

MAA308/309 Mathematical Methods

Complex numbers

A number whose square is negative

The square of a real number is never negative: $x^2 \geq 0$ for every $x \in \mathbb{R}$.

So the equation $x^2 = -4$ has **no real solutions**.

A number whose square is negative

The square of a real number is never negative: $x^2 \geq 0$ for every $x \in \mathbb{R}$.

So the equation $x^2 = -4$ has **no real solutions**.

Definition

We introduce a new symbol i with the property

$$i^2 = -1 \quad (\text{so one may write } i = \sqrt{-1}).$$

A number whose square is negative

The square of a real number is never negative: $x^2 \geq 0$ for every $x \in \mathbb{R}$.

So the equation $x^2 = -4$ has **no real solutions**.

Definition

We introduce a new symbol i with the property

$$i^2 = -1 \quad (\text{so one may write } i = \sqrt{-1}).$$

Rule of the game

Work with i exactly as with any other algebraic symbol, and replace i^2 by -1 whenever it appears.

A number whose square is negative

The square of a real number is never negative: $x^2 \geq 0$ for every $x \in \mathbb{R}$.

So the equation $x^2 = -4$ has **no real solutions**.

Definition

We introduce a new symbol i with the property

$$i^2 = -1 \quad (\text{so one may write } i = \sqrt{-1}).$$

Rule of the game

Work with i exactly as with any other algebraic symbol, and replace i^2 by -1 whenever it appears.

In some parts of engineering and physics the letter j is used instead of i (to avoid confusion with electrical current).

First calculations

$$i^3 = i^2 \times i = (-1) \times i = -i$$

replace i^2 by -1

First calculations

$$i^3 = i^2 \times i = (-1) \times i = -i$$

replace i^2 by -1

$$(3 + i)^2 = 3^2 + 2 \times 3 \times i + i^2 = 9 + 6i - 1 = 8 + 6i$$

expand, then $i^2 = -1$

First calculations

$$i^3 = i^2 \times i = (-1) \times i = -i \quad \text{replace } i^2 \text{ by } -1$$

$$(3 + i)^2 = 3^2 + 2 \times 3 \times i + i^2 = 9 + 6i - 1 = 8 + 6i \quad \text{expand, then } i^2 = -1$$

$$(2i)^2 = 2i \times 2i = 4i^2 = -4$$

So $x = 2i$ solves $x^2 + 4 = 0$. Also $(-2i)^2 = 4i^2 = -4$, so $x = -2i$ is a solution too.

First calculations

$$i^3 = i^2 \times i = (-1) \times i = -i \quad \text{replace } i^2 \text{ by } -1$$

$$(3 + i)^2 = 3^2 + 2 \times 3 \times i + i^2 = 9 + 6i - 1 = 8 + 6i \quad \text{expand, then } i^2 = -1$$

$$(2i)^2 = 2i \times 2i = 4i^2 = -4$$

So $x = 2i$ solves $x^2 + 4 = 0$. Also $(-2i)^2 = 4i^2 = -4$, so $x = -2i$ is a solution too.

Where do the two solutions come from? Difference of two squares:

$$x^2 + 4 = x^2 - (2i)^2 \quad \text{since } 4 = -(2i)^2$$

First calculations

$$i^3 = i^2 \times i = (-1) \times i = -i \quad \text{replace } i^2 \text{ by } -1$$

$$(3 + i)^2 = 3^2 + 2 \times 3 \times i + i^2 = 9 + 6i - 1 = 8 + 6i \quad \text{expand, then } i^2 = -1$$

$$(2i)^2 = 2i \times 2i = 4i^2 = -4$$

So $x = 2i$ solves $x^2 + 4 = 0$. Also $(-2i)^2 = 4i^2 = -4$, so $x = -2i$ is a solution too.

Where do the two solutions come from? Difference of two squares:

$$\begin{aligned} x^2 + 4 &= x^2 - (2i)^2 && \text{since } 4 = -(2i)^2 \\ &= (x - 2i)(x + 2i) = 0 && a^2 - b^2 = (a - b)(a + b) \end{aligned}$$

First calculations

$$i^3 = i^2 \times i = (-1) \times i = -i \quad \text{replace } i^2 \text{ by } -1$$

$$(3 + i)^2 = 3^2 + 2 \times 3 \times i + i^2 = 9 + 6i - 1 = 8 + 6i \quad \text{expand, then } i^2 = -1$$

$$(2i)^2 = 2i \times 2i = 4i^2 = -4$$

So $x = 2i$ solves $x^2 + 4 = 0$. Also $(-2i)^2 = 4i^2 = -4$, so $x = -2i$ is a solution too.

Where do the two solutions come from? Difference of two squares:

$$x^2 + 4 = x^2 - (2i)^2$$

$$= (x - 2i)(x + 2i) = 0$$

$$x = 2i \quad \text{or} \quad x = -2i$$

$$\text{since } 4 = -(2i)^2$$

$$a^2 - b^2 = (a - b)(a + b)$$

a product is zero only if a factor is zero

Example: a quadratic with no real roots

Example. Solve $x^2 - 6x + 34 = 0$.

Example: a quadratic with no real roots

Example. Solve $x^2 - 6x + 34 = 0$.

Strategy: quadratic formula with $a = 1$, $b = -6$, $c = 34$.

Example: a quadratic with no real roots

Example. Solve $x^2 - 6x + 34 = 0$.

Strategy: quadratic formula with $a = 1$, $b = -6$, $c = 34$.

$$x = \frac{6 \pm \sqrt{36 - 4 \times 1 \times 34}}{2} = \frac{6 \pm \sqrt{-100}}{2} \quad b^2 - 4ac < 0: \text{ no real solutions}$$

Example: a quadratic with no real roots

Example. Solve $x^2 - 6x + 34 = 0$.

Strategy: quadratic formula with $a = 1$, $b = -6$, $c = 34$.

$$x = \frac{6 \pm \sqrt{36 - 4 \times 1 \times 34}}{2} = \frac{6 \pm \sqrt{-100}}{2}$$

$b^2 - 4ac < 0$: no real solutions

$$\sqrt{-100} = \sqrt{100} \sqrt{-1} = 10i$$

the new symbol lets us continue

Example: a quadratic with no real roots

Example. Solve $x^2 - 6x + 34 = 0$.

Strategy: quadratic formula with $a = 1$, $b = -6$, $c = 34$.

$$x = \frac{6 \pm \sqrt{36 - 4 \times 1 \times 34}}{2} = \frac{6 \pm \sqrt{-100}}{2}$$

$b^2 - 4ac < 0$: no real solutions

$$\sqrt{-100} = \sqrt{100} \sqrt{-1} = 10i$$

the new symbol lets us continue

$$x = \frac{6 \pm 10i}{2} = 3 \pm 5i$$

two complex solutions

Example: a quadratic with no real roots

Example. Solve $x^2 - 6x + 34 = 0$.

Strategy: quadratic formula with $a = 1$, $b = -6$, $c = 34$.

$$x = \frac{6 \pm \sqrt{36 - 4 \times 1 \times 34}}{2} = \frac{6 \pm \sqrt{-100}}{2} \quad b^2 - 4ac < 0: \text{ no real solutions}$$

$$\sqrt{-100} = \sqrt{100} \sqrt{-1} = 10i \quad \text{the new symbol lets us continue}$$

$$x = \frac{6 \pm 10i}{2} = 3 \pm 5i \quad \text{two complex solutions}$$

Check $x = 3 + 5i$:

$$(3 + 5i)^2 - 6(3 + 5i) + 34 = (9 + 30i - 25) - (18 + 30i) + 34 = 0 \quad \checkmark$$

Same equation, completing the square

Example. Solve $x^2 - 6x + 34 = 0$ again.

Same equation, completing the square

Example. Solve $x^2 - 6x + 34 = 0$ again.

$$x^2 - 6x + 34 = (x - 3)^2 - 9 + 34 = (x - 3)^2 + 25 \quad \text{completed square form}$$

Same equation, completing the square

Example. Solve $x^2 - 6x + 34 = 0$ again.

$$x^2 - 6x + 34 = (x - 3)^2 - 9 + 34 = (x - 3)^2 + 25$$

completed square form

$$(x - 3)^2 = -25$$

set equal to 0, rearrange

Same equation, completing the square

Example. Solve $x^2 - 6x + 34 = 0$ again.

$$x^2 - 6x + 34 = (x - 3)^2 - 9 + 34 = (x - 3)^2 + 25$$

completed square form

$$(x - 3)^2 = -25$$

set equal to 0, rearrange

$$x - 3 = \pm\sqrt{-25} = \pm 5i$$

square root: two signs

Same equation, completing the square

Example. Solve $x^2 - 6x + 34 = 0$ again.

$$x^2 - 6x + 34 = (x - 3)^2 - 9 + 34 = (x - 3)^2 + 25$$

completed square form

$$(x - 3)^2 = -25$$

set equal to 0, rearrange

$$x - 3 = \pm\sqrt{-25} = \pm 5i$$

square root: two signs

$$x = 3 \pm 5i$$

as before ✓

Same equation, completing the square

Example. Solve $x^2 - 6x + 34 = 0$ again.

$$x^2 - 6x + 34 = (x - 3)^2 - 9 + 34 = (x - 3)^2 + 25$$

completed square form

$$(x - 3)^2 = -25$$

set equal to 0, rearrange

$$x - 3 = \pm\sqrt{-25} = \pm 5i$$

square root: two signs

$$x = 3 \pm 5i$$

as before ✓

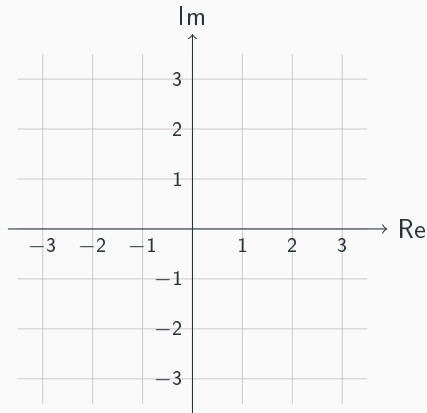
Observation

The two solutions always come as a pair $3 + 5i$ and $3 - 5i$: same real part, opposite imaginary parts.

Cartesian form and the Argand diagram

Cartesian form

A **complex number** is $z = x + iy$ with x, y **real**:
 $x = \operatorname{Re}(z)$, the **real part**; $y = \operatorname{Im}(z)$, the
imaginary part.

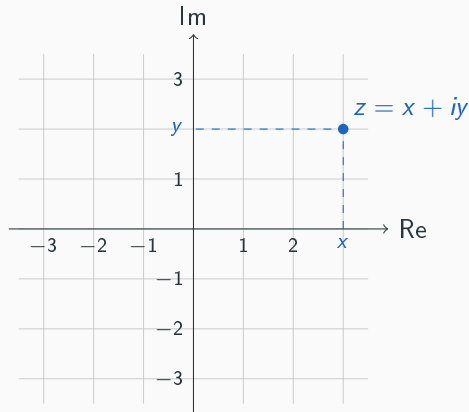


Cartesian form and the Argand diagram

Cartesian form

A **complex number** is $z = x + iy$ with x, y **real**:
 $x = \operatorname{Re}(z)$, the **real part**; $y = \operatorname{Im}(z)$, the
imaginary part.

i is not on the number line, so we add a second axis
at right angles: the **complex plane (Argand
diagram)**. $z = x + iy$ is the point (x, y) .



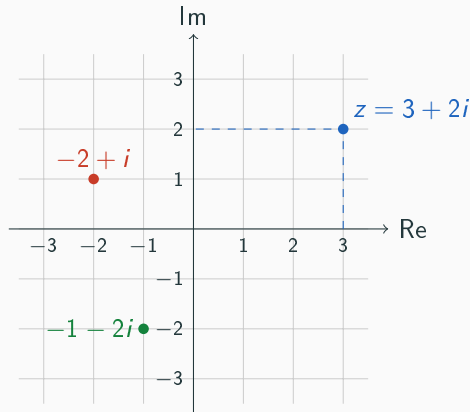
Cartesian form and the Argand diagram

Cartesian form

A **complex number** is $z = x + iy$ with x, y **real**:
 $x = \operatorname{Re}(z)$, the **real part**; $y = \operatorname{Im}(z)$, the **imaginary part**.

i is not on the number line, so we add a second axis at right angles: the **complex plane (Argand diagram)**. $z = x + iy$ is the point (x, y) .

Examples: $3 + 2i \rightarrow (3, 2)$, $-2 + i \rightarrow (-2, 1)$,
 $-1 - 2i \rightarrow (-1, -2)$.



Cartesian form and the Argand diagram

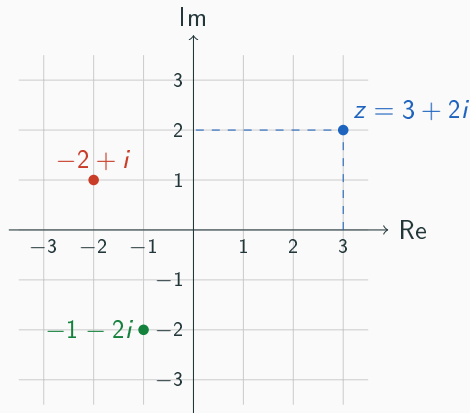
Cartesian form

A **complex number** is $z = x + iy$ with x, y **real**:
 $x = \operatorname{Re}(z)$, the **real part**; $y = \operatorname{Im}(z)$, the **imaginary part**.

i is not on the number line, so we add a second axis at right angles: the **complex plane (Argand diagram)**. $z = x + iy$ is the point (x, y) .

Examples: $3 + 2i \rightarrow (3, 2)$, $-2 + i \rightarrow (-2, 1)$,
 $-1 - 2i \rightarrow (-1, -2)$.

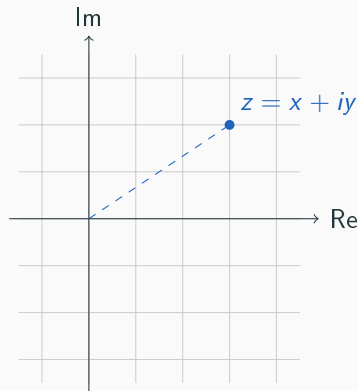
$z_1 = z_2$ exactly when $\operatorname{Re} z_1 = \operatorname{Re} z_2$ **and**
 $\operatorname{Im} z_1 = \operatorname{Im} z_2$.



The complex conjugate

Definition

The **conjugate** of $z = x + iy$ is $z^* = x - iy$ (also written $\bar{z}$).

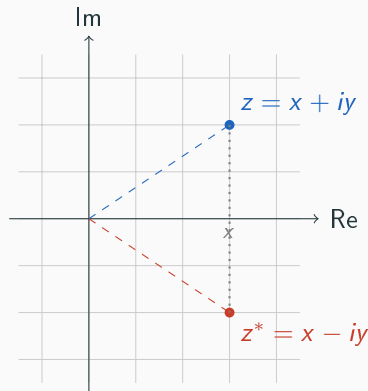


The complex conjugate

Definition

The **conjugate** of $z = x + iy$ is $z^* = x - iy$ (also written $\bar{z}$).

On the Argand diagram: **reflect** z in the real axis.



The complex conjugate

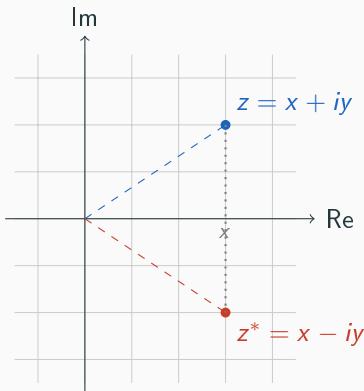
Definition

The **conjugate** of $z = x + iy$ is $z^* = x - iy$ (also written $\bar{z}$).

On the Argand diagram: **reflect** z in the real axis.

Key property: the product zz^* is **real**.

$$zz^* = (x + iy)(x - iy)$$



The complex conjugate

Definition

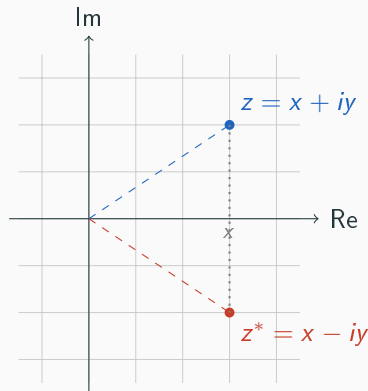
The **conjugate** of $z = x + iy$ is $z^* = x - iy$ (also written $\bar{z}$).

On the Argand diagram: **reflect** z in the real axis.

Key property: the product zz^* is **real**.

$$\begin{aligned} zz^* &= (x + iy)(x - iy) \\ &= x^2 - (iy)^2 \end{aligned}$$

difference of two squares



The complex conjugate

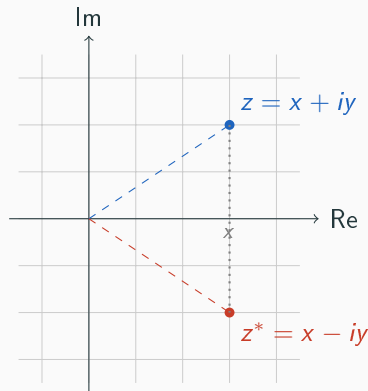
Definition

The **conjugate** of $z = x + iy$ is $z^* = x - iy$ (also written $\bar{z}$).

On the Argand diagram: **reflect** z in the real axis.

Key property: the product zz^* is **real**.

$$\begin{aligned} zz^* &= (x + iy)(x - iy) \\ &= x^2 - (iy)^2 && \text{difference of two squares} \\ &= x^2 - i^2 y^2 = x^2 + y^2 && i^2 = -1 \end{aligned}$$



The complex conjugate

Definition

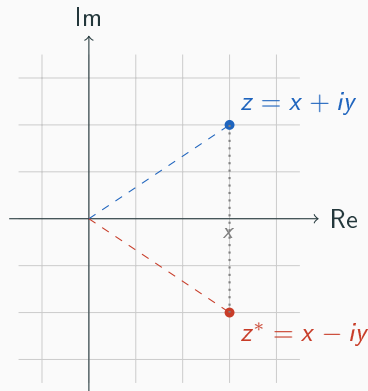
The **conjugate** of $z = x + iy$ is $z^* = x - iy$ (also written $\bar{z}$).

On the Argand diagram: **reflect** z in the real axis.

Key property: the product zz^* is **real**.

$$\begin{aligned} zz^* &= (x + iy)(x - iy) \\ &= x^2 - (iy)^2 && \text{difference of two squares} \\ &= x^2 - i^2 y^2 = x^2 + y^2 && i^2 = -1 \end{aligned}$$

$x^2 + y^2$ is the squared distance of z from the origin, $|z|^2$ (next slide).



Adding and multiplying

Rule

Treat i as an algebraic symbol; gather real parts and imaginary parts; replace i^2 by -1 .

Example. $z_1 = 5 + 3i$, $z_2 = 7 - i$. Find $z_1 + z_2$ and $z_1 z_2$.

Adding and multiplying

Rule

Treat i as an algebraic symbol; gather real parts and imaginary parts; replace i^2 by -1 .

Example. $z_1 = 5 + 3i$, $z_2 = 7 - i$. Find $z_1 + z_2$ and $z_1 z_2$.

$$z_1 + z_2 = (5 + 7) + (3 - 1)i = 12 + 2i \quad \text{add real parts, add imaginary parts}$$

Adding and multiplying

Rule

Treat i as an algebraic symbol; gather real parts and imaginary parts; replace i^2 by -1 .

Example. $z_1 = 5 + 3i$, $z_2 = 7 - i$. Find $z_1 + z_2$ and $z_1 z_2$.

$$z_1 + z_2 = (5 + 7) + (3 - 1)i = 12 + 2i \quad \text{add real parts, add imaginary parts}$$

$$z_1 z_2 = 5(7 - i) + 3i(7 - i) \quad \text{expand the first bracket}$$

Adding and multiplying

Rule

Treat i as an algebraic symbol; gather real parts and imaginary parts; replace i^2 by -1 .

Example. $z_1 = 5 + 3i$, $z_2 = 7 - i$. Find $z_1 + z_2$ and $z_1 z_2$.

$$z_1 + z_2 = (5 + 7) + (3 - 1)i = 12 + 2i \quad \text{add real parts, add imaginary parts}$$

$$z_1 z_2 = 5(7 - i) + 3i(7 - i) \quad \text{expand the first bracket}$$

$$= 35 - 5i + 21i - 3i^2 \quad \text{expand}$$

Adding and multiplying

Rule

Treat i as an algebraic symbol; gather real parts and imaginary parts; replace i^2 by -1 .

Example. $z_1 = 5 + 3i$, $z_2 = 7 - i$. Find $z_1 + z_2$ and $z_1 z_2$.

$$z_1 + z_2 = (5 + 7) + (3 - 1)i = 12 + 2i \quad \text{add real parts, add imaginary parts}$$

$$z_1 z_2 = 5(7 - i) + 3i(7 - i) \quad \text{expand the first bracket}$$

$$= 35 - 5i + 21i - 3i^2 \quad \text{expand}$$

$$= 35 + 16i + 3 = 38 + 16i \quad -3i^2 = +3; \text{gather}$$

Dividing: rationalising a complex fraction

Rule

Multiply top and bottom by the **conjugate of the denominator**: the denominator becomes the real number $x^2 + y^2$.

Example. Find $\frac{5 + 3i}{7 - i}$ in Cartesian form.

Dividing: rationalising a complex fraction

Rule

Multiply top and bottom by the **conjugate of the denominator**: the denominator becomes the real number $x^2 + y^2$.

Compare with surds (Lecture 2): for $\frac{1}{3 + 2\sqrt{2}}$ we multiplied top and bottom by $3 - 2\sqrt{2}$.

Example. Find $\frac{5 + 3i}{7 - i}$ in Cartesian form.

Dividing: rationalising a complex fraction

Rule

Multiply top and bottom by the **conjugate of the denominator**: the denominator becomes the real number $x^2 + y^2$.

Compare with surds (Lecture 2): for $\frac{1}{3 + 2\sqrt{2}}$ we multiplied top and bottom by $3 - 2\sqrt{2}$.

Example. Find $\frac{5 + 3i}{7 - i}$ in Cartesian form.

$$\frac{5 + 3i}{7 - i} = \frac{5 + 3i}{7 - i} \times \frac{7 + i}{7 + i}$$

conjugate of $7 - i$ is $7 + i$

Dividing: rationalising a complex fraction

Rule

Multiply top and bottom by the **conjugate of the denominator**: the denominator becomes the real number $x^2 + y^2$.

Compare with surds (Lecture 2): for $\frac{1}{3 + 2\sqrt{2}}$ we multiplied top and bottom by $3 - 2\sqrt{2}$.

Example. Find $\frac{5 + 3i}{7 - i}$ in Cartesian form.

$$\frac{5 + 3i}{7 - i} = \frac{5 + 3i}{7 - i} \times \frac{7 + i}{7 + i}$$

conjugate of $7 - i$ is $7 + i$

$$= \frac{35 + 5i + 21i + 3i^2}{7^2 - i^2} = \frac{32 + 26i}{50}$$

top: expand; bottom: $x^2 + y^2 = 49 + 1$

Dividing: rationalising a complex fraction

Rule

Multiply top and bottom by the **conjugate of the denominator**: the denominator becomes the real number $x^2 + y^2$.

Compare with surds (Lecture 2): for $\frac{1}{3 + 2\sqrt{2}}$ we multiplied top and bottom by $3 - 2\sqrt{2}$.

Example. Find $\frac{5 + 3i}{7 - i}$ in Cartesian form.

$$\frac{5 + 3i}{7 - i} = \frac{5 + 3i}{7 - i} \times \frac{7 + i}{7 + i}$$

conjugate of $7 - i$ is $7 + i$

$$= \frac{35 + 5i + 21i + 3i^2}{7^2 - i^2} = \frac{32 + 26i}{50}$$

top: expand; bottom: $x^2 + y^2 = 49 + 1$

$$= \frac{16}{25} + \frac{13}{25}i$$

split into real and imaginary parts

Dividing: rationalising a complex fraction

Rule

Multiply top and bottom by the **conjugate of the denominator**: the denominator becomes the real number $x^2 + y^2$.

Compare with surds (Lecture 2): for $\frac{1}{3+2\sqrt{2}}$ we multiplied top and bottom by $3-2\sqrt{2}$.

Example. Find $\frac{5+3i}{7-i}$ in Cartesian form.

$$\frac{5+3i}{7-i} = \frac{5+3i}{7-i} \times \frac{7+i}{7+i}$$

conjugate of $7-i$ is $7+i$

$$= \frac{35 + 5i + 21i + 3i^2}{7^2 - i^2} = \frac{32 + 26i}{50}$$

top: expand; bottom: $x^2 + y^2 = 49 + 1$

$$= \frac{16}{25} + \frac{13}{25}i$$

split into real and imaginary parts

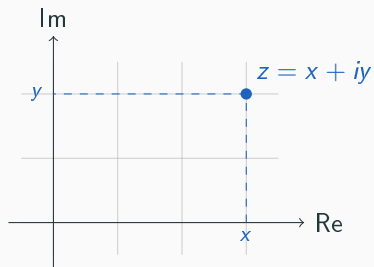
Check: $\left(\frac{16}{25} + \frac{13}{25}i\right)(7-i) = \frac{112}{25} - \frac{16}{25}i + \frac{91}{25}i - \frac{13}{25}i^2 = \frac{125}{25} + \frac{75}{25}i = 5 + 3i \quad \checkmark$

Modulus and argument

Instead of (x, y) , locate z by its **distance** from the origin and its **angle** from the real axis.

Definition

$r = |z| = \sqrt{x^2 + y^2}$ the **modulus**,
 $\theta = \arg z$ the **argument** (in radians).



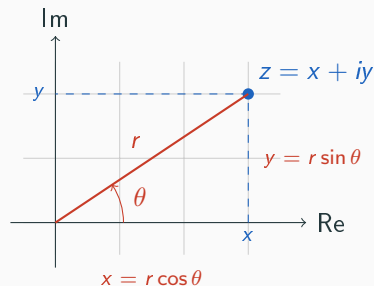
Modulus and argument

Instead of (x, y) , locate z by its **distance** from the origin and its **angle** from the real axis.

Definition

$r = |z| = \sqrt{x^2 + y^2}$ the **modulus**,
 $\theta = \arg z$ the **argument** (in radians).

Polar $\rightarrow$ **Cartesian**: $x = r \cos \theta$, $y = r \sin \theta$.



Modulus and argument

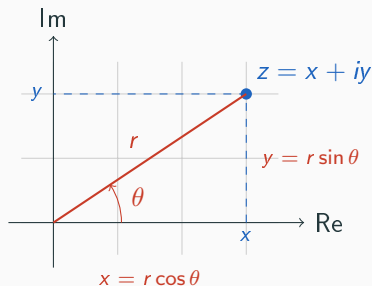
Instead of (x, y) , locate z by its **distance** from the origin and its **angle** from the real axis.

Definition

$r = |z| = \sqrt{x^2 + y^2}$ the **modulus**,
 $\theta = \arg z$ the **argument** (in radians).

Polar $\rightarrow$ **Cartesian**: $x = r \cos \theta$, $y = r \sin \theta$.

Cartesian $\rightarrow$ **polar**: $r = \sqrt{x^2 + y^2}$, $\tan \theta = \frac{y}{x}$.



Modulus and argument

Instead of (x, y) , locate z by its **distance** from the origin and its **angle** from the real axis.

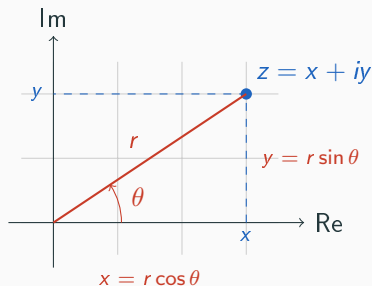
Definition

$r = |z| = \sqrt{x^2 + y^2}$ the **modulus**,
 $\theta = \arg z$ the **argument** (in radians).

Polar $\rightarrow$ Cartesian: $x = r \cos \theta$, $y = r \sin \theta$.

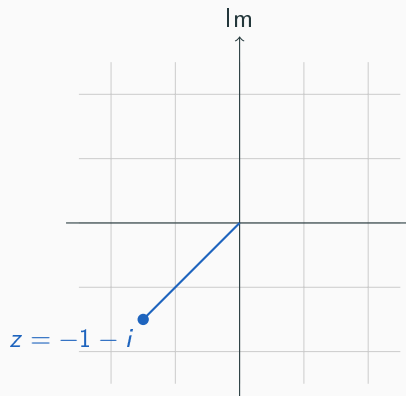
Cartesian $\rightarrow$ polar: $r = \sqrt{x^2 + y^2}$, $\tan \theta = \frac{y}{x}$.

Polar form: $z = r \angle \theta$ (same number, different description).



Warning: θ is not always $\tan^{-1}(y/x)$

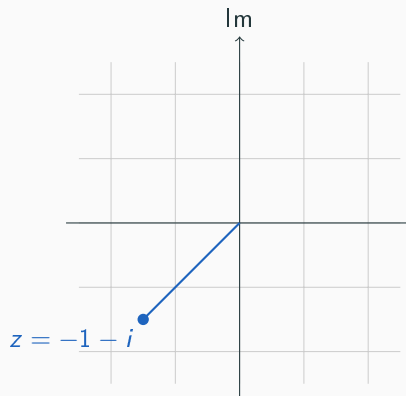
Example. Find the argument of $z = -1 - i$.



Warning: θ is not always $\tan^{-1}(y/x)$

Example. Find the argument of $z = -1 - i$.

$$\tan \theta = \frac{y}{x} = \frac{-1}{-1} = 1$$



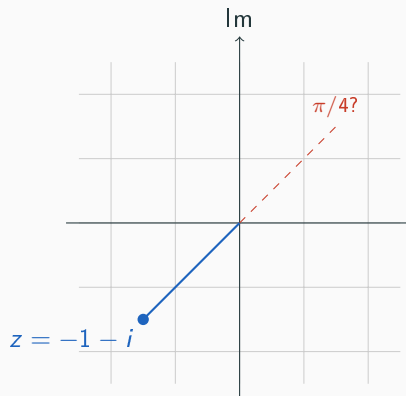
Warning: θ is not always $\tan^{-1}(y/x)$

Example. Find the argument of $z = -1 - i$.

$$\tan \theta = \frac{y}{x} = \frac{-1}{-1} = 1$$

$$\tan^{-1}(1) = \frac{\pi}{4}$$

but $\pi/4$ points into the first quadrant!



Warning: θ is not always $\tan^{-1}(y/x)$

Example. Find the argument of $z = -1 - i$.

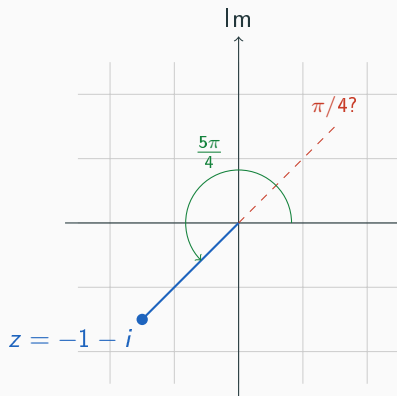
$$\tan \theta = \frac{y}{x} = \frac{-1}{-1} = 1$$

$$\tan^{-1}(1) = \frac{\pi}{4}$$

but $\pi/4$ points into the first quadrant!

$$\arg z = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$$

z is in the third quadrant



Warning: θ is not always $\tan^{-1}(y/x)$

Example. Find the argument of $z = -1 - i$.

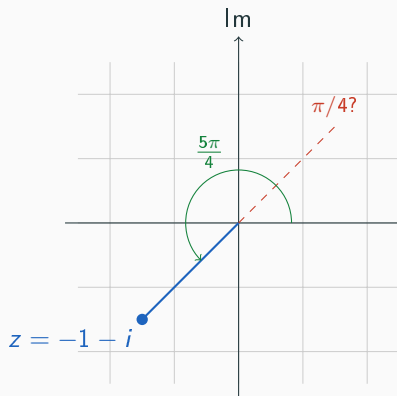
$$\tan \theta = \frac{y}{x} = \frac{-1}{-1} = 1$$

$$\tan^{-1}(1) = \frac{\pi}{4}$$

but $\pi/4$ points into the first quadrant!

$$\arg z = \pi + \frac{\pi}{4} = \frac{5\pi}{4} \quad z \text{ is in the third quadrant}$$

Indeed $\tan \frac{5\pi}{4} = 1$ too: $\tan$ repeats every π , so $\tan^{-1}$ only ever returns an angle in $(-\frac{\pi}{2}, \frac{\pi}{2})$.



Warning: θ is not always $\tan^{-1}(y/x)$

Example. Find the argument of $z = -1 - i$.

$$\tan \theta = \frac{y}{x} = \frac{-1}{-1} = 1$$

$$\tan^{-1}(1) = \frac{\pi}{4}$$

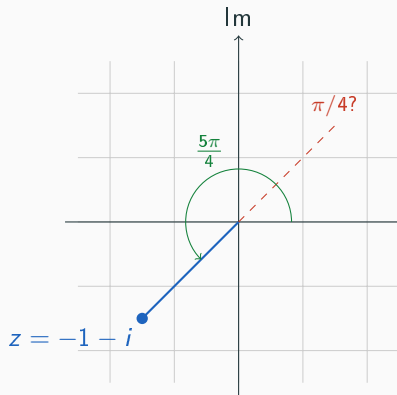
but $\pi/4$ points into the first quadrant!

$$\arg z = \pi + \frac{\pi}{4} = \frac{5\pi}{4} \quad z \text{ is in the third quadrant}$$

Indeed $\tan \frac{5\pi}{4} = 1$ too: $\tan$ repeats every π , so $\tan^{-1}$ only ever returns an angle in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Rule

Sketch the point first, then choose the angle in the right quadrant. Angles differing by 2π give the same direction ($\frac{5\pi}{4}$ or $-\frac{3\pi}{4}$); the one in $(-\pi, \pi]$ is the **principal argument**.



Example: converting between the two forms

Example. Write $1 + i$ and $-\sqrt{3} + i$ in polar form.

Example. Write $z = 2\angle\frac{\pi}{3}$ in Cartesian form.

Example: converting between the two forms

Example. Write $1 + i$ and $-\sqrt{3} + i$ in polar form.

$$1 + i: \quad r = \sqrt{1^2 + 1^2} = \sqrt{2}, \quad \tan \theta = \frac{1}{1} = 1, \quad \text{1st quadrant} \Rightarrow \theta = \frac{\pi}{4}$$

Example. Write $z = 2\angle\frac{\pi}{3}$ in Cartesian form.

Example: converting between the two forms

Example. Write $1 + i$ and $-\sqrt{3} + i$ in polar form.

$$1 + i: \quad r = \sqrt{1^2 + 1^2} = \sqrt{2}, \quad \tan \theta = \frac{1}{1} = 1, \quad \text{1st quadrant} \Rightarrow \theta = \frac{\pi}{4}$$

$$-\sqrt{3} + i: \quad r = \sqrt{3 + 1} = 2, \quad \tan \theta = -\frac{1}{\sqrt{3}}, \quad \text{2nd quadrant} \Rightarrow \theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

(Here $\tan^{-1}(-1/\sqrt{3}) = -\pi/6$ would be wrong: that angle points into the 4th quadrant.)

Example. Write $z = 2\angle\frac{\pi}{3}$ in Cartesian form.

Example: converting between the two forms

Example. Write $1 + i$ and $-\sqrt{3} + i$ in polar form.

$$1 + i: \quad r = \sqrt{1^2 + 1^2} = \sqrt{2}, \quad \tan \theta = \frac{1}{1} = 1, \quad \text{1st quadrant} \Rightarrow \theta = \frac{\pi}{4}$$

$$-\sqrt{3} + i: \quad r = \sqrt{3 + 1} = 2, \quad \tan \theta = -\frac{1}{\sqrt{3}}, \quad \text{2nd quadrant} \Rightarrow \theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

(Here $\tan^{-1}(-1/\sqrt{3}) = -\pi/6$ would be wrong: that angle points into the 4th quadrant.)

Example. Write $z = 2\angle\frac{\pi}{3}$ in Cartesian form.

$$x = r \cos \theta = 2 \cos \frac{\pi}{3} = 2 \times \frac{1}{2} = 1, \quad y = r \sin \theta = 2 \sin \frac{\pi}{3} = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$$

Example: converting between the two forms

Example. Write $1 + i$ and $-\sqrt{3} + i$ in polar form.

$$1 + i: \quad r = \sqrt{1^2 + 1^2} = \sqrt{2}, \quad \tan \theta = \frac{1}{1} = 1, \quad \text{1st quadrant} \Rightarrow \theta = \frac{\pi}{4}$$

$$-\sqrt{3} + i: \quad r = \sqrt{3 + 1} = 2, \quad \tan \theta = -\frac{1}{\sqrt{3}}, \quad \text{2nd quadrant} \Rightarrow \theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

(Here $\tan^{-1}(-1/\sqrt{3}) = -\pi/6$ would be wrong: that angle points into the 4th quadrant.)

Example. Write $z = 2\angle\frac{\pi}{3}$ in Cartesian form.

$$x = r \cos \theta = 2 \cos \frac{\pi}{3} = 2 \times \frac{1}{2} = 1, \quad y = r \sin \theta = 2 \sin \frac{\pi}{3} = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$$

$$z = 1 + \sqrt{3}i$$

Euler's formula and the exponential form

Euler's formula

$$e^{i\theta} = \cos \theta + i \sin \theta$$

Euler's formula and the exponential form

Euler's formula

$$e^{i\theta} = \cos \theta + i \sin \theta$$

So $z = r \cos \theta + i r \sin \theta = r(\cos \theta + i \sin \theta)$ becomes

$$z = re^{i\theta}$$

the **exponential form** (the usual way to write the polar form).

Euler's formula and the exponential form

Euler's formula

$$e^{i\theta} = \cos \theta + i \sin \theta$$

So $z = r \cos \theta + i r \sin \theta = r(\cos \theta + i \sin \theta)$ becomes

$\boxed{z = r e^{i\theta}}$ the **exponential form** (the usual way to write the polar form).

Examples:

$$e^{i\pi/2} = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = i, \quad e^{i\pi} = -1, \quad e^{2\pi i} = 1, \quad 1+i = \sqrt{2} e^{i\pi/4}, \quad -\sqrt{3}+i = 2e^{5i\pi/6}$$

Euler's formula and the exponential form

Euler's formula

$$e^{i\theta} = \cos \theta + i \sin \theta$$

So $z = r \cos \theta + i r \sin \theta = r(\cos \theta + i \sin \theta)$ becomes

$z = r e^{i\theta}$ the **exponential form** (the usual way to write the polar form).

Examples:

$$e^{i\pi/2} = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = i, \quad e^{i\pi} = -1, \quad e^{2\pi i} = 1, \quad 1+i = \sqrt{2} e^{i\pi/4}, \quad -\sqrt{3}+i = 2e^{5i\pi/6}$$

Why it is useful

Exponentials multiply by adding exponents: $e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$. Multiplication, powers and roots become easy.

Multiplying in exponential form

Rule

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} : \quad \text{multiply the moduli, add the arguments.}$$

Example. Compute $(1 + i)^2$ both ways.

Multiplying in exponential form

Rule

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} : \quad \text{multiply the moduli, add the arguments.}$$

A number $t = e^{i\alpha}$ with $|t| = 1$ (a **turn**) leaves the modulus alone: tz is z **rotated** by α . E.g. multiplying by $i = e^{i\pi/2}$ rotates by a quarter turn.

Example. Compute $(1 + i)^2$ both ways.

Multiplying in exponential form

Rule

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} : \quad \text{multiply the moduli, add the arguments.}$$

A number $t = e^{i\alpha}$ with $|t| = 1$ (a **turn**) leaves the modulus alone: tz is z **rotated** by α . E.g. multiplying by $i = e^{i\pi/2}$ rotates by a quarter turn.

Example. Compute $(1 + i)^2$ both ways.

$$\text{Cartesian: } (1 + i)^2 = 1 + 2i + i^2 = 2i$$

Multiplying in exponential form

Rule

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} : \quad \text{multiply the moduli, add the arguments.}$$

A number $t = e^{i\alpha}$ with $|t| = 1$ (a **turn**) leaves the modulus alone: tz is z **rotated** by α . E.g. multiplying by $i = e^{i\pi/2}$ rotates by a quarter turn.

Example. Compute $(1 + i)^2$ both ways.

$$\text{Cartesian: } (1 + i)^2 = 1 + 2i + i^2 = 2i$$

$$\text{Exponential: } (\sqrt{2} e^{i\pi/4})^2 = \sqrt{2} \cdot \sqrt{2} e^{i(\pi/4 + \pi/4)} = 2e^{i\pi/2} = 2i \quad \text{same answer } \checkmark$$

Multiplying in exponential form

Rule

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} : \quad \text{multiply the moduli, add the arguments.}$$

A number $t = e^{i\alpha}$ with $|t| = 1$ (a **turn**) leaves the modulus alone: tz is z **rotated** by α . E.g. multiplying by $i = e^{i\pi/2}$ rotates by a quarter turn.

Example. Compute $(1 + i)^2$ both ways.

$$\text{Cartesian: } (1 + i)^2 = 1 + 2i + i^2 = 2i$$

$$\text{Exponential: } (\sqrt{2} e^{i\pi/4})^2 = \sqrt{2} \cdot \sqrt{2} e^{i(\pi/4 + \pi/4)} = 2e^{i\pi/2} = 2i \quad \text{same answer } \checkmark$$

Compare: $\log(t_1 t_2) = \log t_1 + \log t_2$. Arguments behave like logarithms.

Powers: de Moivre's theorem

De Moivre's theorem

$$z^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n (\cos n\theta + i \sin n\theta)$$

To raise z to the power n : **modulus to the power n , argument multiplied by n .**
(Squaring: square the modulus, double the argument.)

Example. Find $(1 + i)^3$ in Cartesian form.

Powers: de Moivre's theorem

De Moivre's theorem

$$z^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n (\cos n\theta + i \sin n\theta)$$

To raise z to the power n : **modulus to the power n , argument multiplied by n .**
(Squaring: square the modulus, double the argument.)

Example. Find $(1 + i)^3$ in Cartesian form.

$$(1 + i)^3 = (\sqrt{2} e^{i\pi/4})^3 = (\sqrt{2})^3 e^{3i\pi/4} = 2\sqrt{2} e^{3i\pi/4} \quad \text{de Moivre}$$

Powers: de Moivre's theorem

De Moivre's theorem

$$z^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n (\cos n\theta + i \sin n\theta)$$

To raise z to the power n : **modulus to the power n , argument multiplied by n .**
(Squaring: square the modulus, double the argument.)

Example. Find $(1 + i)^3$ in Cartesian form.

$$\begin{aligned}(1 + i)^3 &= (\sqrt{2} e^{i\pi/4})^3 = (\sqrt{2})^3 e^{3i\pi/4} = 2\sqrt{2} e^{3i\pi/4} && \text{de Moivre} \\ &= 2\sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right) = 2\sqrt{2} \left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right) && \text{Euler}\end{aligned}$$

Powers: de Moivre's theorem

De Moivre's theorem

$$z^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n (\cos n\theta + i \sin n\theta)$$

To raise z to the power n : **modulus to the power n , argument multiplied by n .**
(Squaring: square the modulus, double the argument.)

Example. Find $(1 + i)^3$ in Cartesian form.

$$\begin{aligned}(1 + i)^3 &= (\sqrt{2} e^{i\pi/4})^3 = (\sqrt{2})^3 e^{3i\pi/4} = 2\sqrt{2} e^{3i\pi/4} && \text{de Moivre} \\ &= 2\sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right) = 2\sqrt{2} \left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right) && \text{Euler} \\ &= -2 + 2i\end{aligned}$$

Powers: de Moivre's theorem

De Moivre's theorem

$$z^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n (\cos n\theta + i \sin n\theta)$$

To raise z to the power n : **modulus to the power n , argument multiplied by n .**
(Squaring: square the modulus, double the argument.)

Example. Find $(1 + i)^3$ in Cartesian form.

$$\begin{aligned}(1 + i)^3 &= (\sqrt{2} e^{i\pi/4})^3 = (\sqrt{2})^3 e^{3i\pi/4} = 2\sqrt{2} e^{3i\pi/4} && \text{de Moivre} \\ &= 2\sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right) = 2\sqrt{2} \left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right) && \text{Euler} \\ &= -2 + 2i\end{aligned}$$

Check directly: $(1 + i)^3 = (1 + i)(1 + i)^2 = (1 + i)(2i) = 2i + 2i^2 = -2 + 2i$ ✓

Roots of unity: algebraically

The n th **roots of unity** are the solutions of

$$x^n - 1 = 0.$$

$x = 1$ is always one of them. What are the others?

Roots of unity: algebraically

The n th **roots of unity** are the solutions of

$$x^n - 1 = 0.$$

$x = 1$ is always one of them. What are the others?

$$n = 2 : \quad x^2 - 1 = (x - 1)(x + 1) = 0 \Rightarrow x = 1, -1$$

Roots of unity: algebraically

The n th **roots of unity** are the solutions of

$$x^n - 1 = 0.$$

$x = 1$ is always one of them. What are the others?

$$n = 2 : \quad x^2 - 1 = (x - 1)(x + 1) = 0 \Rightarrow x = 1, -1$$

$$n = 3 : \quad x^3 - 1 = (x - 1)(x^2 + x + 1) = 0 \quad \text{(factorising: next week)}$$

Roots of unity: algebraically

The n th **roots of unity** are the solutions of

$$x^n - 1 = 0.$$

$x = 1$ is always one of them. What are the others?

$$n = 2: \quad x^2 - 1 = (x - 1)(x + 1) = 0 \Rightarrow x = 1, -1$$

$$n = 3: \quad x^3 - 1 = (x - 1)(x^2 + x + 1) = 0$$

(factorising: next week)

$$x^2 + x + 1 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{1 - 4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

quadratic formula

Roots of unity: algebraically

The n th **roots of unity** are the solutions of

$$x^n - 1 = 0.$$

$x = 1$ is always one of them. What are the others?

$$n = 2: \quad x^2 - 1 = (x - 1)(x + 1) = 0 \Rightarrow x = 1, -1$$

$$n = 3: \quad x^3 - 1 = (x - 1)(x^2 + x + 1) = 0 \quad \text{(factorising: next week)}$$

$$x^2 + x + 1 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{1 - 4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i \quad \text{quadratic formula}$$

$$\text{Cube roots of unity: } x = 1, \quad x = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad x = -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

Roots of unity: algebraically

The n th **roots of unity** are the solutions of

$$x^n - 1 = 0.$$

$x = 1$ is always one of them. What are the others?

$$n = 2: \quad x^2 - 1 = (x - 1)(x + 1) = 0 \Rightarrow x = 1, -1$$

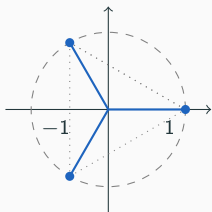
$$n = 3: \quad x^3 - 1 = (x - 1)(x^2 + x + 1) = 0 \quad \text{(factorising: next week)}$$

$$x^2 + x + 1 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{1 - 4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i \quad \text{quadratic formula}$$

$$\text{Cube roots of unity: } x = 1, \quad x = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad x = -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

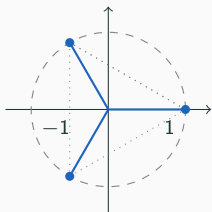
$$n = 4: \quad x^4 - 1 = (x^2 - 1)(x^2 + 1) \text{ gives } x = 1, -1, i, -i.$$

Roots of unity on the Argand diagram

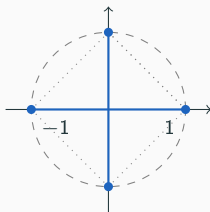


$$z^3 = 1$$

Roots of unity on the Argand diagram

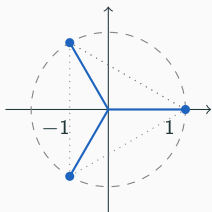


$$z^3 = 1$$

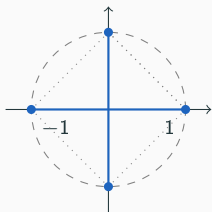


$$z^4 = 1$$

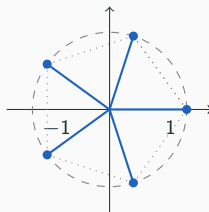
Roots of unity on the Argand diagram



$$z^3 = 1$$

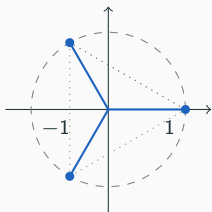


$$z^4 = 1$$

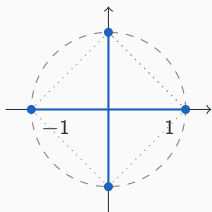


$$z^5 = 1$$

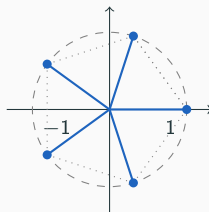
Roots of unity on the Argand diagram



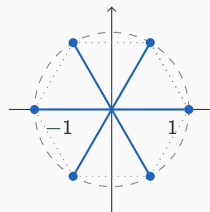
$$z^3 = 1$$



$$z^4 = 1$$

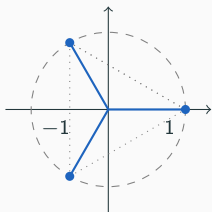


$$z^5 = 1$$

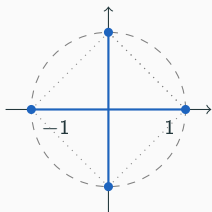


$$z^6 = 1$$

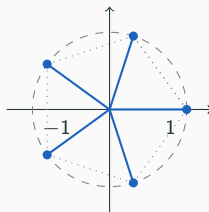
Roots of unity on the Argand diagram



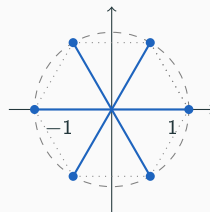
$$z^3 = 1$$



$$z^4 = 1$$



$$z^5 = 1$$



$$z^6 = 1$$

Pattern

The n roots of $z^n = 1$ all have modulus 1 and are **equally spaced** around the unit circle, an angle $\frac{2\pi}{n}$ apart, starting from $z = 1$: the vertices of a regular n -gon.

Roots via the exponential form

Since $\cos$ and $\sin$ repeat every 2π , the same number has **many** arguments:

$$z = re^{i\theta} = re^{i(\theta+2\pi k)}, \quad k = 0, \pm 1, \pm 2, \dots$$

Example. Solve $z^5 = 1$.

Roots via the exponential form

Since $\cos$ and $\sin$ repeat every 2π , the same number has **many** arguments:

$$z = re^{i\theta} = re^{i(\theta+2\pi k)}, \quad k = 0, \pm 1, \pm 2, \dots$$

*n*th roots

Taking the *n*th root, each *k* gives a different answer:

$$z^{1/n} = r^{1/n} e^{i(\theta+2\pi k)/n}, \quad k = 0, 1, 2, \dots, n-1.$$

n distinct roots, all of modulus $r^{1/n}$, spaced $2\pi/n$ apart: a regular polygon. ($k = n$ repeats $k = 0$.)

Example. Solve $z^5 = 1$.

Roots via the exponential form

Since $\cos$ and $\sin$ repeat every 2π , the same number has **many** arguments:

$$z = re^{i\theta} = re^{i(\theta+2\pi k)}, \quad k = 0, \pm 1, \pm 2, \dots$$

n th roots

Taking the n th root, each k gives a different answer:

$$z^{1/n} = r^{1/n} e^{i(\theta+2\pi k)/n}, \quad k = 0, 1, 2, \dots, n-1.$$

n distinct roots, all of modulus $r^{1/n}$, spaced $2\pi/n$ apart: a regular polygon. ($k = n$ repeats $k = 0$.)

Example. Solve $z^5 = 1$.

$$z^5 = 1 = e^{i(0+2\pi k)}$$

1 has modulus 1 and argument 0

Roots via the exponential form

Since $\cos$ and $\sin$ repeat every 2π , the same number has **many** arguments:

$$z = re^{i\theta} = re^{i(\theta+2\pi k)}, \quad k = 0, \pm 1, \pm 2, \dots$$

n th roots

Taking the n th root, each k gives a different answer:

$$z^{1/n} = r^{1/n} e^{i(\theta+2\pi k)/n}, \quad k = 0, 1, 2, \dots, n-1.$$

n distinct roots, all of modulus $r^{1/n}$, spaced $2\pi/n$ apart: a regular polygon. ($k = n$ repeats $k = 0$.)

Example. Solve $z^5 = 1$.

$$z^5 = 1 = e^{i(0+2\pi k)}$$

$$z = e^{2\pi i k/5}, \quad k = 0, 1, 2, 3, 4$$

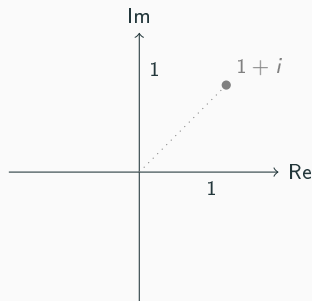
1 has modulus 1 and argument 0

five roots, 72° apart (previous slide)

Example: all values of $(1 + i)^{1/3}$

$$1 + i = \sqrt{2} e^{i\pi/4} = 2^{1/2} e^{i(\pi/4 + 2\pi k)}$$

all arguments of $1 + i$

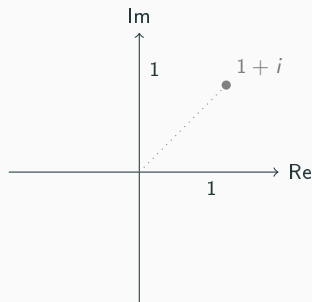


Example: all values of $(1 + i)^{1/3}$

$$1 + i = \sqrt{2} e^{i\pi/4} = 2^{1/2} e^{i(\pi/4 + 2\pi k)}$$

all arguments of $1 + i$

$$(1 + i)^{1/3} = 2^{1/6} e^{i(\pi/12 + 2\pi k/3)}, \quad k = 0, 1, 2 \quad \text{cube root}$$



Example: all values of $(1 + i)^{1/3}$

$$1 + i = \sqrt{2} e^{i\pi/4} = 2^{1/2} e^{i(\pi/4 + 2\pi k)}$$

all arguments of $1 + i$

$$(1 + i)^{1/3} = 2^{1/6} e^{i(\pi/12 + 2\pi k/3)}, \quad k = 0, 1, 2 \quad \text{cube root}$$

$$z_1 = 2^{1/6} e^{i\pi/12}$$

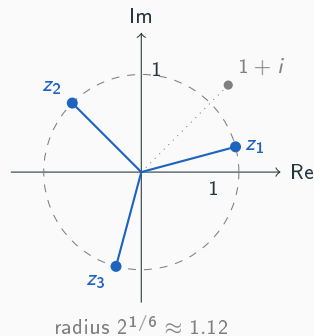
$k = 0$

$$z_2 = 2^{1/6} e^{i(\pi/12 + 2\pi/3)} = 2^{1/6} e^{3i\pi/4}$$

$k = 1$

$$z_3 = 2^{1/6} e^{i(\pi/12 + 4\pi/3)} = 2^{1/6} e^{17i\pi/12}$$

$k = 2$



Example: all values of $(1 + i)^{1/3}$

$$1 + i = \sqrt{2} e^{i\pi/4} = 2^{1/2} e^{i(\pi/4 + 2\pi k)}$$

all arguments of $1 + i$

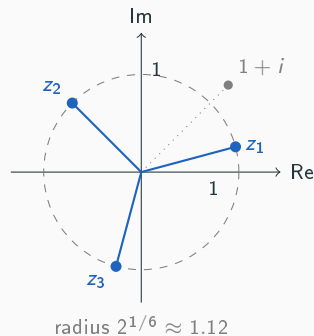
$$(1 + i)^{1/3} = 2^{1/6} e^{i(\pi/12 + 2\pi k/3)}, \quad k = 0, 1, 2 \quad \text{cube root}$$

$$z_1 = 2^{1/6} e^{i\pi/12} \quad k = 0$$

$$z_2 = 2^{1/6} e^{i(\pi/12 + 2\pi/3)} = 2^{1/6} e^{3i\pi/4} \quad k = 1$$

$$z_3 = 2^{1/6} e^{i(\pi/12 + 4\pi/3)} = 2^{1/6} e^{17i\pi/12} \quad k = 2$$

Three roots, 120° apart, on a circle of radius $2^{1/6}$; $k = 3$ repeats z_1 .



Example: all values of $(1 + i)^{1/3}$

$$1 + i = \sqrt{2} e^{i\pi/4} = 2^{1/2} e^{i(\pi/4 + 2\pi k)} \quad \text{all arguments of } 1 + i$$

$$(1 + i)^{1/3} = 2^{1/6} e^{i(\pi/12 + 2\pi k/3)}, \quad k = 0, 1, 2 \quad \text{cube root}$$

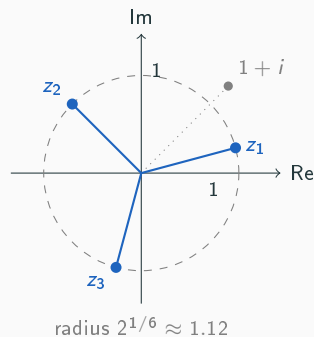
$$z_1 = 2^{1/6} e^{i\pi/12} \quad k = 0$$

$$z_2 = 2^{1/6} e^{i(\pi/12 + 2\pi/3)} = 2^{1/6} e^{3i\pi/4} \quad k = 1$$

$$z_3 = 2^{1/6} e^{i(\pi/12 + 4\pi/3)} = 2^{1/6} e^{17i\pi/12} \quad k = 2$$

Three roots, 120° apart, on a circle of radius $2^{1/6}$; $k = 3$ repeats z_1 .

Check z_1 : $(2^{1/6})^3 = 2^{1/2} = \sqrt{2}$ and $3 \times \frac{\pi}{12} = \frac{\pi}{4}$, so
 $z_1^3 = \sqrt{2} e^{i\pi/4} = 1 + i$ ✓



Summary

- $i^2 = -1$; otherwise i behaves like any algebraic symbol. Quadratics with $b^2 - 4ac < 0$ have two complex solutions.
- **Cartesian form** $z = x + iy$, the point (x, y) ; conjugate $z^* = x - iy$;
 $zz^* = x^2 + y^2 = |z|^2$.
- Divide by multiplying top and bottom by the conjugate of the denominator.
- **Exponential form** $z = re^{i\theta}$: $r = \sqrt{x^2 + y^2}$, $\tan \theta = y/x$ **with the quadrant checked by a sketch**; Euler: $e^{i\theta} = \cos \theta + i \sin \theta$.
- Multiply: multiply moduli, add arguments. De Moivre: $z^n = r^n e^{in\theta}$.
- n th roots: $z^{1/n} = r^{1/n} e^{i(\theta + 2\pi k)/n}$, $k = 0, \dots, n-1$: n points equally spaced on a circle.