The Green of Green Functions

In 1828, an English miller from Nottingham published a mathematical essay that generated little response. George Green's analysis, however, has since found applications in areas ranging from classical electrostatics to modern quantum field theory.

Lawrie Challis and Fred Sheard

Nottingham, an attractive and thriving town in the English Midlands, is famous for its association with Robin Hood, whose statue stands in the shadow of the castle wall. The Sheriff of Nottingham still has a special role in the city government although happily no longer strikes terror into the hearts of the good citizens.

Recently a new attraction, a windmill, has appeared on the Nottingham skyline (see figure 1). The sails turn on windy days and the adjoining mill shop sells packets of stone ground flour but also, more surprisingly, tracts on mathematical physics. The connection between the flour and the physics is part of the mill's unique character and is explained by a plaque once attached to the side of the mill tower that said,

HERE LIVED AND LABOURED
GEORGE GREEN
MATHEMATICIAN
B.1793–D.1841.

That is the Green of Green's theorem, which is familiar to physics undergraduate students worldwide, and of the Green functions that are used in many branches of both classical and quantum physics.

Early life and education

George Green's father had a bakery near the center of Nottingham, then a town with a population of about 30 000. His only son, George, was baptized on 14 July 1793. In March 1801, George was enrolled as pupil number 255 in Robert Goodacre's Academy, where he stayed for only 18 months. When he was nine years old, he was put to work in his father's bakery. The period spent at Goodacre's Academy was Green's only formal education before, at the age of 40, he went to the University of Cambridge. He was fortunate to have had even that amount of schooling. Most children had none, although some were taught to read and write at Sunday schools. Green was also fortunate that his father could afford to educate him privately and chose the academy of Robert Goodacre, a man known for his enthusiasm for astronomy and natural science.

Green spent five years in his father's bakery and was then sent to learn to be a miller in the tower mill that his father had built on a hill in the village of Sneinton, about a mile away from Nottingham. The mill was five stories high and stood in a yard with granaries, stabling for eight horses, and a cottage for the miller John Smith, his wife, and his daughter Jane. The streets of Nottingham were not safe places to walk in after dark so, for many years before his family built a house next to the mill, Green spent most of his days and many of his nights working and indeed living in the mill. When he was 31, Jane Smith bore him a daughter. They had seven children in all but never married. It was said that Green's father felt that Jane was not a suitable wife for the son of a prosperous tradesman and landowner and threatened to disinherit him.

Little is known about Green's life from 1802 until 1823. In particular, it is not known whether he received any help in his mathematical development or if he was entirely self-taught. He may have received help from John Toplis, a fellow of Queens' College in the University of Cambridge and headmaster of the Nottingham Grammar School. Toplis's translation of Pierre-Simon Laplace's book *Mécanique Céleste*, published in Nottingham in 1814, seems a likely source of Green's interest in potential theory. The work was unusual in Britain at that time inasmuch as Toplis used Gottfried Leibniz's more convenient notation for differentials rather than Isaac Newton's. Because Green adapted the Leibniz notation, it seems plausible that Green was influenced by Toplis, but there is no evidence that Toplis acted in any way as his tutor.

In 1823, Green joined the Nottingham Subscription Library, the center of intellectual activity in the town. The library was situated in Bromley House (see figure 2). Library membership provided Green with encouragement, support, and access to the *Philosophical Transactions of the Royal Society* and other scientific journals. These did not include overseas journals, but the *Transactions* listed the contents of those journals, and that would have allowed Green to obtain reprints directly from the authors.

Green's essay of 1828

Green's first published work, in 1828, was *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism.* This major work, some 70 pages long, contains the derivation of Green's theorem and applies the theorem, in conjunction with Green functions, to electrostatic problems. Its title page is reproduced in figure 3.

The customary route to publication of scientific papers in England was then through one of the journals of the two learned societies, the Royal Society and the Cambridge Philosophical Society. But Green, with no qualifications and no contacts with the scientific establishment, felt that it would be presumptuous to submit his paper to a journal. So he paid for his paper to be published privately in Nottingham. The tentative way in which he approached publication is apparent in his foreword. There, he expressed his hope that "the difficulty of the subject will incline mathematicians to read this work with indulgence, more particularly when they are informed that it was written by a young man, who has been obliged to obtain the little knowledge he possesses, at such intervals and by such means, as other indispensable avocations which offer but few opportunities of mental improvement, afforded."

Each copy of Green's essay cost 7½ shillings, roughly the weekly wage of a Nottingham stocking maker. Purchasers included local doctors, schoolmasters, clergymen, and manufacturers of lace and hosiery, Nottingham's main industry at the time. Nearly half of those who bought the work were fellow members of the Nottingham Subscription

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Figure 1. Green's mill, restored to working order in 1985, is located in Sneinton, England, within a mile of Nottingham's city center. The George Green Science Center in the mill yard features hands-on science exhibits and multimedia displays that illustrate Green's life. (Photograph by Jake Matchett, Media Centre, Nottingham University.)

Library. Few if any could have understood the essay, so they must evidently have had considerable confidence in its author's abilities.

Green's main purpose in publishing would have been to bring his work to the attention of other mathematicians in the UK and overseas. It appears, however, that with one exception, there was little or no response. That must have been very disheartening.

The exception was Edward Bromhead, who was evidently impressed by Green's paper. He replied immediately, offering to help Green get any future papers published in a journal. Bromhead was a wealthy and influential person, a public figure and a benefactor to the city of Lincoln, 35 miles northeast of Nottingham. He had studied mathematics at Glasgow University and then at Cambridge, and although his position in society took him away from an academic career, he was in close contact with a number of leading British mathematicians and scientists including Charles Babbage, John Herschel, and William Whewell.

So Bromhead was well placed to bring Green into contact with those and other mathematicians, and must have been surprised and disappointed that he received no reply from Green. Eventually one arrived, some 20 months later. Green's long letter explained how pleased and grateful he had been to hear from Bromhead and how he had originally intended to write and accept his offer. But he had been advised that Bromhead's letter to him was written out of politeness and that, given the differences in their social positions, it would have been inappropriate to reply. Only later had he discovered how poor that advice had been.

Bromhead quickly responded and the exchange was followed by many others and by meetings at Bromhead's house near Lincoln. With Bromhead's help, in 1833, Green published his first paper in a journal. That paper was again on electricity, but on Bromhead's advice, Green then left the topic, which was of little interest to mathematicians in the UK at the time. He moved on to the more fashionable topics of hydrodynamics, wave motion, and optics, the subjects of his eight papers published between 1835 and 1839.

Soon after they met, Green told Bromhead of his dream of going to Cambridge. He returned to the subject in a letter in April 1833, in which he wrote, "You are aware that I have an inclination for Cambridge if there were a fair prospect for success. Unfortunately, I possess little Latin, less Greek, have seen too many winters, and am thus held in a state of suspense by counteracting motives." Green must also have been aware that the existence of Jane and his children—four at that time—was somewhat at odds with the university requirement that members be celibate! But it seemed that a lack of celibacy could be tolerated so long as the member was not actually married. With Bromhead's influence, Green entered Cambridge in October 1833 as an undergraduate, becoming a member of Gonville & Caius College, the same college Bromhead went to. Green graduated in 1837 and in November 1839 became a college fellow. Unfortunately, he soon became ill, returned to Nottingham in the spring of 1840, and died of influenza on 31 May 1841. He and Jane are buried next to each other in St. Stephen's Church yard a few hundred yards from the mill. There appears to be no portrait of him, and he died shortly after photography was invented.

Rediscovery of Green's essay

By the early 1840s, much of Green's work was available in the open literature, but his most important contribution, his essay, had still not been published in a journal. It might have been undiscovered for many years had it not been for

Figure 2. Bromley House was, and still is, the location of the Nottingham Subscription Library. In this photograph, circa 1880, the building is third from the left. (Reproduced from D. M. Cannell, *George Green: Mathematician & Physicist, 1793–1841: The Background to His Life and Work,* Society for Industrial and Applied Mathematics, Philadelphia, 2001. Courtesy of SIAM.)

William Thomson, shown in figure 4. Thomson, later Lord Kelvin, went to Cambridge soon after Green's death. His father was a professor of mathematics at Glasgow, and Thomson already had a degree from Glasgow before going to Cambridge. He was interested in electricity and had seen a reference to Green's essay in a footnote to a paper by Robert Murphy. In a letter about Green written shortly before his death in 1907, Thomson wrote to Joseph Larmor:

> When I went up to Cambridge as a freshman, I asked at all the book shops in Cambridge for Green's Essay on Electricity and Magnetism, and could hear nothing of it.
>
> The day before I left Cambridge for Paris after taking my degree, in Jan. 1845, I met [William] Hopkins on what I believe was then called the Senior Wranglers' Walk, and I told him I had enquired in vain for Green's Essay. . . . He said "I have some copies of it." He turned with me and took me to his house, and there, in his chief coaching room in which I had been day after day for two years, he found three copies of Green's Essay in his bookcase and gave them to me.
>
> I had only time that evening to look at some pages of it, which astonished me. Next day, if I remember right, on the top of a diligence [stagecoach] on my way to Paris, I managed to read some more of it.

Thomson went to Paris for four months to get some experience in experimental physics. But he also wanted to meet theorists such as Michel Chasles, Joseph Liouville, and Charles Sturm. His letter to Larmor goes on to say that Liouville "gave great attention" to Green's essay when Thomson showed it to him at his home. Later, while Thomson was with a Cambridge colleague, Sturm came along, panting with exertion. He eagerly asked to see Green's essay. "So I handed it to him. He sat down and turned over the pages with avidity. He stopped at one place calling out, 'Ah voila mon affaire [There's my work].' " Inevitably, many of Green's findings had been rediscovered over the 17 years since the essay appeared. Thomson notes that, in addition to Sturm, Chasles found his own results and demonstrations in the essay.

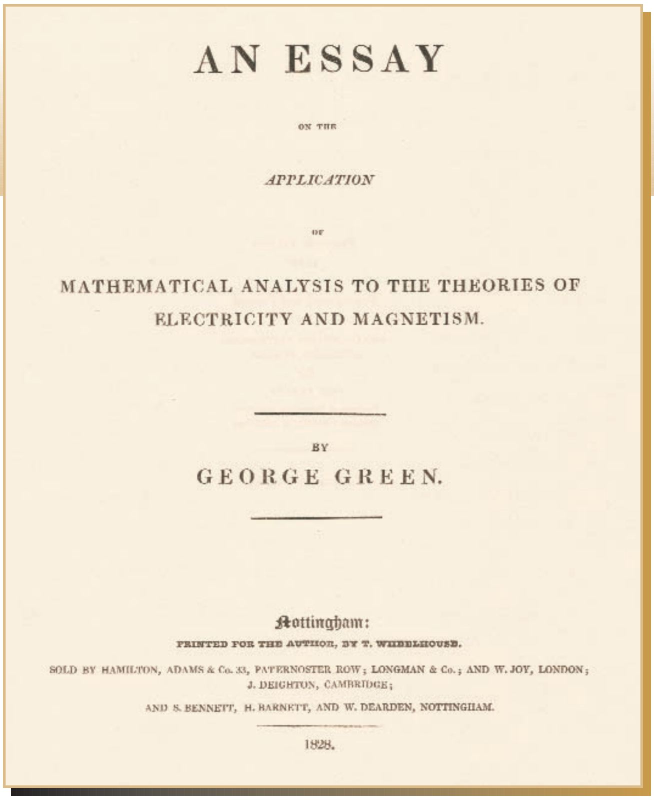
When he returned to England, Thomson arranged for Green's essay to be republished in *Crelle's Journal*. Edmund Whittaker, in his *A History of the Theories of Aether and Electricity* (Dover, 1989), says,

> It is impossible to avoid noticing throughout all Kelvin's work evidences of the deep impression which was made on him by the writings of Green. The same may be said of Kelvin's friend and contemporary, [George] Stokes, and indeed it is no exaggeration to describe Green as the founder of that "Cambridge School" of natural philosophers of which Kelvin, Stokes, [Lord] Rayleigh, [James] Clerk Maxwell, [Horace] Lamb, J. J. Thomson, Larmor and [Augustus] Love were the most illustrious members in the latter half of the nineteenth century.

Green's contribution to science

The inspiration for Green's essay came from France, from Laplace and Siméon Poisson. The inverse-square law for the forces between two charges had recently been established experimentally and Poisson had shown how this determined the charge distribution over the surfaces of conductors. The techniques he used were only applicable to surfaces with simple geometry, so Green devised powerful techniques to obtain the distributions for any surface. He made great use of the electric potential and gave it that name. One of the theorems he developed, now called Green's theorem, readily simplifies to what is often called the divergence theorem or Gauss's theorem. Many early textbooks, though, also call this simplification Green's theorem, presumably to emphasize his claims to precedence. The other powerful tool developed in the essay is now called a Green or Green's function.

Green's subsequent work on elasticity is remembered because of Green's tensor. Green became interested in elasticity through considerations of the "aether," which, of course, had to be a solid because light waves are transverse.



AN ESSAY

ON THE

APPLICATION

OF

MATHEMATICAL ANALYSIS TO THE THEORIES OF

ELECTRICITY AND MAGNETISM.

BY

GEORGE GREEN.

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J. DEIGHTON, CAMBRIDGE;
AND S. BENNETT, H. BARNETT, AND W. DEARDEN, NOTTINGHAM.

1828.

Figure 3. George Green's essay introducing Green's theorem and Green functions was published at the author's expense in 1828.

He showed that 21 moduli are generally required to account for the elastic properties of an anisotropic medium, and he explained how symmetry can reduce that number.

In another paper, Green made the first correct calculations of the proportions of energy reflected and refracted at an interface and explained the phenomenon of total internal reflection; included in that explanation was a description of the evanescent wave that exists in the medium of higher refractive index. In that work, he became one of the first to write down the principle of the conservation of energy. Green's later work contained a number of other mathematical firsts. These include the derivation of an approximate method for solving differential equations from his paper on the motion of water waves in a canal of variable width and depth. His approach reappeared more than a century later as the Wentzel-Kramers-Brillouin (WKB) method. He was also the first to state Dirichlet's principle of functional minimization, although Bernhard Riemann gave it the name it is usually known by.

Green's theorem and Green functions

The concept of a Green function is most easily illustrated by considering the dynamics of a particle initially at rest under the influence of a time-dependent force $F(t)$. One first considers a force acting for a very short time: a sharp blow or impulse. The impulse is chosen to induce a unit change in momentum at a time t'. At a later time t, the displacement $s(t)$ of the particle is defined to be the Green function $G(t,t')$. But a force $F(t')$ acting for an infinitesimal time interval $\Delta t'$ is an impulse of magnitude $F(t')\,\Delta t'$, and a force applied continuously over time can be regarded as generating a sequence of such impulses. One can find the particle's motion by summing the effects of all impulses applied from the initial time t_0 up to the time t. Thus

$$s(t) = \int_{t_o}^{t} G(t,t')F(t')\,\mathrm{d}t', \qquad (1)$$

which obeys the initial conditions $s = 0$, $\mathrm{d}s/\mathrm{d}t = 0$. The system's response to arbitrary forces may be readily calculated once the Green function has been found. Note that the Green function depends on the dynamical system but not on the form of the applied force. Box 1 gives an explicit sample calculation.

The superposition shown in equation 1 of the effects of successive impulses is only valid for a linear system, in which the response is proportional to the applied force. Nevertheless, Green's technique clearly has wide application to a variety of systems, both mechanical and electrical, for which the linear response to forces or voltages is important in practical applications.

Green's original work was directed toward the solution of electrostatic problems in bounded regions. In that case, the Green function $G(\mathbf{r},\mathbf{r}')$ is the potential at the point $\mathbf{r}$ produced by a unit point charge at $\mathbf{r}'$. The point charge is the spatial analogue of the impulsive force that only acts at a single instant in time. The Green function is not the same as the electrostatic Coulomb potential generated by the point charge, because a point source also induces charges on the boundaries. With what is now known as Green's theorem, the electrostatic potential $\phi(\mathbf{r})$ can be expressed in modern notation (see box 2 for further details) as

$$\phi(\mathbf{r}) = \int_\tau G(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')\mathrm{d}\tau' + \int_S \left[\phi(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\nabla'\phi(\mathbf{r}')\right]\cdot \mathrm{d}\mathbf{S}'. \qquad (2)$$

The potential $\phi(\mathbf{r})$ is a superposition of the effects due to the spatial charge density $\rho(\mathbf{r}')$ in the volume τ and induced charges on the volume's bounding surface S, the influences of which are transmitted by the Green function $G(\mathbf{r},\mathbf{r}')$. The importance of Green's work lay in its generality. There is no restriction on the geometry of the surface. It is simple to show, for example, that if any surface is maintained at zero potential (grounded) and encloses no charge ($\rho = 0$), then the interior solution is $\phi = 0$ everywhere. That is to say, the internal volume is completely screened from external electrostatic influences.

Green functions in scattering theory

Green-function techniques were increasingly used in the latter part of the 19th century to solve the partial differential equations, all very similar, describing electrical, magnetic, mechanical, and thermal phenomena. Green functions can also be used to formulate the theory of classical wave scattering. Indeed, because the Schrödinger equation has a similar form to the wave equation, Green functions can also be used to describe the nonrelativistic scattering of a single particle by an external potential energy $V(\mathbf{r})$. For a particle of total energy E and mass m, the time-independent part of the wavefunction $\psi(\mathbf{r})$ obeys

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(\mathbf{r})\,\psi(\mathbf{r}) = E\,\psi(\mathbf{r}). \qquad (3)$$

By treating the term $V\psi$ in a similar manner to the imposed charge density in an electrostatic problem, one can write a formal solution of equation 3:

Figure 4. William Thomson, later Lord Kelvin (1824–1907) was largely responsible for the rediscovery of George Green's 1828 essay and its republication (1850–54) in *Crelle's Journal*. This photograph of Thomson was taken in 1846. (From the Burndy Library. Courtesy of AIP Emilio Segrè Visual Archives.)

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) + \int G(\mathbf{r},\mathbf{r}')\frac{2m}{\hbar^2}V(\mathbf{r}')\psi(\mathbf{r}')\mathrm{d}\tau', \qquad (4)$$

where $\psi_0(\mathbf{r})$ is an incident wave for a particle of energy E and the Green function $G(\mathbf{r},\mathbf{r}')$ is the wave amplitude at $\mathbf{r}$ induced by a given point source at $\mathbf{r}'$. But unlike equation 1, equation 4 is not an explicit solution, because the unknown wavefunction appears inside the integral. The physical reason is that the source of the scattered wave, the $V\psi$ term in equation 3, only exists when an incident wave is present, unlike the force in equation 1 that is determined by external influences.

The usual way to solve equation 4 is by iteration. The first iterative term, with $\psi(\mathbf{r}')$ inside the integral replaced by $\psi_0(\mathbf{r}')$, is the familiar Born approximation of scattering theory. Successive terms in the iteration involve multiple integrals of the form

$$\int G(\mathbf{r},\mathbf{r}')V(\mathbf{r}')\psi_0(\mathbf{r}')G(\mathbf{r}',\mathbf{r}'')V(\mathbf{r}'')\psi_0(\mathbf{r}'')\ldots\mathrm{d}\tau''\mathrm{d}\tau'. \qquad (5)$$

Such terms correspond to multiple scattering events in which the incident wave is scattered at points $\mathbf{r}'$, $\mathbf{r}''$, and so on, before arriving at $\mathbf{r}$. Because there is a momentum change at each scattering point, $G(\mathbf{r},\mathbf{r}')$ is the response to an impulse just as is the time-dependent Green function $G(t,t')$.

Later developments

In elementary particle physics, scattering is of crucial importance because the only way we have of investigating the properties of elementary particles is through their interactions with each other. But those interactions must be described in terms of quantized fields that transmit forces between particles via exchange of virtual quanta of energy and momentum. For charged particles interacting electromagnetically, those quanta are the photons of quantum electrodynamics (QED). The theory of the interaction of electrons (and positrons) with the quantized electromagnetic field was developed in the late 1940s to explain the Lamb shift of the hydrogen atom's 1*s* energy level and the deviation of the magnetic moment of the electron from the Dirac value. The two phenomena were attributed to fluctuations in the fields, but early theory was troubled by infinite corrections until it was shown that the infinities could be removed by a renormalization of electron mass and charge.

A consistent formulation of QED was achieved independently by Julian Schwinger and Richard Feynman. Schwinger, who earlier had used Green functions to describe microwave propagation when working on radar, gave a formal field-theoretic treatment in which Green functions appeared to propagate fields between spacetime points. Some aspects of his theory were very similar to work by Sin-Itiro Tomonaga in Japan, who was developing a quantum field-theory approach to renormalization. Feynman used his intuitive spacetime approach to quantum mechanics. In that formulation, the probability amplitude for a given process is the sum of amplitudes for each available spacetime path. For any particular path, the amplitude is the product of factors corresponding to either free propagation or scattering interactions as in equation 5. Each sequence of propagators and interactions can be represented as a so-called Feynman diagram that gives a helpful physical picture of the process. Later, Freeman Dyson demonstrated the equivalence of the Feynman and Schwinger theories and systematized the calculation of higher-order effects in QED.

In Feynman's treatment, it is clear why Green functions play such a natural and fundamental role in the theory of particle physics. Interparticle interactions are complicated multiple scattering events in which the forces are transmitted by quantum fields. But the propagation of fields between points is precisely what Green functions were originally devised to describe. So Green functions,

Box 1. Simple Example of a Green Function

Suppose one wishes to find the velocity $v(t)$ of a particle starting from rest and acted on by a viscous force αv and an arbitrary force $F(t)$. The equation of motion is

$$m\frac{\mathrm{d}v}{\mathrm{d}t} + \alpha v = F(t).$$

For a unit impulsive force applied at time $t = t'$, the solution is the Green function $G(t,t')$. The velocity immediately after the impulse is $1/m$ and thereafter decays exponentially, so that, for $t > t'$,

$$G(t,t') = \frac{1}{m}e^{-\alpha(t-t')/m} \qquad (t \geq t').$$

For $t < t'$, the Green's function vanishes.

Thus, the general solution of the dynamical problem for arbitrary force is

$$v(t) = \int_0^t \frac{1}{m}e^{-\alpha(t-t')/m}F(t')\mathrm{d}t'.$$

For this simple example, the general solution may also be obtained by direct integration of the equation of motion using an integrating factor.

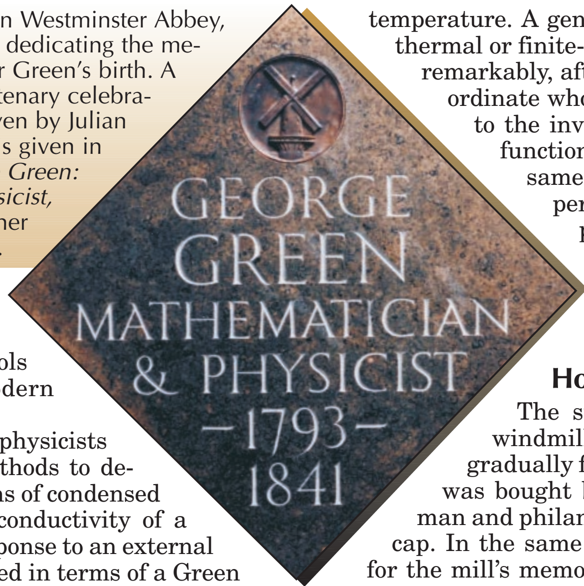

Figure 5. Green's memorial in Westminster Abbey, London. In 1993, the ceremony dedicating the memorial was held, 200 years after Green's birth. A full account of the Green bicentenary celebrations, including the lectures given by Julian Schwinger and Freeman Dyson, is given in Mary Cannell's book, *George Green: Mathematician & Physicist, 1793–1841* (see further reading below).

often called Feynman propagators in particle physics, are among the standard working tools of theoretical analysis in modern quantum physics.

In the 1950s and 1960s, physicists began using Green-function methods to describe the many-body interactions of condensed matter physics. The electrical conductivity of a solid is essentially the linear response to an external field. It can therefore be expressed in terms of a Green function, as was shown by Ryogo Kubo and others. The sources of electrical resistance in a solid are scattering processes, which are naturally described by means of Green functions. But unlike the processes usually described in particle physics, condensed matter processes occur at finite temperature. A generalization of the theory leads to thermal or finite-temperature Green functions. But remarkably, after introducing a complex time coordinate whose imaginary part is proportional to the inverse temperature, thermal Green functions can be calculated with much the same techniques as ordinary, zero-temperature Green functions. So today, physicists apply Green functions in areas not even conceived of in Green's time, and are likely to continue doing so whatever may develop in the future.

Box 2. Green Functions, Delta Functions, and Boundary Conditions

Green functions are usually used to solve for a classical field $\phi(\mathbf{r})$ in a partial differential equation of the general form

$$\nabla^2\phi(\mathbf{r}) + k^2\phi(\mathbf{r}) = \rho(\mathbf{r}),$$

where $k = 0$ for electrostatic fields but is nonzero for wave fields, and $\rho(\mathbf{r})$ is a source function. The Green function $G(\mathbf{r},\mathbf{r}')$ is the solution for a point source at $\mathbf{r}'$ represented by a Dirac delta function $\delta(\mathbf{r} - \mathbf{r}')$. That is, the Green function satisfies the equation

$$\nabla^2 G(\mathbf{r},\mathbf{r}') + k^2 G(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}').$$

Applying Green's theorem to the functions ϕ and G,

$$\int_\tau \left(\phi\boldsymbol{\nabla}^2 G - G\boldsymbol{\nabla}^2\phi\right)\mathrm{d}\tau = \int_S \left(\phi\boldsymbol{\nabla}G - G\boldsymbol{\nabla}\phi\right)\cdot\mathrm{d}\mathbf{S},$$

leads to equation 2 after one interchanges $\mathbf{r}$ and $\mathbf{r}'$ and exploits the symmetry of $G(\mathbf{r},\mathbf{r}')$.

To determine a unique solution for $\phi(\mathbf{r})$, one must specify boundary conditions. Suitable boundary conditions must also be chosen for $G(\mathbf{r},\mathbf{r}')$. Thus, the Green function is the field due to a point source but modified by the effects of boundaries. In electrostatics, the boundary effects arise from surface charges. In wave problems, the boundaries give rise to reflected waves that may be calculated by solving the appropriate source-free equation.

In the Green-function treatment of particle motion, a unit impulse is represented by a force $F(t) = \delta(t - t')$ and is the analogue of the unit point source in spatial problems. The initial conditions play the role of boundary conditions.

In scattering problems, the Green function is defined as the solution of equation 3 with the $V\psi$ term brought to the right-hand side and replaced by a delta function source $\delta(\mathbf{r} - \mathbf{r}')$. For such problems, the boundary conditions require that the solution be an outgoing wave at large distances from the source because incoming waves would imply unphysical reflections at infinity.

Honor in his own country

The steam engine brought the era of windmills to an end, and Green's Mill gradually fell into disrepair. In 1920, the mill was bought by Oliver Hind, a local businessman and philanthropist, who repaired its wooden cap. In the same year, the Holbrook Bequest paid for the mill's memorial plaque. For a while, the mill became a furniture polish factory, but in 1947, it caught fire and was completely gutted. Hind sealed the shell by boarding up the doors and windows and replacing the cap with a concrete slab. Local interest in Green had been maintained over the years at a modest level, notably as a result of activity by members of Nottingham University. In 1945, H. Gwynedd Green (no relative), a member of the university's mathematics department, wrote a biography of Green. Mary Cannell published a substantially longer work in 1993; an enlarged second edition appeared in 2001.

The plan to restore the mill was initially prompted by a rumor that it would be knocked down for a new development. That led to the formation of the George Green Memorial Fund by Nottingham University physicists and others, and eventually, as a fitting memorial to Green, the mill was restored by Nottingham City. It is now one of Nottingham's principal museums and tourist attractions. Green's second major memorial is in Westminster Abbey, London. That is the resting place of British kings and queens and, in more recent times, the place where many famous literary and scientific figures are remembered. In 1993, on the bicentenary of Green's birth, Michael Atiyah, president of the Royal Society, unveiled a memorial. The plaque, shown in figure 5, lies on the Abbey floor next to memorials to Newton, Michael Faraday, Maxwell, and Kelvin. The congregation included a number of Green's descendants and many other scientists and mathematicians including Schwinger and Dyson. Kelvin's memorial had to be moved sideways to accommodate Green's. But he surely would not have objected.

Further reading

▶ *The Scientific Papers of George Green*, compiled in 1995 into three volumes, available on the George Green Society Web site at http://www.nottingham.ac.uk/physics/gg. Click on "The George Green Memorial Fund" for ordering information.

▶ N. M. Ferrers, ed., *Mathematical Papers of George Green*, (1871; reprint, Chelsea, New York, 1970).

▶ H. G. Green, "A Biography of Geroge Green," in M. F. A. Montagu, ed., *Studies and Essays in the History of Science and Learning*, Schuman, New York (1946; New York: Arno Press, 1975).

▶ D. M. Cannell, *George Green: Mathematician & Physicist, 1793–1841: The Background to His Life and Work*, Society for Industrial and Applied Mathematics, Philadelphia (2001). ■