HG1M12

Chapter 2
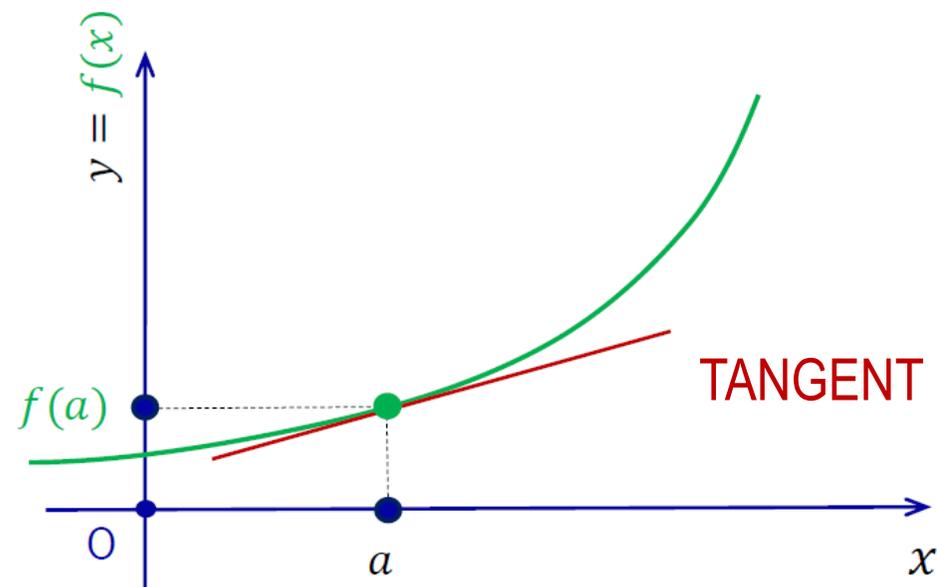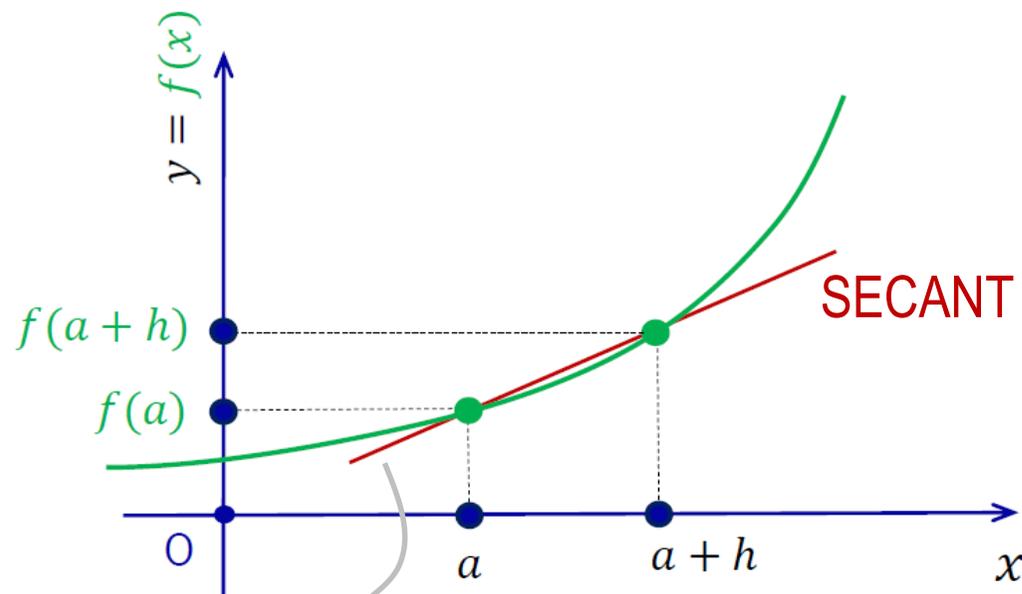## (multivariate functions)

Outline

TODAY (22.02.2019):

- Functions of two or more variables

- **Partial derivatives**

- **Stationary points** for functions of two variables

  (minima, maxima, saddle points)

(Some of the examples on these slides will be discussed in the examples class at **16:00 today**)
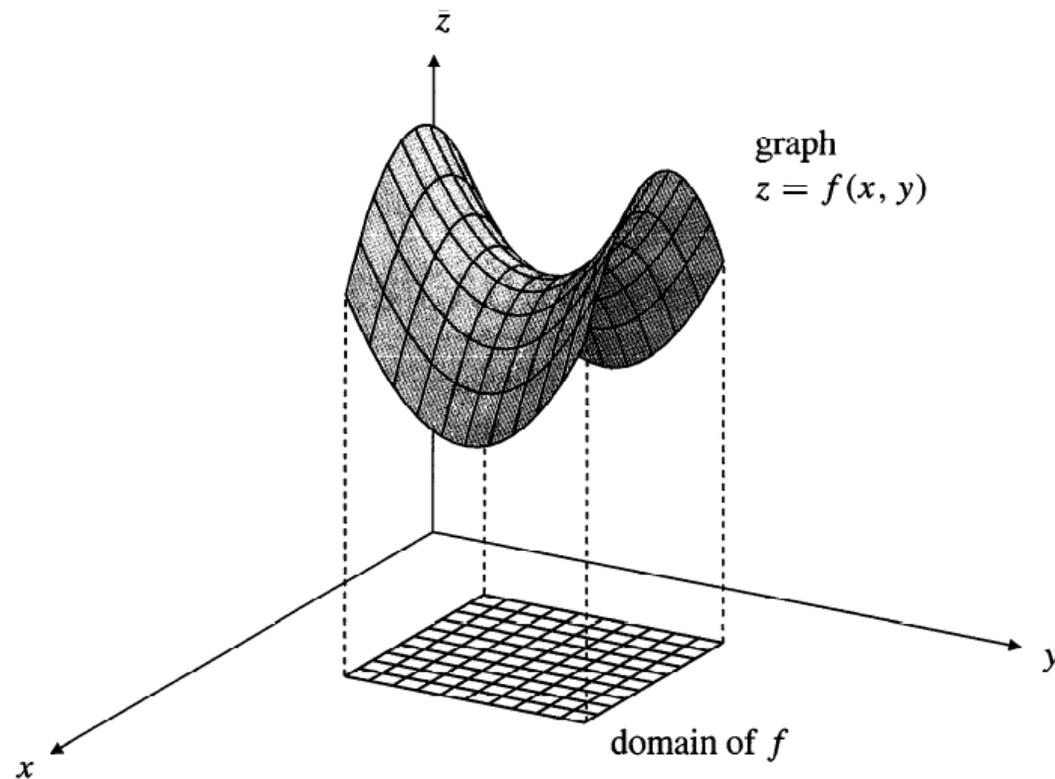
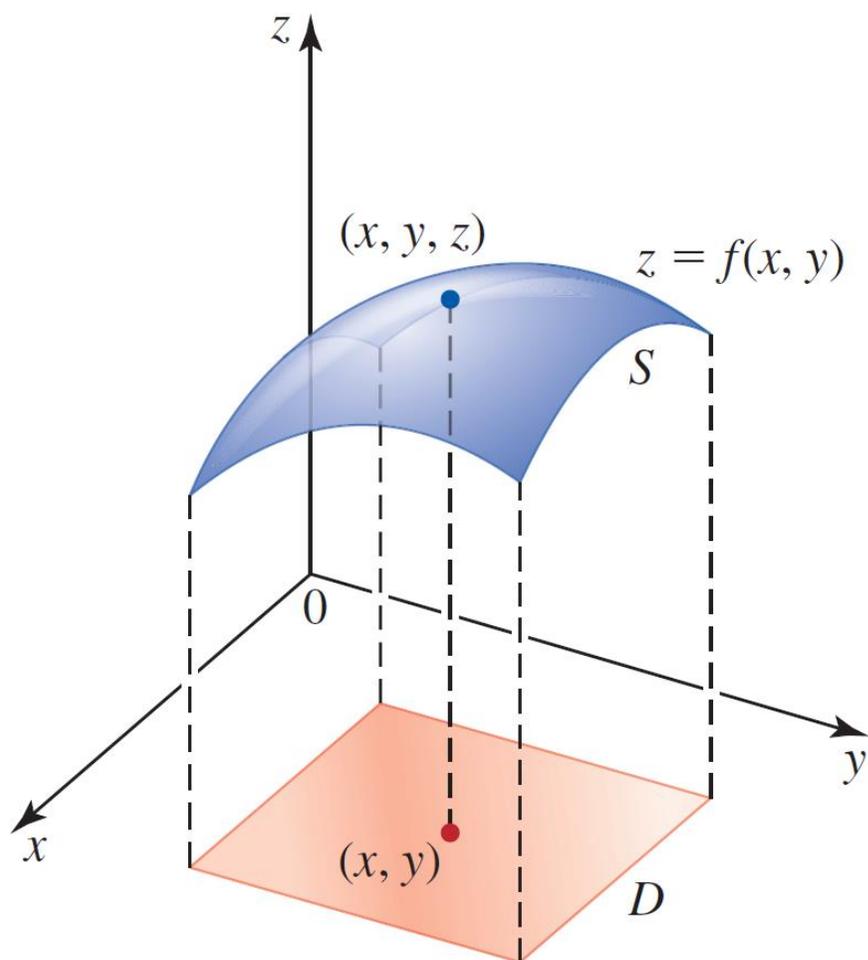# ASIDE (derivative for functions of one variable)

SECANT

TANGENT

SLOPE:

$$\frac{f(a+h) - f(a)}{h}$$

DIFFERENCE QUOTIENT

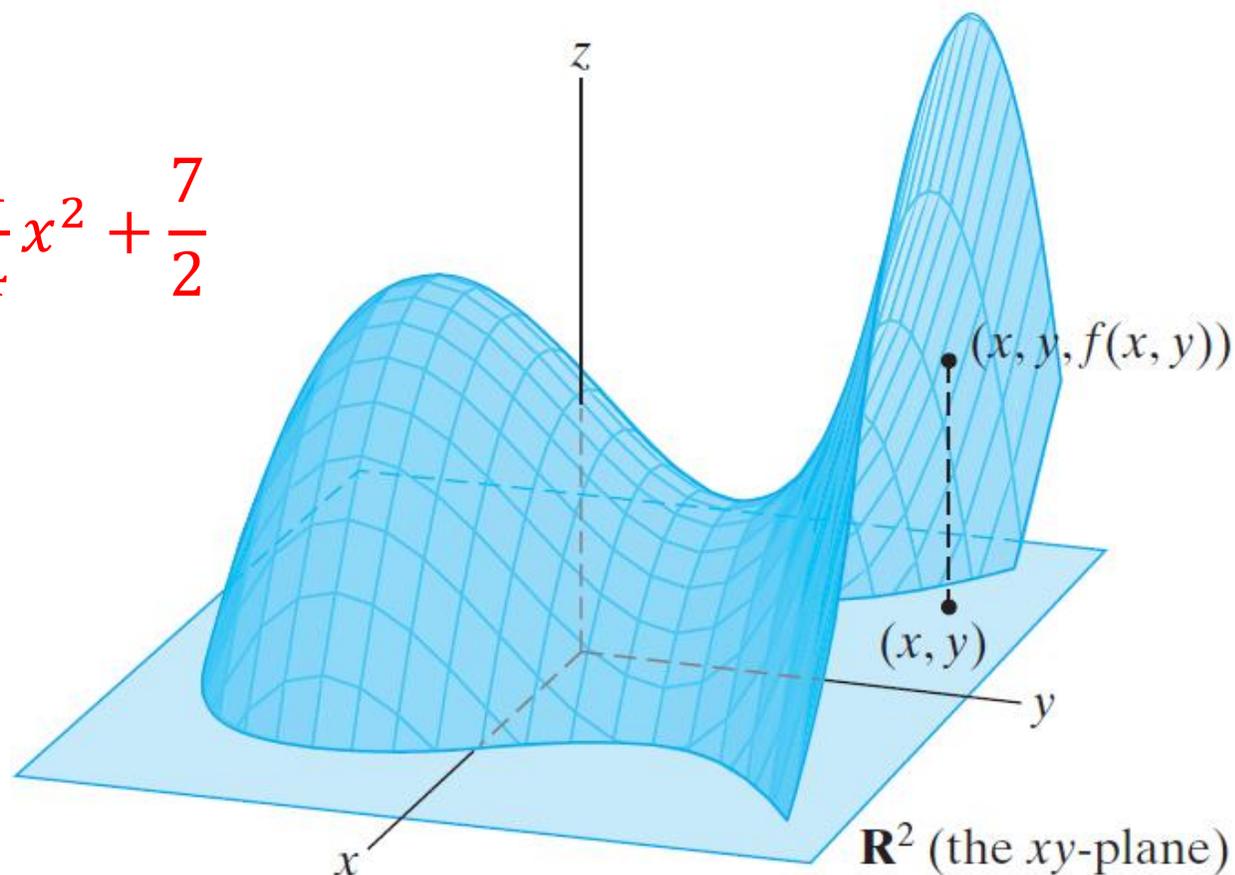$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Graph of a function $z = f(x, y)$

Generally, just as $y = f(x)$ represents a 1-D curve, $z = f(x, y)$ represents a 2-D surface.

Graph of a function $z = f(x, y)$

$$f : \boldsymbol{R}^2 \to \boldsymbol{R}$$

$$f(x, y) = \frac{1}{12}y^3 - y - \frac{1}{4}x^2 + \frac{7}{2}$$

$(x, y, f(x, y))$

(x, y)

$\boldsymbol{R}^2$ (the xy-plane)

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$$f(x, y) = 32 - x^2 - y^2$$

z

$$z = f(x, y)$$

paraboloid

$$-10 \leq x \leq 10$$

$$-10 \leq y \leq 10$$

y

x

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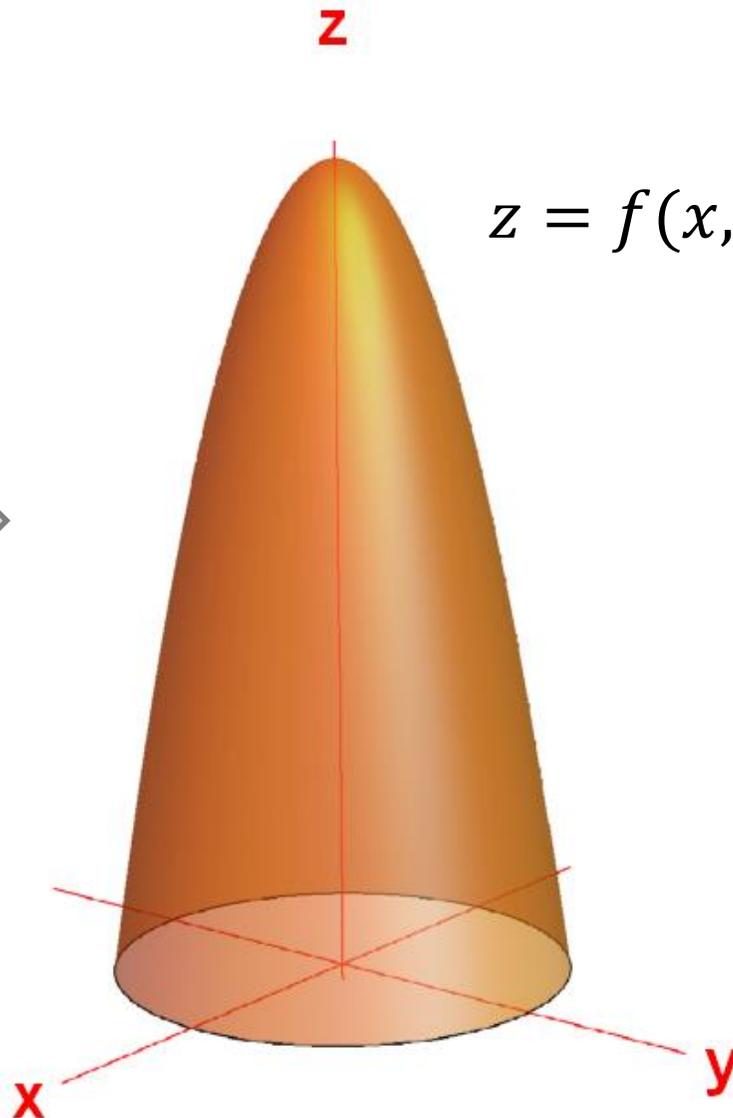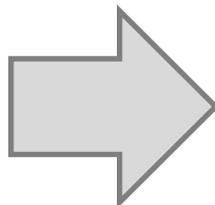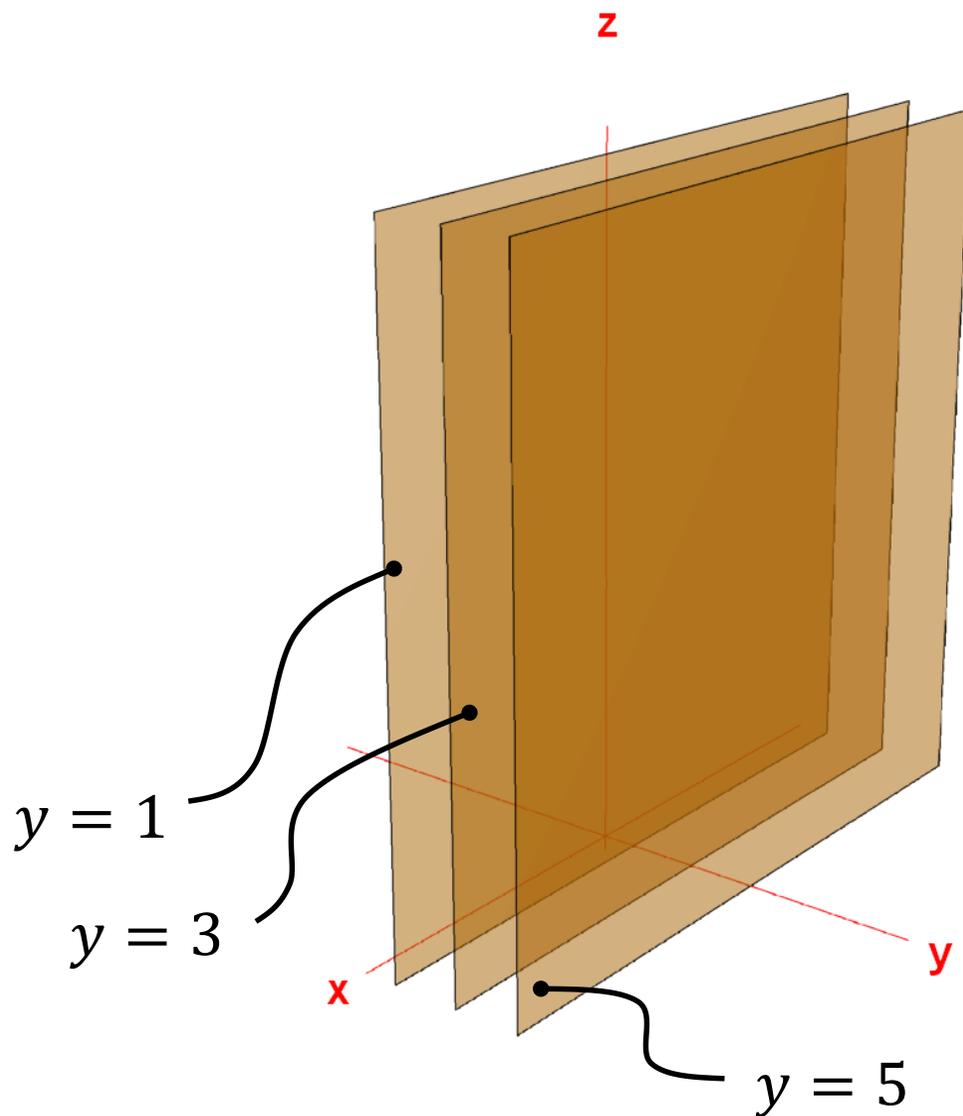
$$Ax + By + Cz = D$$

$$A = C = 0$$
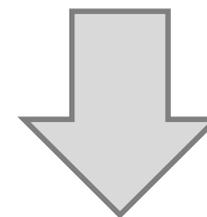
family of planes

$$y = const.$$

(**plane** perpendicular to the y-axis)

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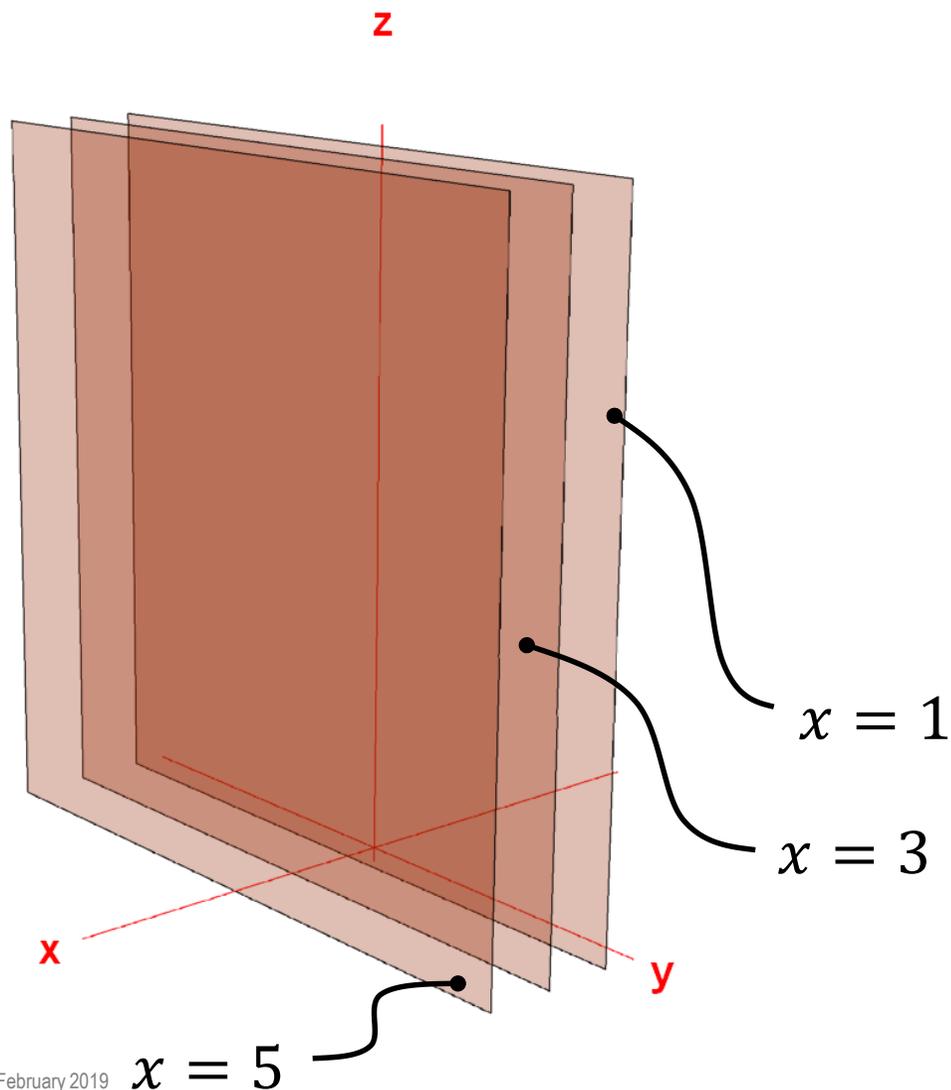
$$Ax + By + Cz = D$$

$$B = C = 0$$
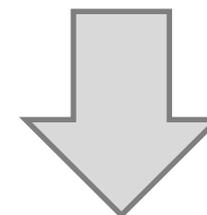
family of planes

$$x = const.$$

(**plane** perpendicular to the *x*-axis)

$x = 1$

$x = 3$

$x = 5$
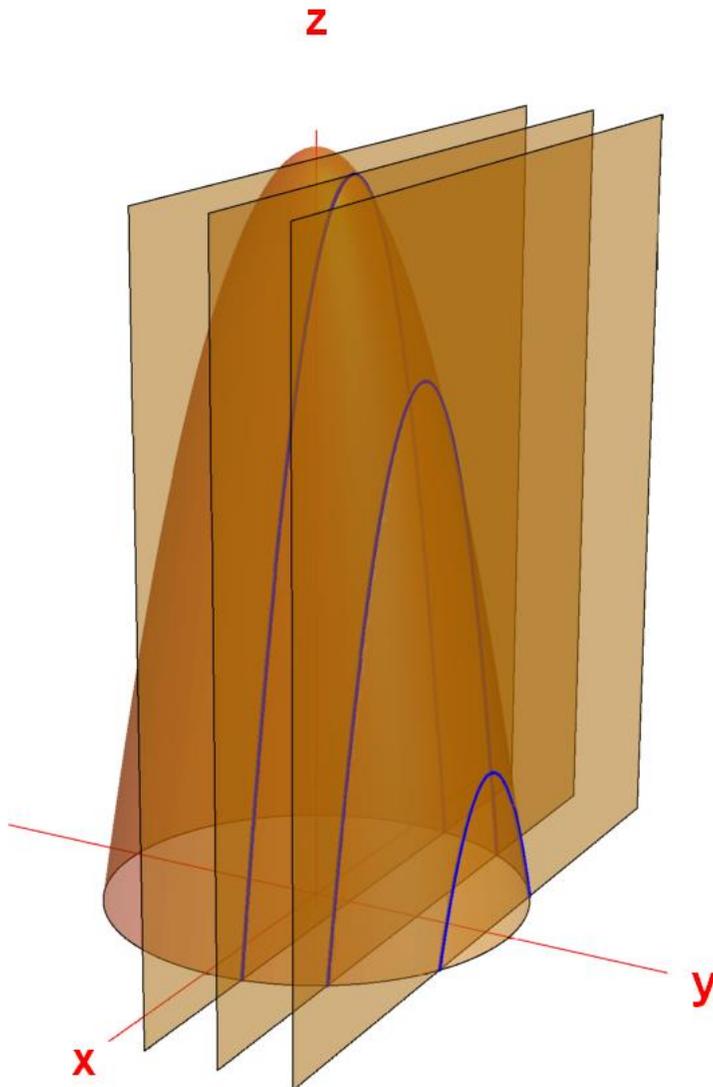
Tangent to the intersection curve?
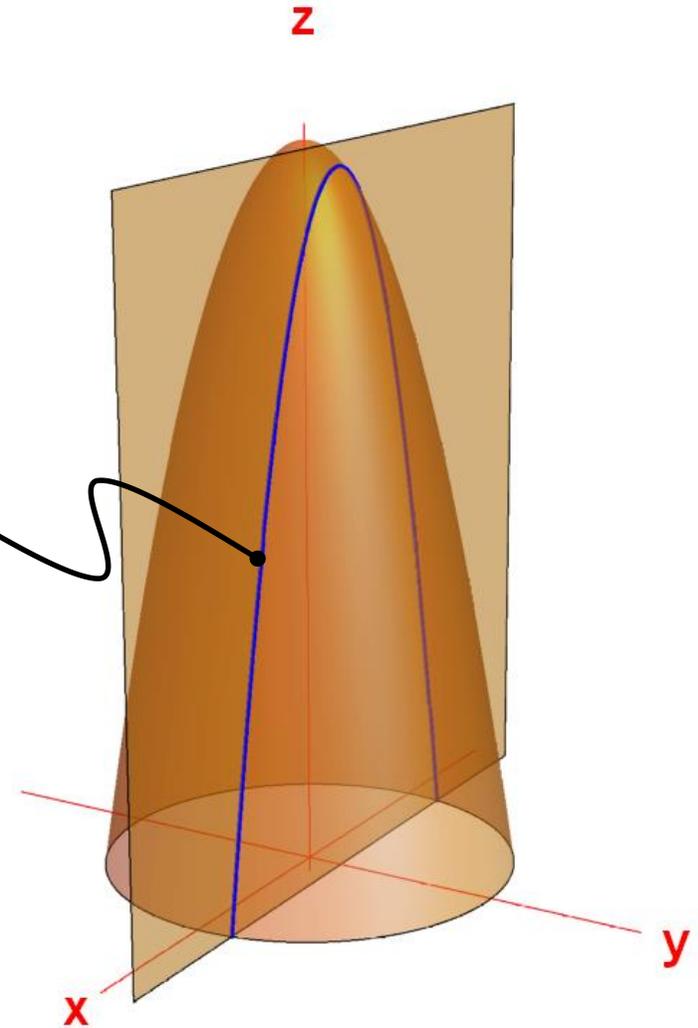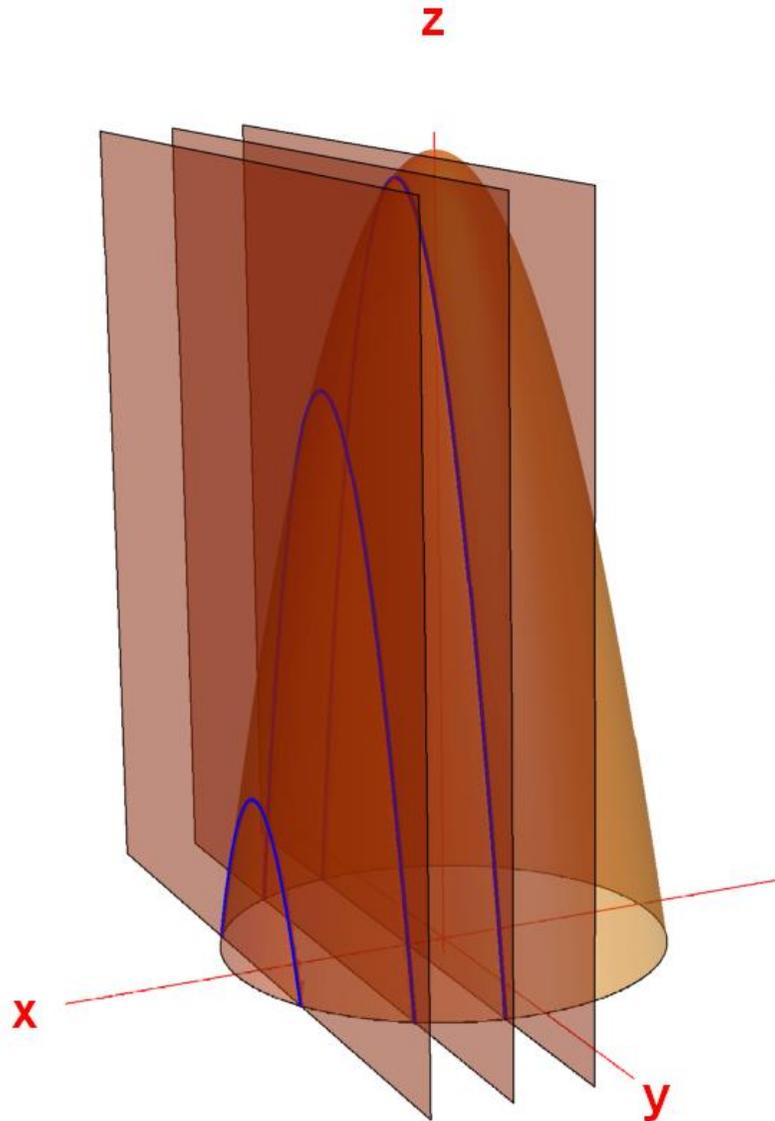
Tangent to the intersection curve?
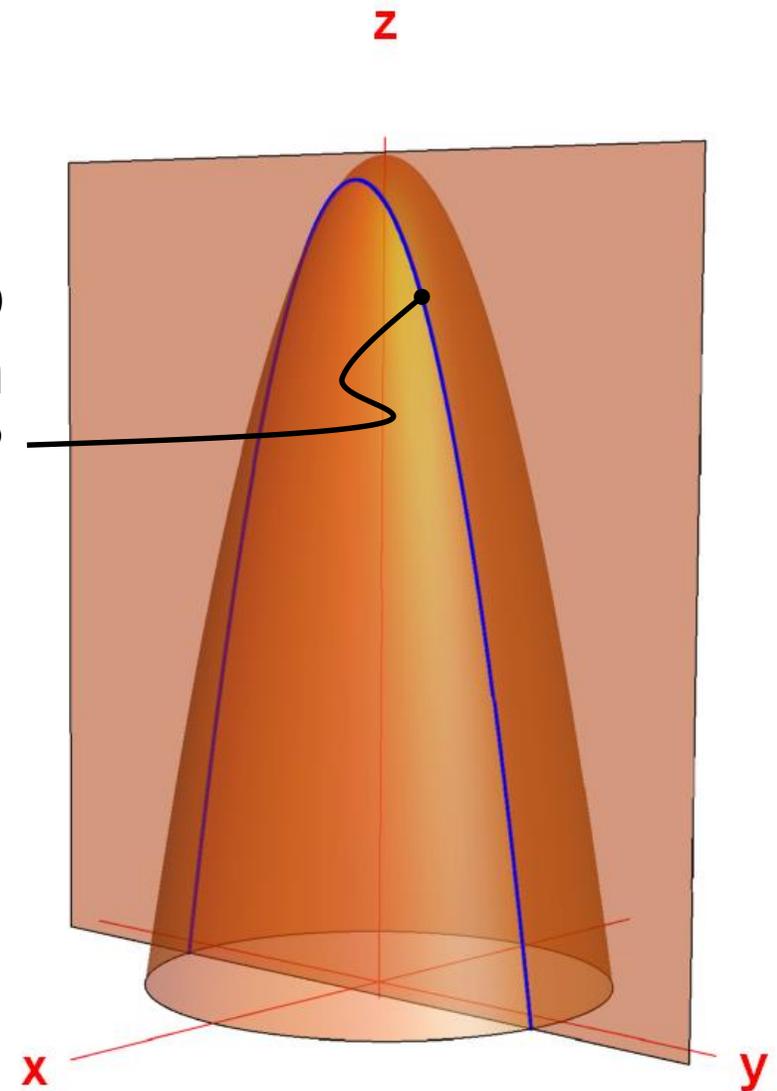
Partial differentiation: definitions

Let's look at the function $z = f(x, y)$, which is a function of **two variables** x and y

$$\frac{\partial f}{\partial x} = \text{ \textbf{partial derivative} of } f \text{ with respect to } x$$

f_x

To calculate this we keep all other variables **fixed** and only differentiate w.r.t. x

$$\frac{\partial f}{\partial y} = \text{ \textbf{partial derivative} of } f \text{ with respect to } y$$

f_y

To calculate this we keep all other variables **fixed** and only differentiate w.r.t. y

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Partial differentiation: remarks

It is important to note that when we have a **partial derivative** (i.e., more than one independent variable) we use

$$\partial \qquad \text{i.e.} \qquad \frac{\partial f}{\partial x}$$

When we have an **ordinary derivative** (i.e., only one independent variable, such as $f(x)$), we use

$$d \qquad \text{i.e.} \qquad \frac{df}{dx}$$

If we remember to keep all of the other variables fixed, the partial derivative follows the same differentiation laws as in the case of single-variable functions.

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The **first partial derivative** of the function $f(x, y)$ **with respect to the variables** x and y are defined by

$$\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

and

$$\frac{\partial f}{\partial y}(x, y) = \lim_{k \to 0} \frac{f(x, y + k) - f(x, y)}{k}$$

provided that these limits exist.

Each of the two partial derivatives is the limit of a **difference quotient** in one of the variables.
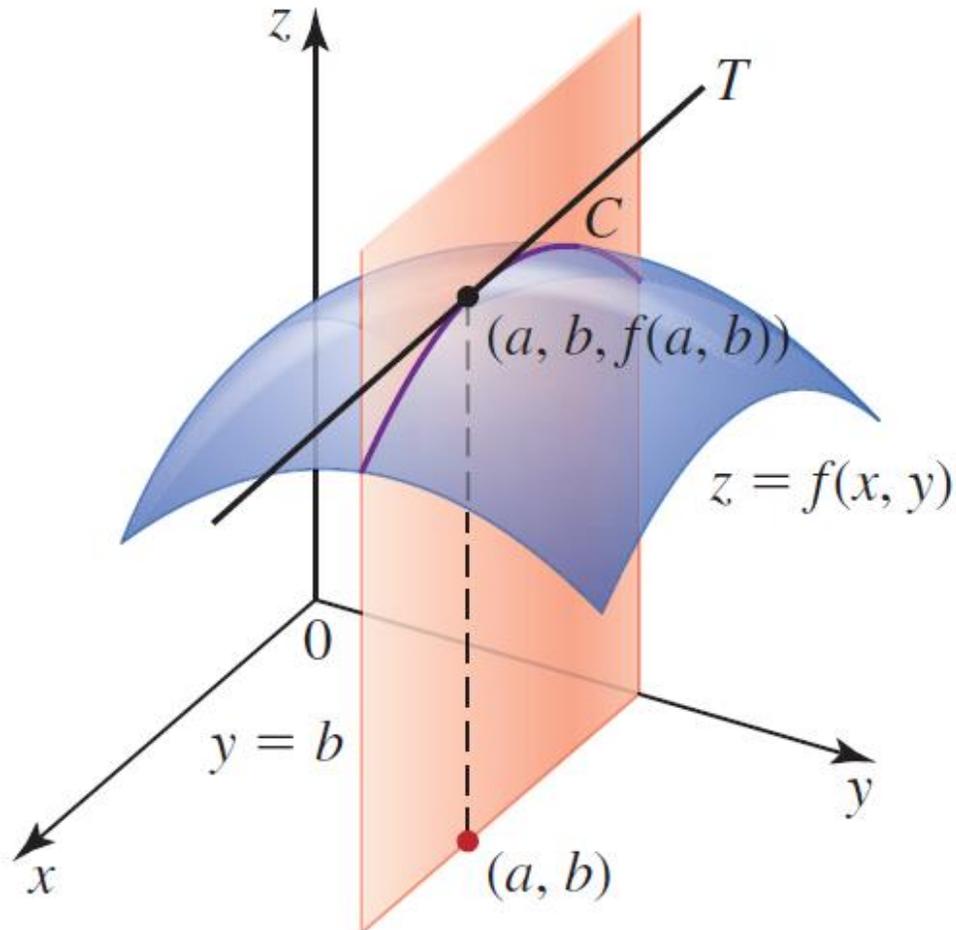
Interpretation

$$\frac{\partial f}{\partial x}(a, b)$$

represents the **slope** at the point $(a, b, f(a, b))$ of the curve obtained by intersecting the surface $z = f(x, y)$ with the plane $y = b$

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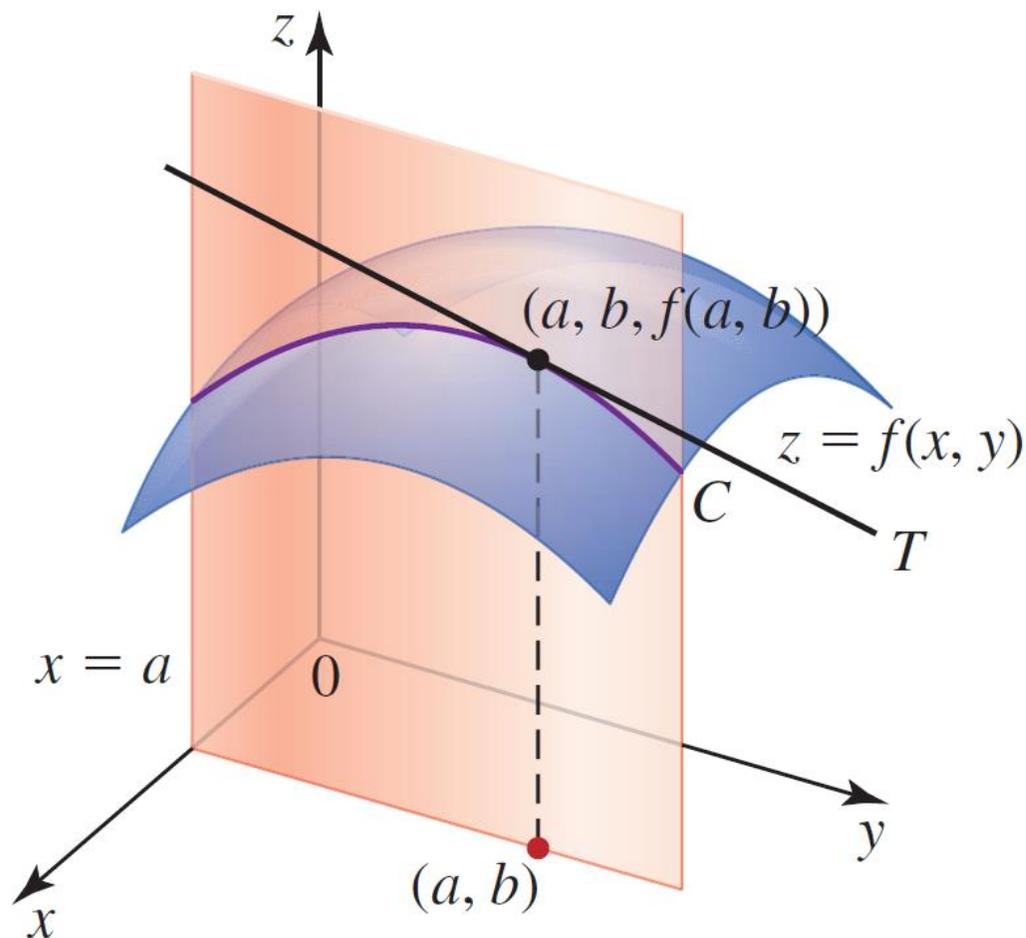
Interpretation

$$\frac{\partial f}{\partial y}(a, b)$$

represents the **slope**
at the point $(a, b, f(a, b))$

of the curve obtained by
intersecting the surface $z = f(x, y)$
with the plane $x = a$

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Interpretation

Tangent plane to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0):

$$z - z_0 = f_x(x_0, y_0)(x - x_0)$$
$$+ f_y(x_0, y_0)(y - y_0)$$

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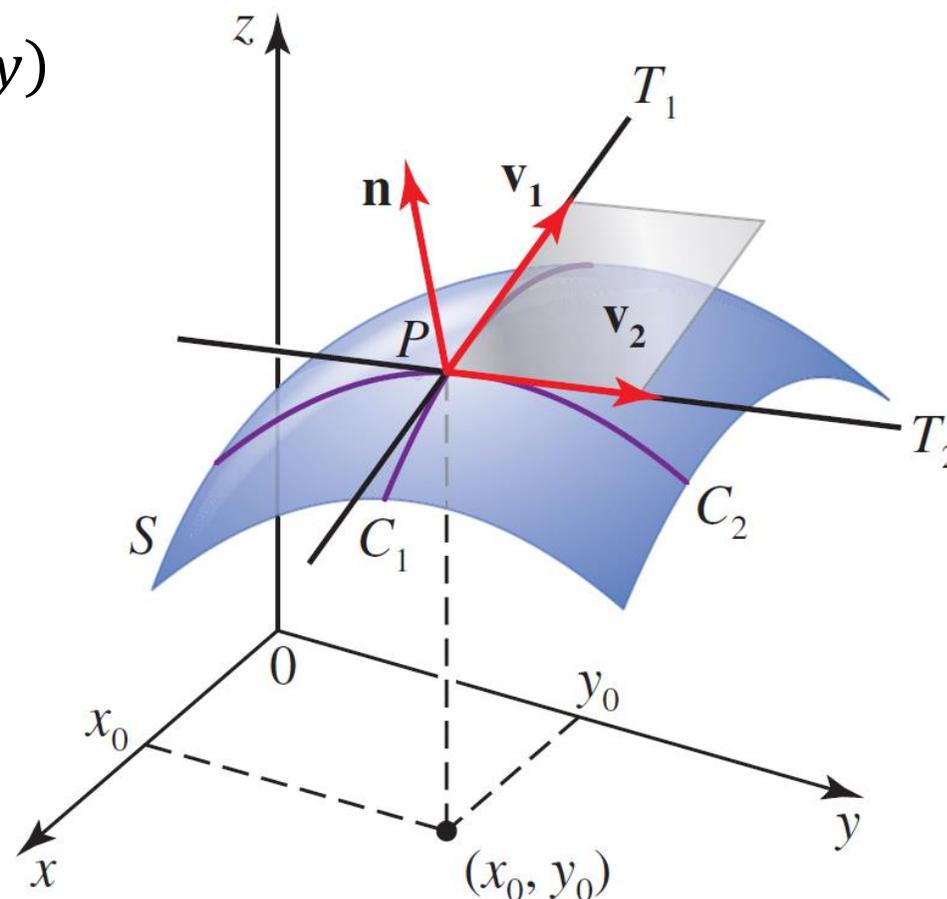
Practical Rules

Most of the previous rules for standard derivatives apply for partial differentiation as well (the **Chain Rule** is more complicated for functions of two variables!)

Product Rule

$$\frac{\partial}{\partial x}(f \cdot g) = \frac{\partial f}{\partial x} \cdot g + f \cdot \frac{\partial g}{\partial x}$$

$$\frac{\partial}{\partial y}(f \cdot g) = \frac{\partial f}{\partial y} \cdot g + f \cdot \frac{\partial g}{\partial y}$$

$$f = f(x, y)$$

$$g = g(x, y)$$

Quotient Rule

$$\frac{\partial}{\partial x}\left(\frac{f}{g}\right) = \frac{1}{g^2}\left(\frac{\partial f}{\partial x} \cdot g - f \cdot \frac{\partial g}{\partial x}\right)$$

etc

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General points

$f(x, y)$ has two first-order partial derivatives,

$$\frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y}.$$

Also,

$$\boxed{\frac{\partial g(y)}{\partial x} = 0,} \qquad \boxed{\frac{\partial h(x)}{\partial y} = 0}$$

and

$$\frac{\partial h(x)}{\partial x} = \frac{\mathrm{d}h(x)}{\mathrm{d}x}; \qquad \frac{\partial g(y)}{\partial y} = \frac{\mathrm{d}g(y)}{\mathrm{d}y}$$

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Let $f(x, y) = x^2 + x^3 y^4 + \sin y$. Find $\partial f / \partial x$ and $\partial f / \partial y$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}(x^3 y^4) + \frac{\partial}{\partial x}(\sin y)$$

$$= 2x + 3x^2 y^4 + 0$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2) + \frac{\partial}{\partial y}(x^3 y^4) + \frac{\partial}{\partial y}(\sin y)$$

$$= 0 + 4y^3 x^3 + \cos y$$

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Chain Rule in 2D (simplest case)

This is related to situation in which we have **composite functions**

$$f(u) \qquad\qquad u = g(x,y) \qquad\longrightarrow\qquad f(g(x,y))$$

function of one
variable $u \in \mathbb{R}$

function of two
variables x, y
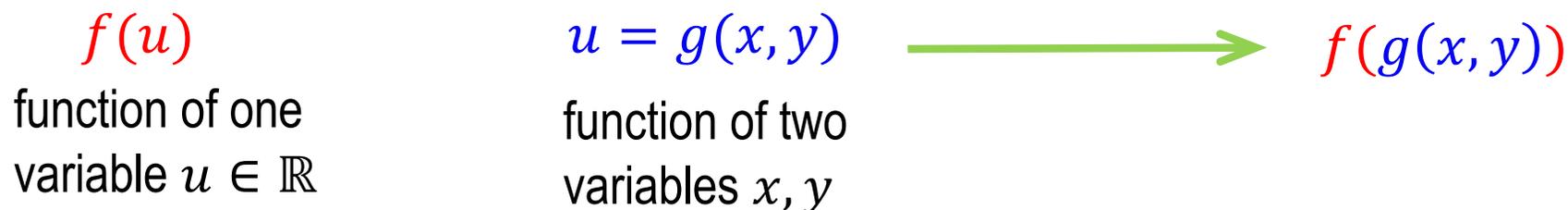
$$\frac{\partial}{\partial x} f\big(g(x,y)\big) = f'\big(g(x,y)\big) \frac{\partial g}{\partial x}(x,y)$$

$$\frac{\partial}{\partial y} f\big(g(x,y)\big) = f'\big(g(x,y)\big) \frac{\partial g}{\partial y}(x,y)$$

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Find f_x and f_y for the function $f(x,y) = \ln(x^2 + y^2)$

$f(x,y) = F(g(x,y))$, where $F(u) = \ln u$ $\qquad g(x,y) = x^2 + y^2$

$$F'(u) = \frac{1}{u}$$

$$g_x = 2x \qquad g_y = 2y$$

$$f_x = F'(g(x,y))g_x = \frac{1}{x^2 + y^2}(2x) = \frac{2x}{x^2 + y^2}$$

$$f_y = F'(g(x,y))g_y = \frac{1}{x^2 + y^2}(2y) = \frac{2y}{x^2 + y^2}$$

Find f_x and f_y for the function $f(x,y) = e^{-x/y}$

$f(x,y) = F(g(x,y))$, where $\begin{cases} F(u) = e^{-u} \\ F'(u) = -e^{-u} \end{cases}$ $\qquad g(x,y) = \frac{x}{y}$

$$g_x = \frac{1}{y} \qquad g_y = -\frac{x}{y^2}$$

$$f_x = F'(g(x,y))g_x = \left(-e^{-\frac{x}{y}}\right)\frac{1}{y} = -\frac{1}{y}e^{-\frac{x}{y}}$$

$$f_y = F'(g(x,y))g_y = \left(-e^{-\frac{x}{y}}\right)\left(-\frac{x}{y^2}\right) = \frac{x}{y^2}e^{-\frac{x}{y}}$$

Find f_x and f_t for the function $\boxed{f(x,t) = t^{-1/2}e^{-x^2/4t}}$

$$f_x = \frac{\partial}{\partial x}\left(t^{-1/2}e^{-x^2/4t}\right) = t^{-1/2}\frac{\partial}{\partial x}\left(e^{-x^2/4t}\right) \leftarrow \text{Use the Chain Rule to calculate this}$$

$$\frac{\partial}{\partial x}\left(e^{-x^2/4t}\right) = \left(-e^{-x^2/4t}\right)\left(\frac{2x}{4t}\right) = -\frac{x}{2t}e^{-x^2/4t}$$

$$\text{Hence } f_x = t^{-1/2}\left(-\frac{x}{2t}e^{-x^2/4t}\right) = \boxed{-\frac{x}{2t^{3/2}}e^{-x^2/4t}} \leftarrow f_x$$

For f_t we must use the product rule:

$$f_t = \frac{\partial}{\partial t}\left(t^{-1/2}\right)e^{-x^2/4t} + t^{-1/2}\frac{\partial}{\partial t}\left(e^{-x^2/4t}\right) \ \circledast$$

$$\frac{d}{dt}\left(t^{-1/2}\right) = -\frac{1}{2}t^{-3/2}$$

Use the Chain Rule

$$\frac{\partial}{\partial t}\left(e^{-x^2/4t}\right) = \left(-e^{-x^2/4t}\right)\left(-\frac{x^2}{4t^2}\right) = \frac{x^2}{4t^2}e^{-x^2/4t}$$

From ✸ :

$$f_t = -\frac{1}{2}t^{-3/2}e^{-x^2/4t} + t^{-1/2}\cdot\frac{x^2}{4t^2}e^{-x^2/4t}$$

$$= \boxed{-\frac{1}{2}t^{-3/2}\left(1 - \frac{x^2}{2t}\right)e^{-x^2/4t}} \quad \leftarrow f_t$$

Further examples

Example A: If the function f is differentiable show that $z = f(x/y)$ satisfies

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 0$$

Example B: Find the first-order partial derivatives of the function

$$h(x, y) = \tan^{-1} \frac{y}{x}$$

Example C: Find the first-order partial derivatives of the function

$$h(x, y) = \sinh \sqrt{x^2 + y^2}$$

Higher-order derivatives

We can take higher partial derivatives in the same way, and also mixed derivatives, so with $f(x, y) = x^2 + x^3 y^4 + \sin y$,
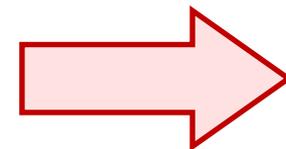
$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial x}(3x^2 y^4)$$

$$= 2 + 6xy^4$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(4y^3 x^3) + \frac{\partial}{\partial y}(\cos y)$$

$$= 12y^2 x^3 - \sin y$$

Mixed partial derivatives

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2x + 3x^2 y^4)$$

$$= \frac{\partial}{\partial y}(2x) + \frac{\partial}{\partial y}(3x^2 y^4)$$

$$= 0 + 12x^2 y^3$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(4y^3 x^3 + \cos y)$$

$$= 12y^3 x^2 + 0$$

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If $f(x, y) = y e^{x/y}$, then $\dfrac{\partial^2 f}{\partial x \partial y}$ is equal to

(I). $\dfrac{1}{y^2} e^{x/y}$

(II). $\dfrac{x}{y} e^{x/y}$

(III). $\left(1 - \dfrac{x}{y^2} \right) e^{x/y}$

(IV). $-\dfrac{x}{y^2} e^{x/y}$

Solution: $f(x,y) = y e^{x/y}$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

will use this expression to calculate
the required mixed partial derivative

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(y e^{\frac{x}{y}}\right) = y \frac{\partial}{\partial x}\left(e^{\frac{x}{y}}\right) = y\left(e^{x/y}\right)\left(\frac{1}{y}\right) = e^{x/y}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(e^{x/y}\right) = e^{x/y}\left(-\frac{x}{y^2}\right) = -\frac{x}{y^2} e^{x/y}$$

Hence, correct answer is Ⅳ

Further examples (higher-order derivatives)

Example: (I) Verify that

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

for the following functions:

$(a)\ f(x,y) = x^2 - xy + y^2;$ $(b)\ f(x,y) = \dfrac{xy}{x+y}$

$$(c)\ f(x,y) = x\sin(x-y)$$

(II) Verify that for any real number k the functions $\begin{cases} z = e^{kx}\cos(ky) \\ z = e^{kx}\sin(ky) \end{cases}$ satisfy

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$ **(Laplace's equation)**

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Suppose we have a function depending on three variables, such as

$$f(x, y, z).$$

We can still calculate partial derivatives **exactly as before**, by treating the other variables as constants when performing the differentiation.

Example: $f(x, y, z) = xy^2 z^3$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(xy^2 z^3\right) = y^2 z^3 \frac{\partial}{\partial x}(x) = y^2 z^3,$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(xy^2 z^3\right) = xz^3 \frac{\partial}{\partial y}(y^2) = 2xyz^3,$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(xy^2 z^3\right) = xy^2 \frac{\partial}{\partial z}(z^3) = 3xy^2 z^2.$$

Find the first order partial derivatives of $f(x, y, z) = (x + 2y - z)^3$.

$$\frac{\partial f}{\partial x} = 3(x + 2y - z)^2 (1) = 3(x + 2y - z)^2$$

$$\frac{\partial f}{\partial y} = 3(x + 2y - z)^2 (2) = 6(x + 2y - z)^2$$

$$\frac{\partial f}{\partial z} = 3(x + 2y - z)^2 (-1) = -3(x + 2y - z)^2$$

Find the first order partial derivatives of $f(x, y, z) = \sqrt{z}e^{xy}$.

$$\frac{\partial f}{\partial x} = \sqrt{z}\,\frac{\partial}{\partial x}\left(e^{xy}\right) = \sqrt{z}\left(e^{xy}\right)(y) = y\sqrt{z}\,e^{xy}$$

$$\frac{\partial f}{\partial y} = x\sqrt{z}\,e^{xy}$$

$$\frac{\partial f}{\partial z} = e^{xy}\,\frac{\partial}{\partial z}\left(\sqrt{z}\right) = \frac{e^{xy}}{2\sqrt{z}}$$