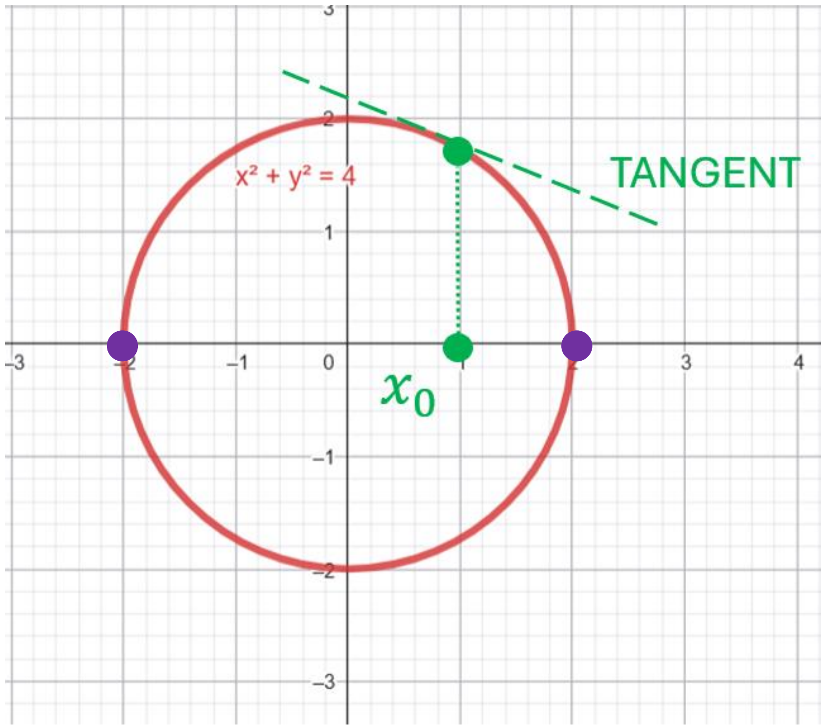
$$2x + 2y(x)\frac{dy}{dx} = 0 \longrightarrow x + y(x)\frac{dy}{dx} = 0$$

Implicit differentiation: how?



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Make dy/dx the subject: $\boxed{\frac{dy}{dx} = -\frac{x}{y}}$ $(y \neq 0)$

the points $(\pm 2,0)$ are excluded;
the tangent at these points is vertical, i.e.
the slope is infinite.

Implicit differentiation: higher-order derivatives

This strategy can be extended to **higher-order derivatives**:

$$\frac{dy}{dx} = -\frac{x}{y}$$

(found on the previous slide)

we will use **Example 2** to illustrate how this works

Implicit differentiation: higher-order derivatives

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$$\frac{dy}{dx} = -\frac{x}{y}$$

(found on the previous slide)

$$\underbrace{\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)}_{\text{by definition}}$$

Implicit differentiation: higher-order derivatives

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$$= -\frac{1}{y^2}\left(y - x\frac{dy}{dx}\right) = -\frac{1}{y^2}\left[y - x\left(-\frac{x}{y}\right)\right]$$

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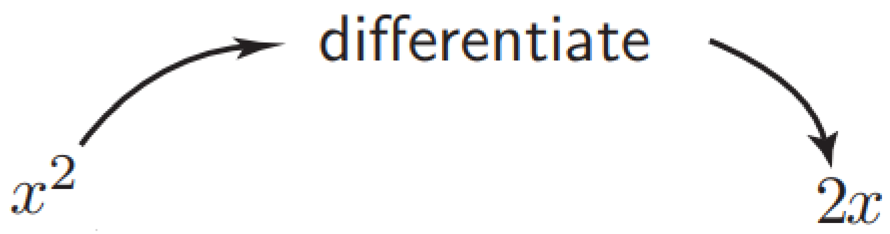
$$= -\frac{1}{y}\left(1 + \frac{x^2}{y^2}\right) \longrightarrow \boxed{\frac{d^2y}{dx^2} = -\frac{1}{y}\left(1 + \frac{x^2}{y^2}\right)}$$

INTEGRATION

HELM 13.1, 13.4,13.5

What is integration?



$$\frac{d}{dx}(x^2) = 2x$$

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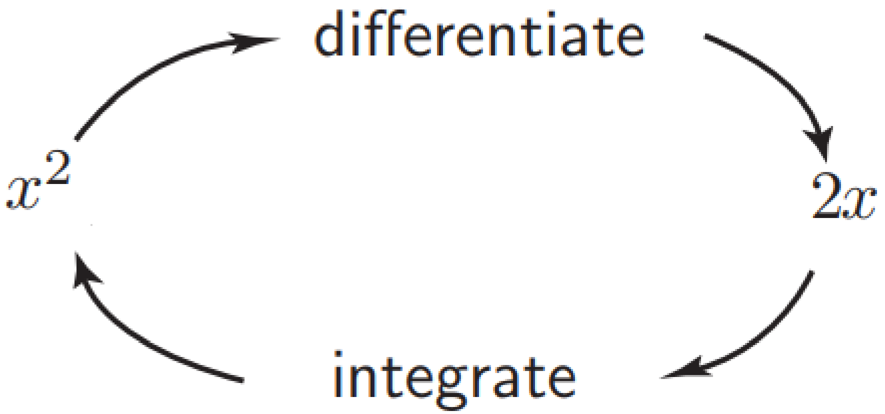

Integration is the **reverse process** of differentiation

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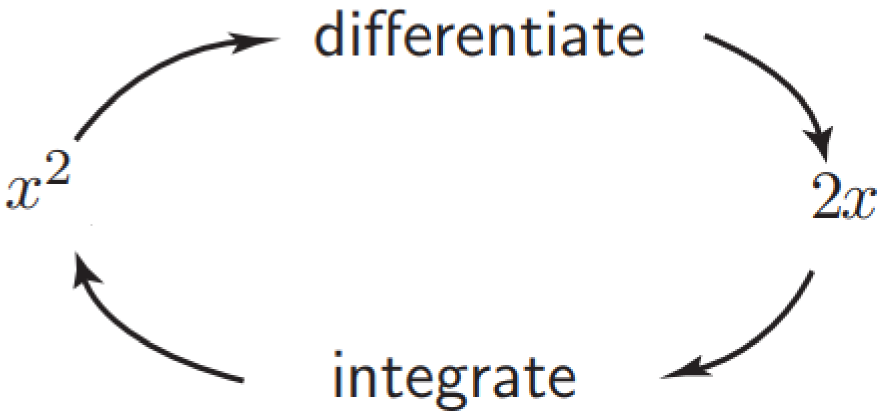
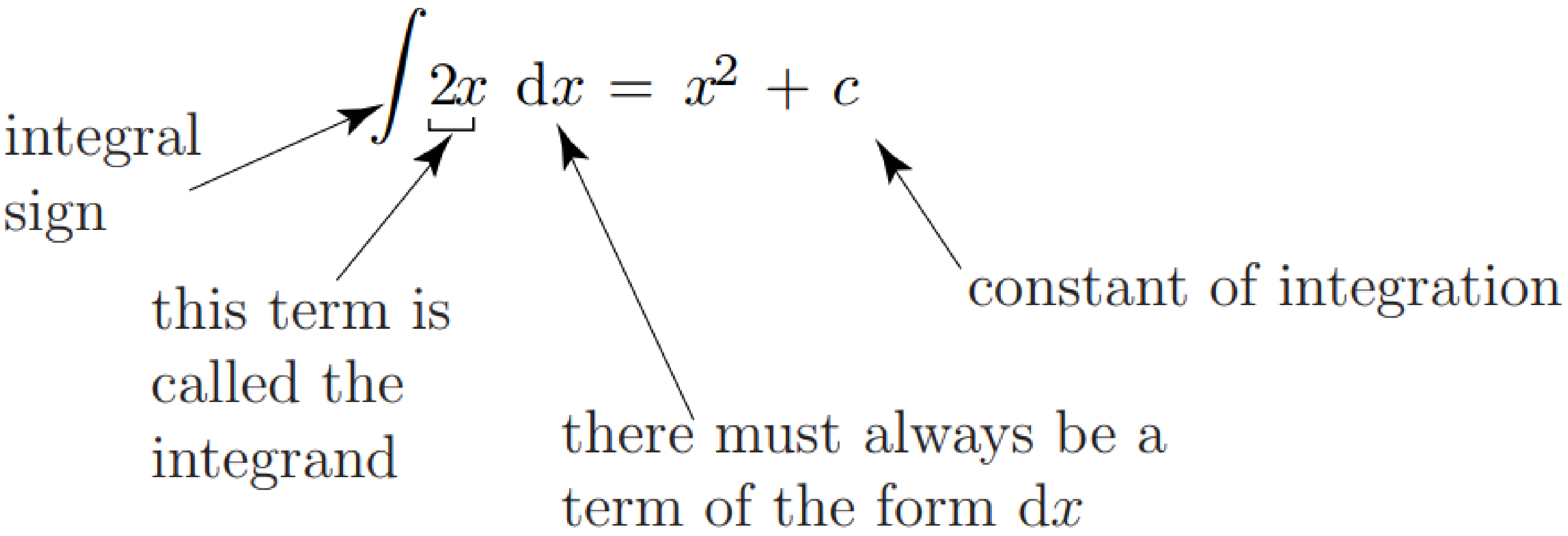


Things are a little more complicated....

$$\frac{d}{dx}(x^2 + 2) = 2x \qquad \frac{d}{dx}(x^2 + 3.5) = 2x$$

$$\frac{d}{dx}(x^2 - 50) = 2x,$$ ETC

Terminology



$$\frac{d}{dx}(x^2 + c) = 2x$$

INTEGRAND

Terminology
(general case)

$$\int f(x)\,dx = F(x) + c \quad \Longleftrightarrow \quad F'(x) = f(x)$$

Terminology
(general case)

$$\int f(x)\,dx = F(x) + c \quad \Longleftrightarrow \quad F'(x) = f(x)$$

- indefinite integral
- anti-derivative
- primitive

of $f(x)$

Table of common integrals

Table 1: Integrals of Common Functions

function $f(x)$	indefinite integral $\int f(x)\,dx$
constant, k	$kx + c$
x	$\frac{1}{2}x^2 + c$
x^2	$\frac{1}{3}x^3 + c$
x^n	$\dfrac{x^{n+1}}{n+1} + c, \quad n \neq -1$
x^{-1} (or $\dfrac{1}{x}$)	$\ln\lvert x\rvert + c$
$\cos x$	$\sin x + c$
$\sin x$	$-\cos x + c$
$\cos kx$	$\dfrac{1}{k}\sin kx + c$
$\sin kx$	$-\dfrac{1}{k}\cos kx + c$

Key properties

$$\int k\, f(x)\, dx = k \int f(x)\, dx$$

$$\int [\, f(x) + g(x) \,]\, dx = \int f(x)\, dx + \int g(x)\, dx$$

$$\int [\, f(x) - g(x) \,]\, dx = \int f(x)\, dx - \int g(x)\, dx$$

indefinite integration
is a **linear** operation

How it works…

Example: $\int (5x^2 + 3x + 7)\, dx \ = ?$

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Example: $$\int (5x^2 + 3x + 7)\,dx = ?$$

$$\int (5x^2 + 3x + 7)\,dx = \int (5x^2)\,dx + \int (3x)\,dx + \int 7\,dx$$

How it works...

Example: $$\int (5x^2 + 3x + 7)\, dx \; = ?$$

$$\int (5x^2 + 3x + 7)\, dx \; = \int (5x^2)\, dx + \int (3x)\, dx + \int 7\, dx$$

$$= 5 \int x^2 dx + 3 \int x\, dx + 7 \int 1\, dx$$

How it works…

Example: $$\int (5x^2 + 3x + 7)\,dx = ?$$

TABLE:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

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$$1 = x^0$$

How it works…

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$$1 = \ x^0$$

$$= 5\left(\frac{x^3}{3}\right) + 3\left(\frac{x^2}{2}\right) + 7\left(\frac{x^1}{1}\right) + C$$

How it works…

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$$= \frac{5x^3}{3} + \frac{3x^2}{2} + 7x + C$$ **answer**

Integration by substitution

Example: $\int x\, e^{x^2} dx = ?$

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Let $u = x^2$ → $\frac{du}{dx} = 2x$ → $du = 2xdx$ → $\frac{1}{2}du = xdx$

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Let $u = x^2$ → $\frac{du}{dx} = 2x$ → $du = 2xdx$ → $\frac{1}{2} du = xdx$

$$\int x\, e^{x^2} dx = \int \frac{1}{2} e^u\, du = \frac{1}{2} \int e^u du$$

$$= \frac{1}{2} e^u = \frac{1}{2} e^{x^2} + c$$

Integration by substitution

Example: $\int x\, e^{x^2} dx = ?$

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$$\int x\, e^{x^2} dx = \int \frac{1}{2} e^u\, du = \frac{1}{2}\int e^u du$$

$$= \frac{1}{2}e^u = \frac{1}{2}e^{x^2} + c$$

Therefore, $\int x\, e^{x^2} dx = \frac{1}{2}e^{x^2} + c$ **answer**

Integration by substitution

Example: $$\int \frac{x}{x^2+1}\,dx = ?$$

Let $u = x^2 + 1$ → $\dfrac{du}{dx} = 2x$ → $du = 2xdx$ → $\dfrac{1}{2}du = xdx$

Integration by substitution

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Therefore, $\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1) + c$ **answer**

Integration by substitution

Example: $$\int e^x \sqrt{1 + e^x}\; dx = ?$$

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Integration by substitution

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$$\int e^x\sqrt{1+e^x}\ dx = \int \sqrt{1+e^x}\,(e^x dx) = \int \sqrt{u}\ du$$

TABLE:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

Integration by substitution

Example: $\int e^x \sqrt{1 + e^x} \; dx = ?$

Let $u = e^x + 1$ → $\dfrac{du}{dx} = e^x$ → $du = e^x dx$

$$\int e^x \sqrt{1 + e^x} \; dx = \int \sqrt{1 + e^x} \, (e^x dx) = \int \sqrt{u} \; du$$

$$= \int u^{1/2} \, du = \frac{2}{3} \, u^{3/2} + c$$

Integration by substitution

TABLE:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

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Let $u = e^x + 1$ $\longrightarrow$ $\frac{du}{dx} = e^x$ $\longrightarrow$ $du = e^x dx$

$$\int e^x \sqrt{1+e^x} \; dx = \int \sqrt{1+e^x} \; (e^x dx) = \int \sqrt{u} \; du$$

$$= \int u^{1/2} \, du = \frac{2}{3} \, u^{3/2} + c = \frac{2}{3} (1+e^x)^{3/2} + c$$

Therefore, $\int e^x \sqrt{1+e^x} \; dx = \frac{2}{3} (1+e^x)^{3/2} + c$ **answer**

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