
MAA308/309

Determinants & Inverses

Ciprian D. Coman

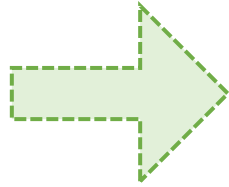
OUTLINE

❖ **Determinants**

❖ **The inverse** of a (square) matrix

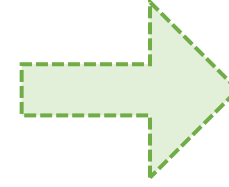
MOTIVATION

$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y = -4 \\ 2x - 5y + 5z = 17 \end{cases}$$



$$AX = b$$

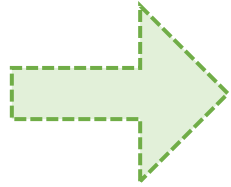
A, X, b
new
mathematical
objects



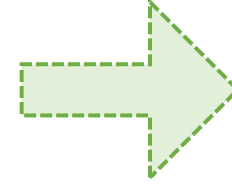
$$X = A^{-1}b$$

MOTIVATION

$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y = -4 \\ 2x - 5y + 5z = 17 \end{cases}$$



$$AX = b$$



$$X = A^{-1}b$$

A, X, b
new
mathematical
objects

ASIDE:

$$5x = 2 \quad \Rightarrow \quad x = \frac{2}{5} = \frac{1}{5} \times 2 = 5^{-1} \times 2 = (5^{-1})(2)$$

THE IDENTITY MATRIX:

For numbers:

$$2^{-1} \cdot 2 = 1$$

$$23^{-1} \cdot 23 = 1$$

THE IDENTITY MATRIX:

For numbers:

$$2^{-1} \cdot 2 = 1$$

$$23^{-1} \cdot 23 = 1$$

The **inverse** of a (square) matrix A is denoted by A^{-1} and has the property that

$$AA^{-1} = A^{-1}A = I$$

THE IDENTITY MATRIX:

For numbers:

$$2^{-1} \cdot 2 = 1$$

$$23^{-1} \cdot 23 = 1$$

The **inverse** of a (square) matrix A is denoted by A^{-1} and has the property that

$$AA^{-1} = A^{-1}A = I$$

identity matrix

$$I = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{2 \times 2}$$

$$I = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{3 \times 3}$$

ETC.

THE IDENTITY MATRIX:

$$(a) \begin{bmatrix} 3 & -2 \\ 4 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & 0 \\ -1 & 1 \end{bmatrix}$$

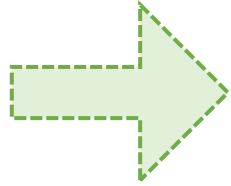
THE IDENTITY MATRIX:

$$(a) \begin{bmatrix} 3 & -2 \\ 4 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & 0 \\ -1 & 1 \end{bmatrix}$$

$$(b) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$$

Back to the start...

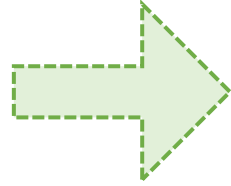
$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y = -4 \\ 2x - 5y + 5z = 17 \end{cases}$$



$$AX = b$$

Back to the start...

$$\begin{cases} 1x - 2y + 3z = 9 \\ -1x + 3y + 0z = -4 \\ 2x - 5y + 5z = 17 \end{cases}$$



$$AX = b$$

Back to the start...

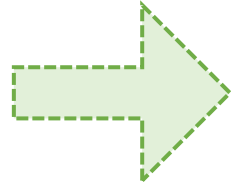
$$\begin{cases} 1x - 2y + 3z = 9 \\ -1x + 3y + 0z = -4 \\ 2x - 5y + 5z = 17 \end{cases} \quad \Rightarrow \quad AX = b$$

$$\underbrace{\begin{bmatrix} 1 & -2 & 3 \\ -1 & 3 & 0 \\ 2 & -5 & 5 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_X = \underbrace{\begin{bmatrix} 9 \\ -4 \\ 17 \end{bmatrix}}_b$$

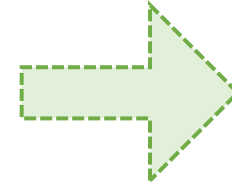
$3 \times 3 \qquad 3 \times 1 \qquad 3 \times 1$

Back to the start...

$$\begin{cases} 1x - 2y + 3z = 9 \\ -1x + 3y + 0z = -4 \\ 2x - 5y + 5z = 17 \end{cases}$$



$$AX = b$$



$$X = A^{-1}b$$

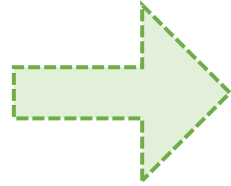
$$\underbrace{\begin{bmatrix} 1 & -2 & 3 \\ -1 & 3 & 0 \\ 2 & -5 & 5 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_X = \underbrace{\begin{bmatrix} 9 \\ -4 \\ 17 \end{bmatrix}}_b$$

$3 \times 3 \qquad 3 \times 1 \qquad 3 \times 1$

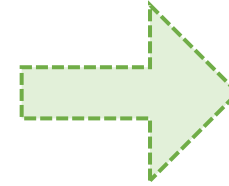
Under what conditions
can we find A^{-1} ?

Back to the start...

$$\begin{cases} 1x - 2y + 3z = 9 \\ -1x + 3y + 0z = -4 \\ 2x - 5y + 5z = 17 \end{cases}$$



$$AX = b$$



$$X = A^{-1}b$$

$$\underbrace{\begin{bmatrix} 1 & -2 & 3 \\ -1 & 3 & 0 \\ 2 & -5 & 5 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_X = \underbrace{\begin{bmatrix} 9 \\ -4 \\ 17 \end{bmatrix}}_b$$

$3 \times 3 \qquad 3 \times 1 \qquad 3 \times 1$

Under what conditions
can we find A^{-1} ?

Short answer: $\det(A) \neq 0$

DETERMINANTS



Each **square** matrix A has a **determinant**, a real number denoted either by $\det(A)$ or $|A|$

Each **square** matrix A has a **determinant**, a real number denoted either by $\det(A)$ or $|A|$

2 x 2 matrices:

The **determinant** of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is given by

$$\det(A) = |A| = a_{11}a_{22} - a_{21}a_{12}.$$

Each **square** matrix A has a **determinant**, a real number denoted either by $\det(A)$ or $|A|$

2 x 2 matrices:

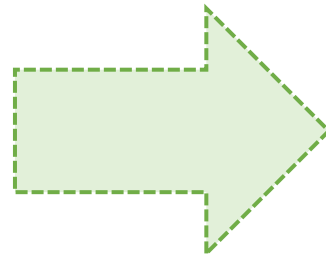
The **determinant** of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is given by

$$\det(A) = |A| = a_{11}a_{22} - a_{21}a_{12}.$$

An easy way for remembering:



$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

The diagram illustrates the calculation of the determinant for a 2x2 matrix. It shows the matrix $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ with blue arrows indicating the products to be subtracted: one arrow from a_{11} to a_{22} and another from a_{21} to a_{12} . The result is shown as $= a_{11}a_{22} - a_{21}a_{12}$.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Examples: Find the determinant of each matrix:

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 3 \\ 2 & 4 \end{bmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Examples: Find the determinant of each matrix:

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 3 \\ 2 & 4 \end{bmatrix}$$

$$\begin{aligned} |A| &= \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} \\ &= 2(2) - 1(-3) \\ &= 4 + 3 = 7 \end{aligned}$$

$$\begin{aligned} |B| &= \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} \\ &= 2(2) - 4(1) \\ &= 4 - 4 = 0 \end{aligned}$$

$$\begin{aligned} |C| &= \begin{vmatrix} 0 & 3 \\ 2 & 4 \end{vmatrix} \\ &= 0(4) - 2(3) \\ &= 0 - 6 = -6 \end{aligned}$$

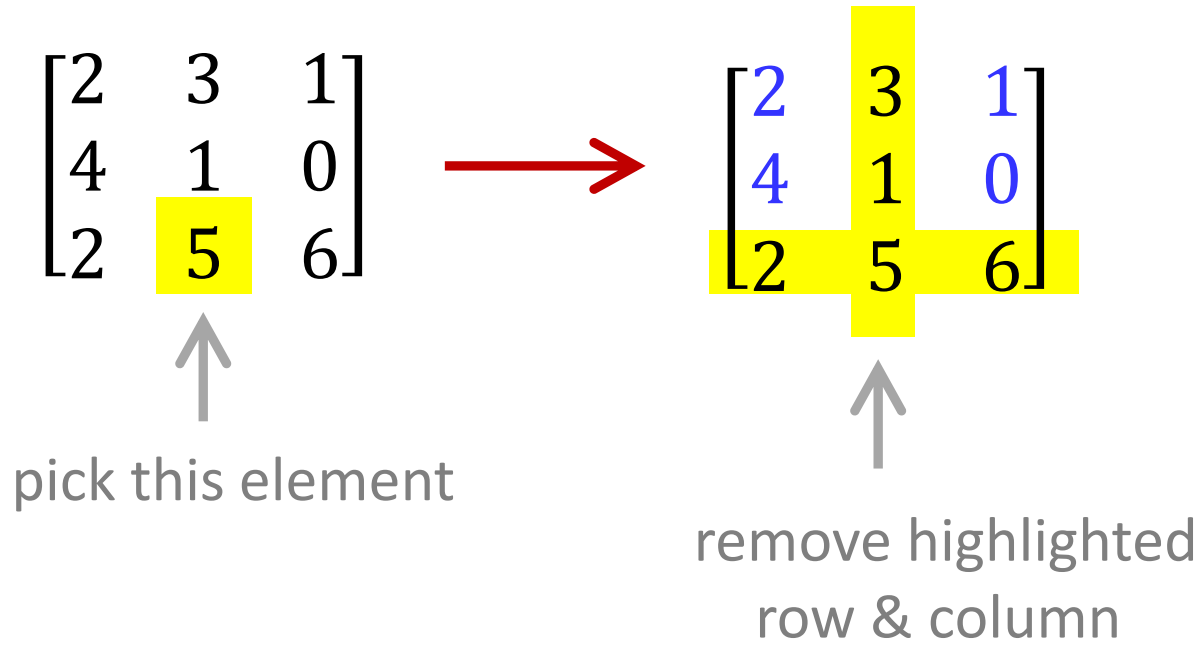
MINORS & COFACTORS

$$\begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 0 \\ 2 & 5 & 6 \end{bmatrix}$$

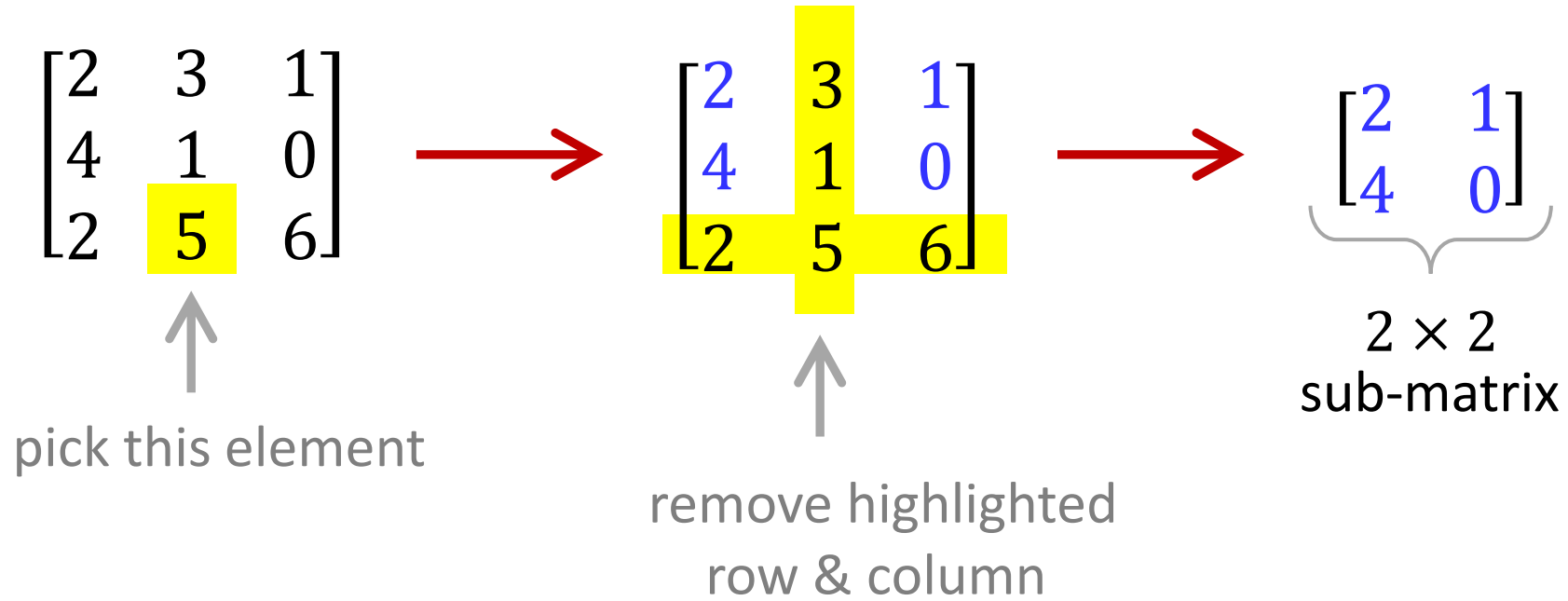


pick this element

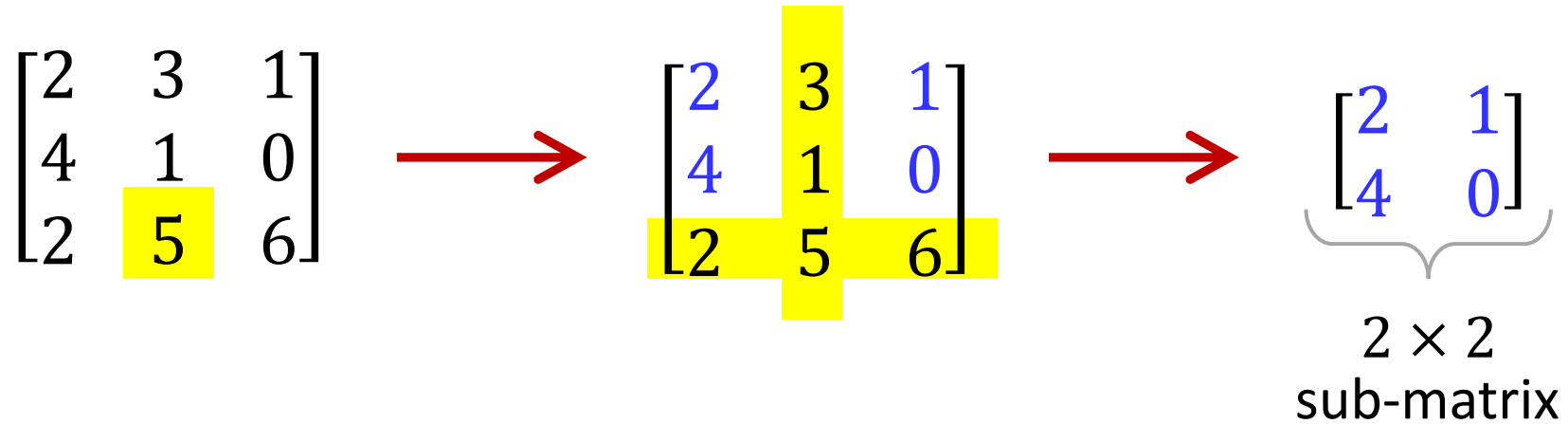
MINORS & COFACTORS



MINORS & COFACTORS



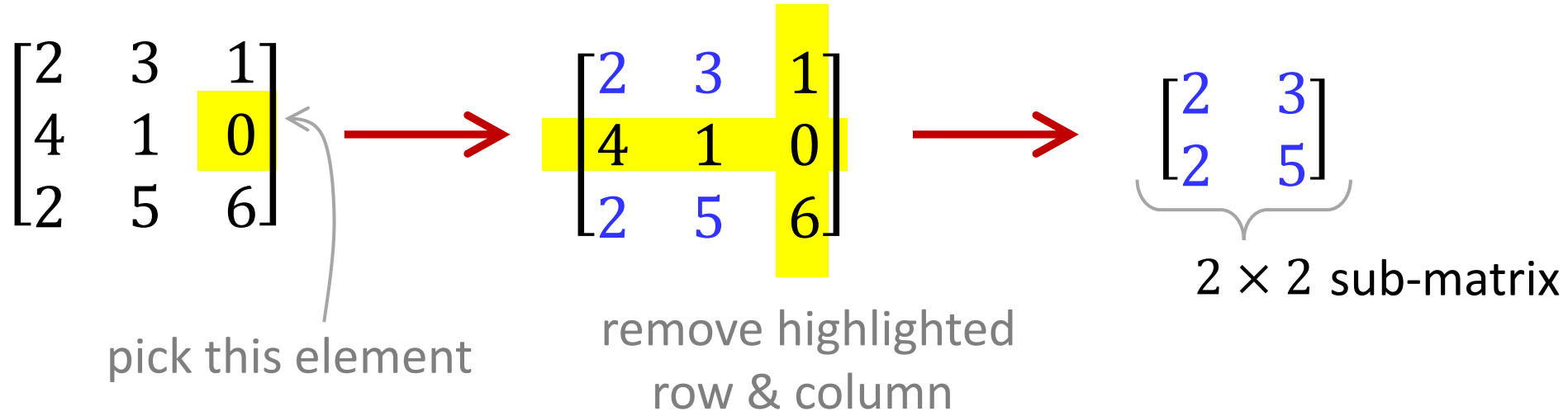
MINORS & COFACTORS



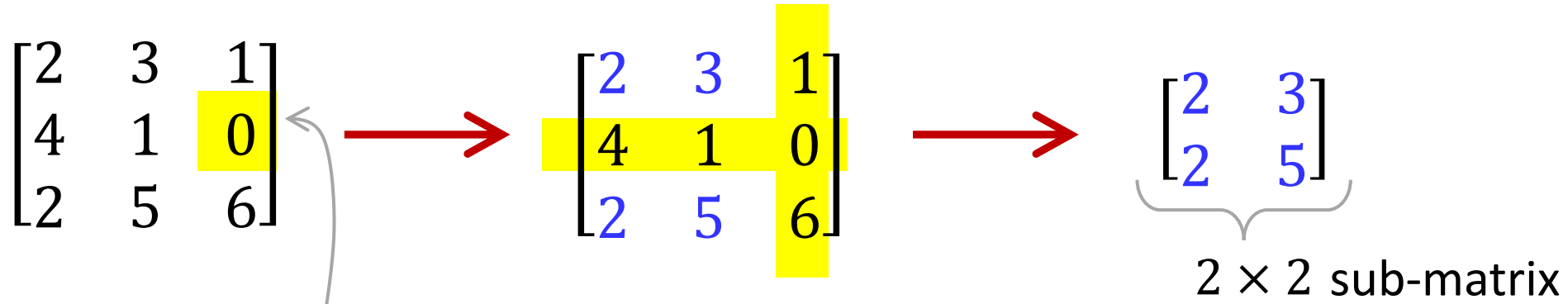
the **determinant** of this is
the **MINOR** of the highlighted element (5)
in the original matrix:

$$\begin{vmatrix} 2 & 1 \\ 4 & 0 \end{vmatrix} = (2)(0) - (4)(1) = 0 - 4 = -4$$

MINORS & COFACTORS



MINORS & COFACTORS

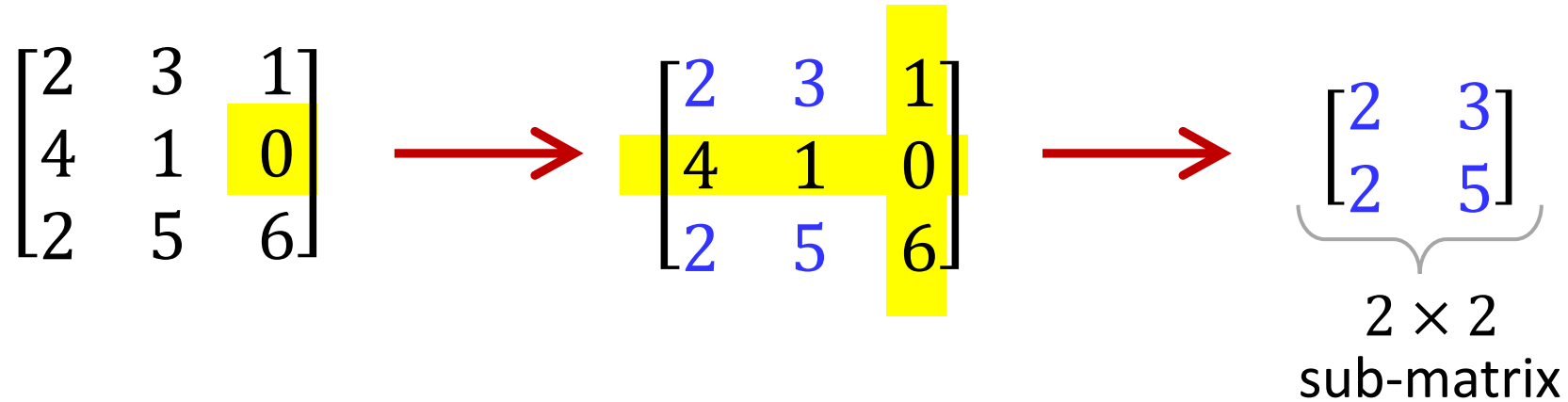


pick this element

remove highlighted
row & column

$$\begin{vmatrix} 2 & 3 \\ 2 & 5 \end{vmatrix} = (2)(5) - (2)(3) = 10 - 6 = 4$$

MINORS & COFACTORS



$$\begin{vmatrix} 2 & 3 \\ 2 & 5 \end{vmatrix} = (2)(5) - (2)(3) = 10 - 6 = 4$$

Observation:

We can repeat this process **for each element** of the original matrix
(9 minors in total)

COFACTOR $\longrightarrow$ $\pm$ MINOR

COFACTOR $\longrightarrow$ $\pm$ MINOR

$$C = (-1)^{ROW \# + COL \#} M$$

COFACTOR $\longrightarrow$ $\pm$ MINOR

$$C = (-1)^{ROW \# + COL \#} M$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 0 \\ 2 & 5 & 6 \end{bmatrix}$$

$$C_{32} = (-1)^{3+2} M_{32} = -M_{32} = -(-4) = 4$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 0 \\ 2 & 5 & 6 \end{bmatrix}$$

$$C_{23} = (-1)^{2+3} M_{23} = -M_{23} = -(4) = -4$$

DETERMINANTS OF 3×3 MATRICES:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

DETERMINANTS OF 3×3 MATRICES:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

C_{11}

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$+ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

C_{12}

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$- \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

C_{13}

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$+ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example:

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

$$\det(A) = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13}$$

Example:

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

$$\det(A) = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13}$$

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix} \rightarrow M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1 \rightarrow c_{11} = M_{11} = -1$$

Example:

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

$$\det(A) = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13}$$

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix} \rightarrow M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1 \rightarrow c_{11} = M_{11} = -1$$

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix} \rightarrow M_{12} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = -5 \rightarrow c_{12} = -M_{12} = 5$$

Example:

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

$$\det(A) = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13}$$

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

$$\rightarrow M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1 \rightarrow c_{11} = M_{11} = -1$$

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

$$\rightarrow M_{12} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = -5 \rightarrow c_{12} = -M_{12} = 5$$

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

$$\rightarrow M_{13} = \begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = 4 \rightarrow c_{13} = M_{13} = 4$$

Example:

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= (0)(-1) + (2)(5) + (1)(4) = 14 \end{aligned}$$

$$\text{or } \det(A) = 14$$

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

$$\rightarrow M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1 \rightarrow C_{11} = M_{11} = -1$$

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

$$\rightarrow M_{12} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = -5 \rightarrow C_{12} = -M_{12} = 5$$

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

$$\rightarrow M_{13} = \begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = 4 \rightarrow C_{13} = M_{13} = 4$$

YOUR TURN!

(10 minutes)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

The determinant of the above matrix is:

a). 0

b). -6

c). 6

d). 12

REMINDER:

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

C_{11}	C_{12}	C_{13}
$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$	$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$	$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$
$+ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$	$- \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$	$+ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Solution:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\det(A) = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \rightarrow M_{11} = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = -3 \rightarrow c_{11} = M_{11} = -3$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \rightarrow M_{12} = \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = -6 \rightarrow c_{12} = -M_{12} = 6$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \rightarrow M_{13} = \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = -3 \rightarrow c_{13} = M_{13} = -3$$

Solution:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= (1)(-3) + (2)(6) + (3)(-3) = 0 \end{aligned}$$

$$\text{or } \det(A) = 0$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\rightarrow M_{11} = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = -3 \rightarrow C_{11} = M_{11} = -3$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\rightarrow M_{12} = \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = -6 \rightarrow C_{12} = -M_{12} = 6$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\rightarrow M_{13} = \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = -3 \rightarrow C_{13} = M_{13} = -3$$