

14 February 2019

VECTORS

(angles, vector projection, applications
to mechanics, etc)

LAST WEEK:

LINES:

$$x = x_0 + \lambda d_1$$

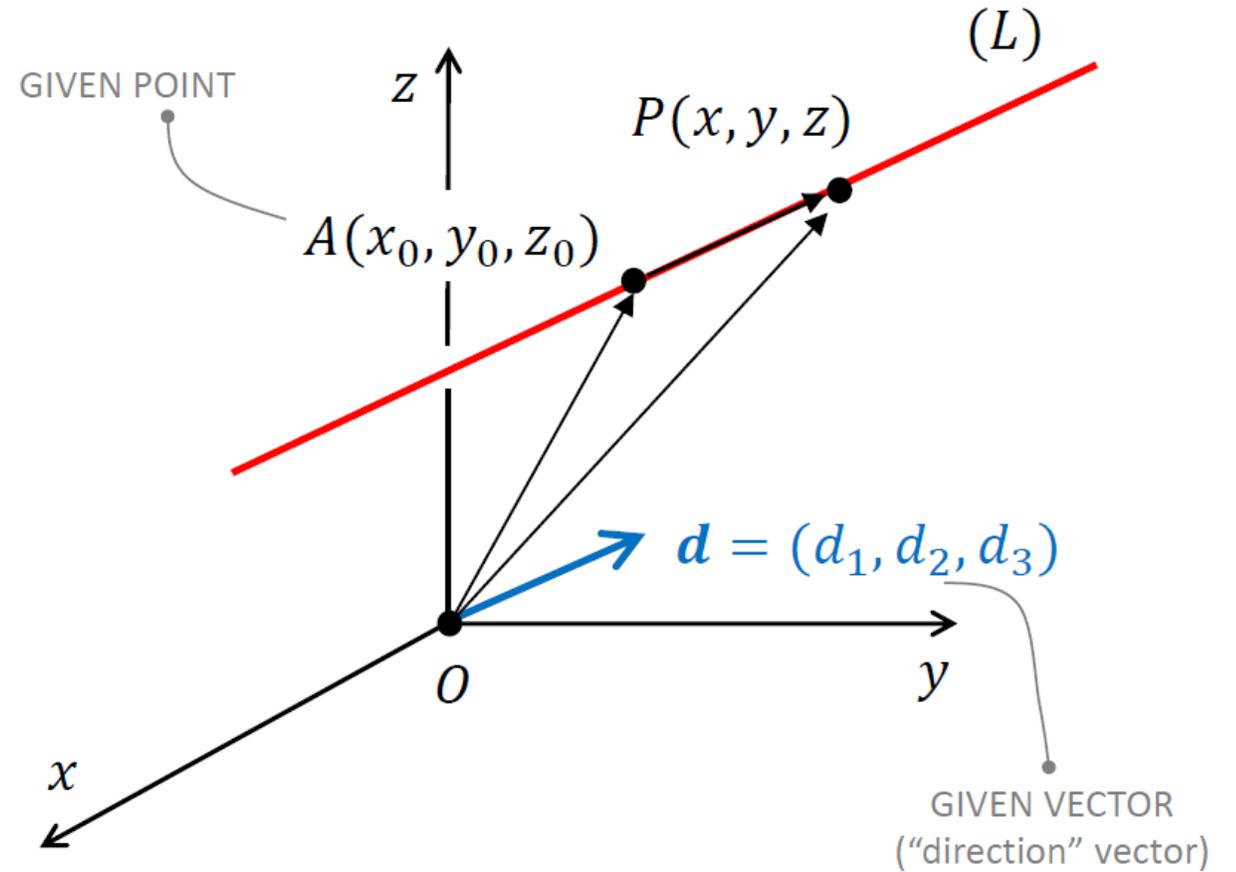
$$y = y_0 + \lambda d_2$$

$$z = z_0 + \lambda d_3$$

PARAMETRIC FORM....

$$\frac{x - x_0}{d_1} = \frac{y - y_0}{d_2} = \frac{z - z_0}{d_3}$$

SYMMETRIC FORM....



LAST WEEK:

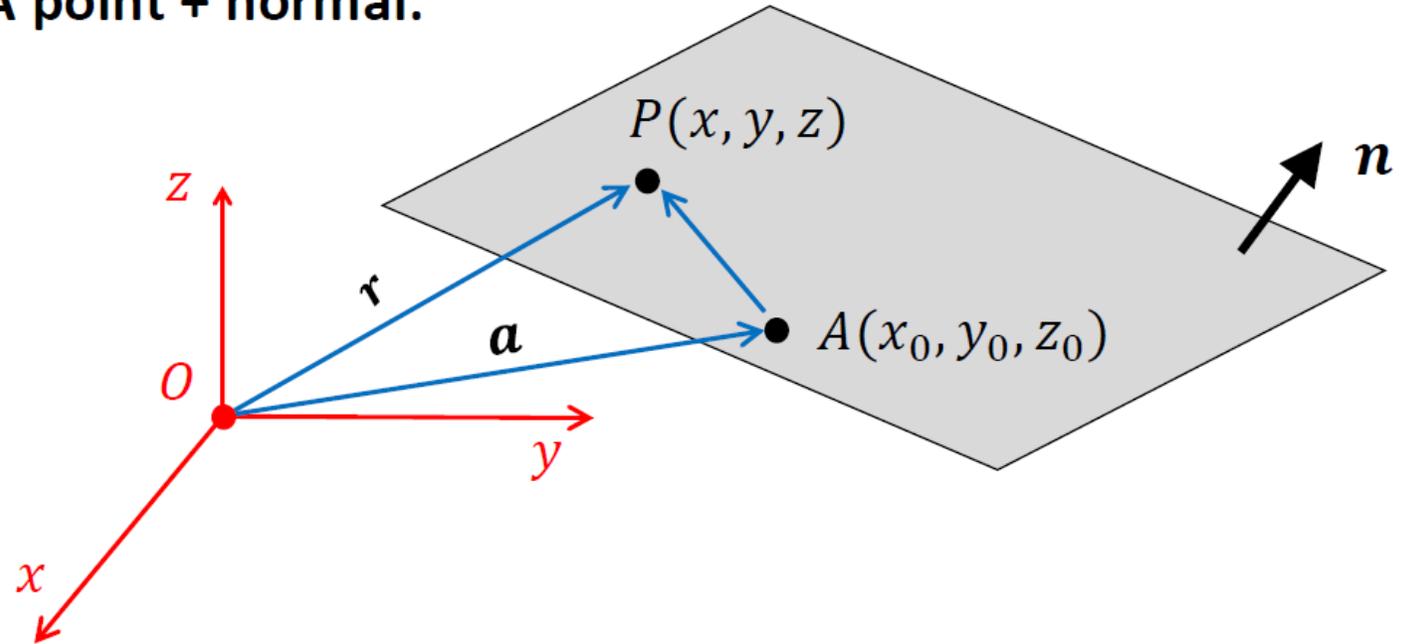
PLANES:

$$Ax + By + Cz = D$$

$$\mathbf{n} = (A, B, C)$$

$$D = Ax_0 + By_0 + Cz_0$$

A point + normal:

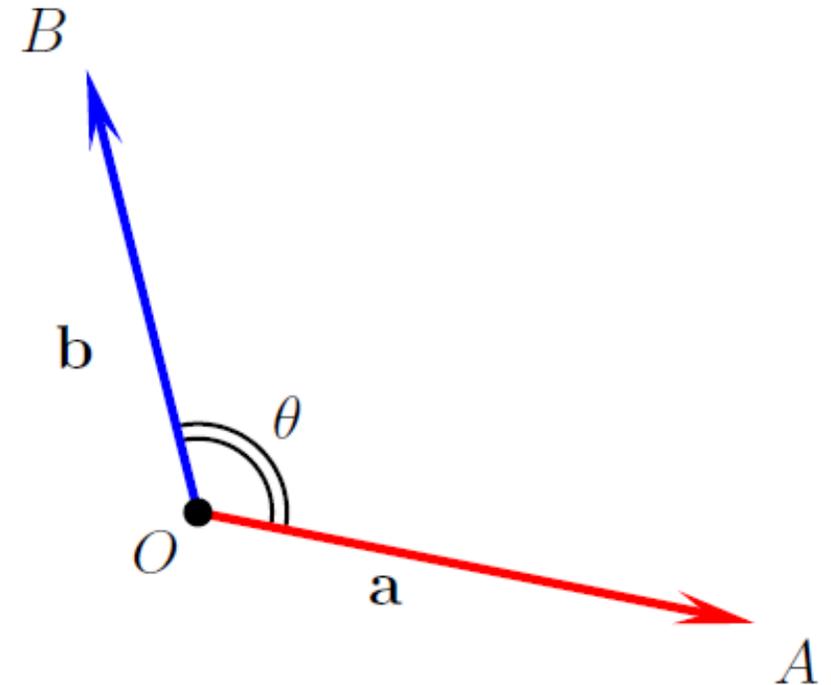
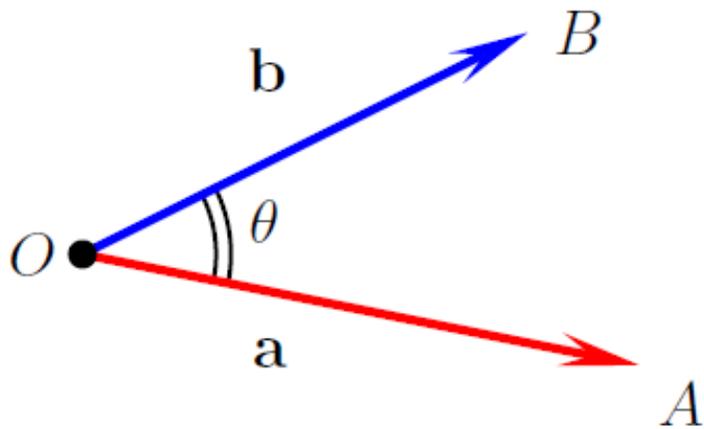


ANGLES:

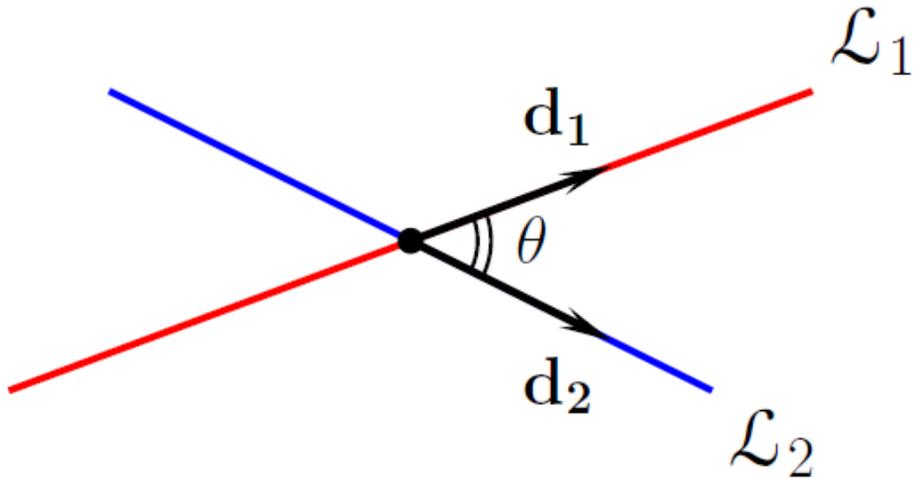
$\mathbf{a}, \mathbf{b} \neq \mathbf{0}$

the **angle** between these vectors is always taken to be between $[0, \pi]$

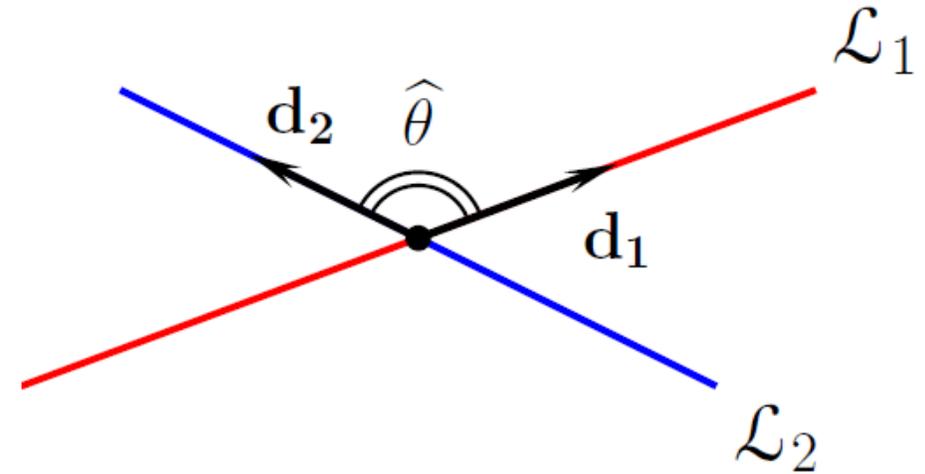
1. $\theta = 0$ means that \mathbf{a} and \mathbf{b} are in the same direction.
2. $\theta = \pi$ means that \mathbf{a} and \mathbf{b} are in the opposite direction.
3. $\theta = \pi/2$ means that \mathbf{a} and \mathbf{b} are perpendicular.



ANGLE BETWEEN
TWO LINES = angles between their direction vectors



$$\theta + \hat{\theta} = \pi$$



ANGLE
BETWEEN THE
TWO LINES ABOVE ==>

$$\cos^{-1} \left(\frac{\mathbf{d}_1 \cdot \mathbf{d}_2}{|\mathbf{d}_1| |\mathbf{d}_2|} \right)$$

EXAMPLE 1: Find the angle between the lines

$$x = -2y = 2z \quad \text{and}$$

$$x = y = \frac{z}{2}$$

$$\frac{x}{2} = \frac{y}{-1} = \frac{z}{1}$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{2}$$

(symmetric forms)

$$\Rightarrow \underline{d}_1 = (2, -1, 1)$$

$$\Rightarrow \underline{d}_2 = (1, 1, 2)$$

(direction vectors)

$$\underline{d}_1 \cdot \underline{d}_2 = (2, -1, 1) \cdot (1, 1, 2) = (2)(1) + (-1)(1) + (1)(2) = 3$$

$$|\underline{d}_1| = \sqrt{4 + 1 + 1} = \sqrt{6}$$

$$|\underline{d}_2| = \sqrt{1 + 1 + 4} = \sqrt{6}$$

$$\theta = \cos^{-1} \left(\frac{\underline{d}_1 \cdot \underline{d}_2}{|\underline{d}_1| |\underline{d}_2|} \right) = \cos^{-1} \left(\frac{3}{\sqrt{6} \sqrt{6}} \right) = \cos^{-1} \left(\frac{1}{2} \right) = 60^\circ$$

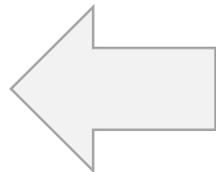
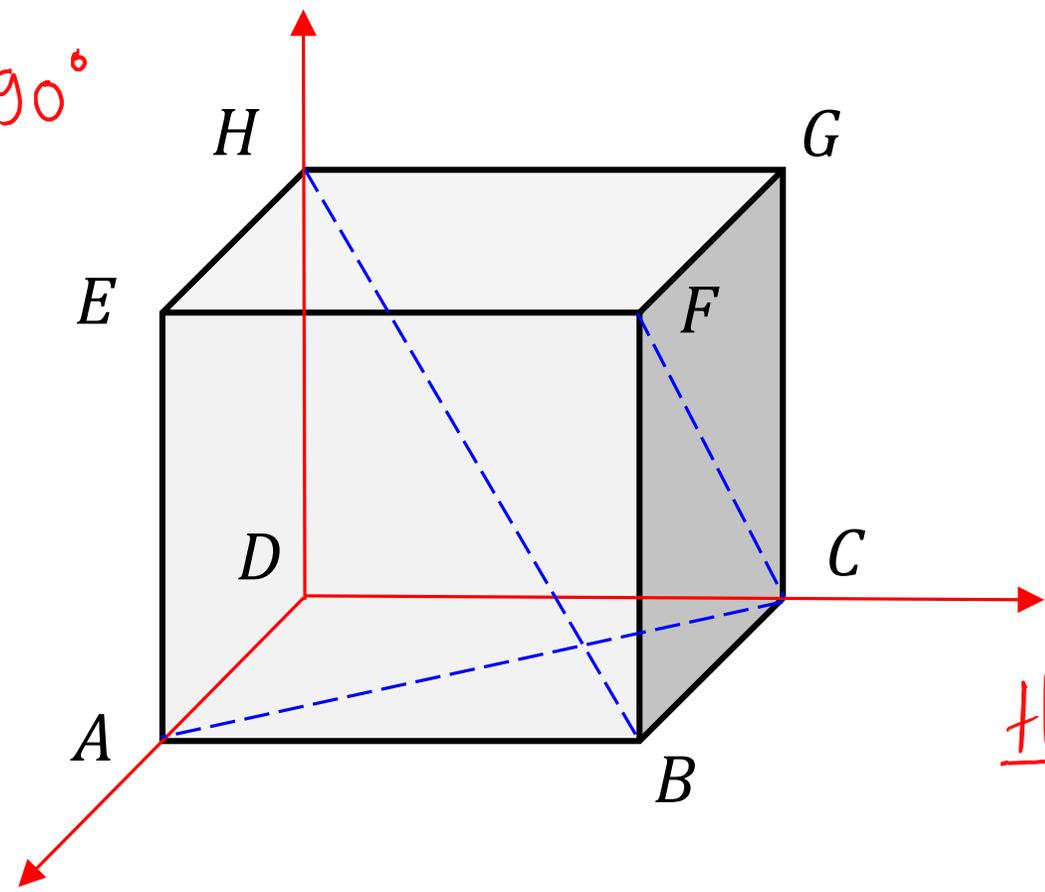
angle between lines = 60°

(OR $\frac{\pi}{3}$)

OPTIONAL QUESTION:

Find the **acute** angle between: (i) AC and BH; (ii) BH and FC.

$0 < \text{angle} < 90^\circ$

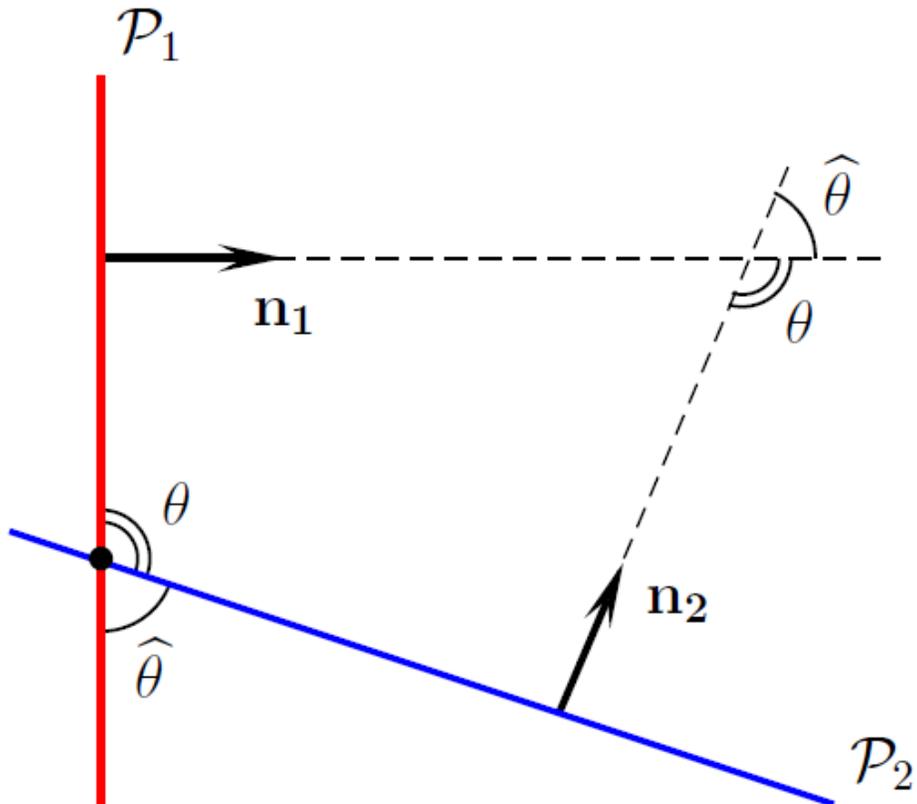


cube of side a
(e.g., take $a = 4$)

Hint: What are the coordinates of A, B, C, D, E, ... ?

ANGLE BETWEEN

TWO PLANES = angle between normal vectors of the planes:



$$\cos^{-1} \left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} \right)$$

$$\theta + \hat{\theta} = \pi$$

EXAMPLE 2: Find the **acute** angle between the planes

$$x + y + 2z = 4 \quad \text{and} \quad x - 2y - z = 5$$

$$\underline{n}_1 = (1, 1, 2)$$

$$\underline{n}_2 = (1, -2, -1)$$

$$\underline{n}_1 \cdot \underline{n}_2 = (1, 1, 2) \cdot (1, -2, -1) = (1)(1) + (1)(-2) + (2)(-1) = -3$$

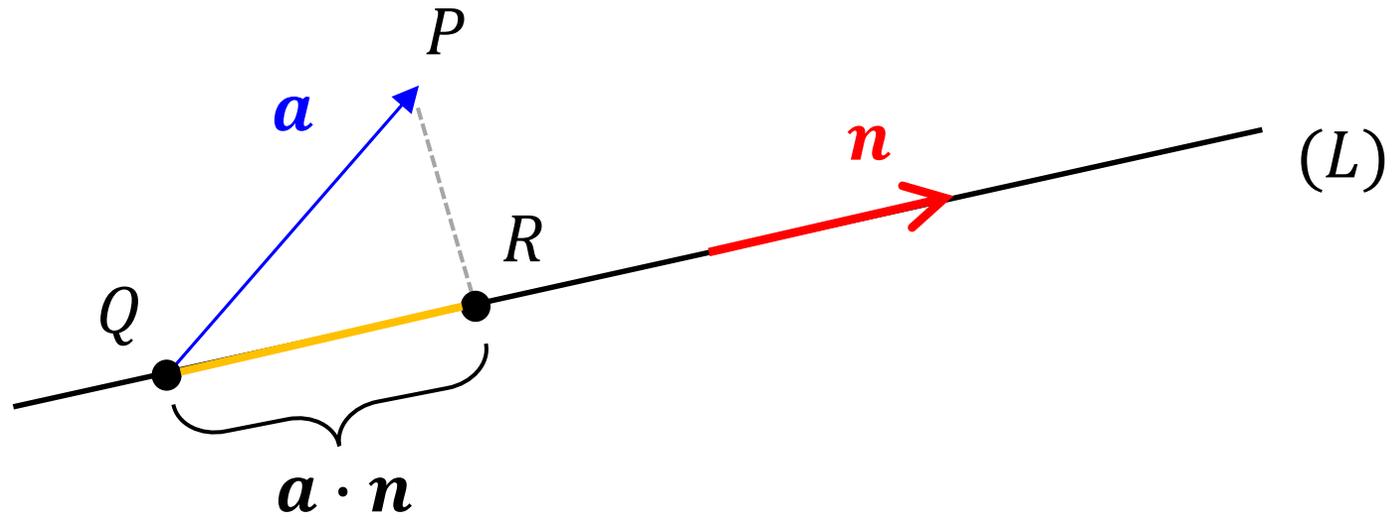
$$|\underline{n}_1| = |\underline{n}_2| = \sqrt{6}$$

$$\theta = \cos^{-1} \left(\frac{\underline{n}_1 \cdot \underline{n}_2}{|\underline{n}_1| |\underline{n}_2|} \right) = \cos^{-1} \left(\frac{-3}{\sqrt{6} \sqrt{6}} \right) = \cos^{-1} \left(-\frac{1}{2} \right) = \frac{2\pi}{3} > \frac{\pi}{2}$$

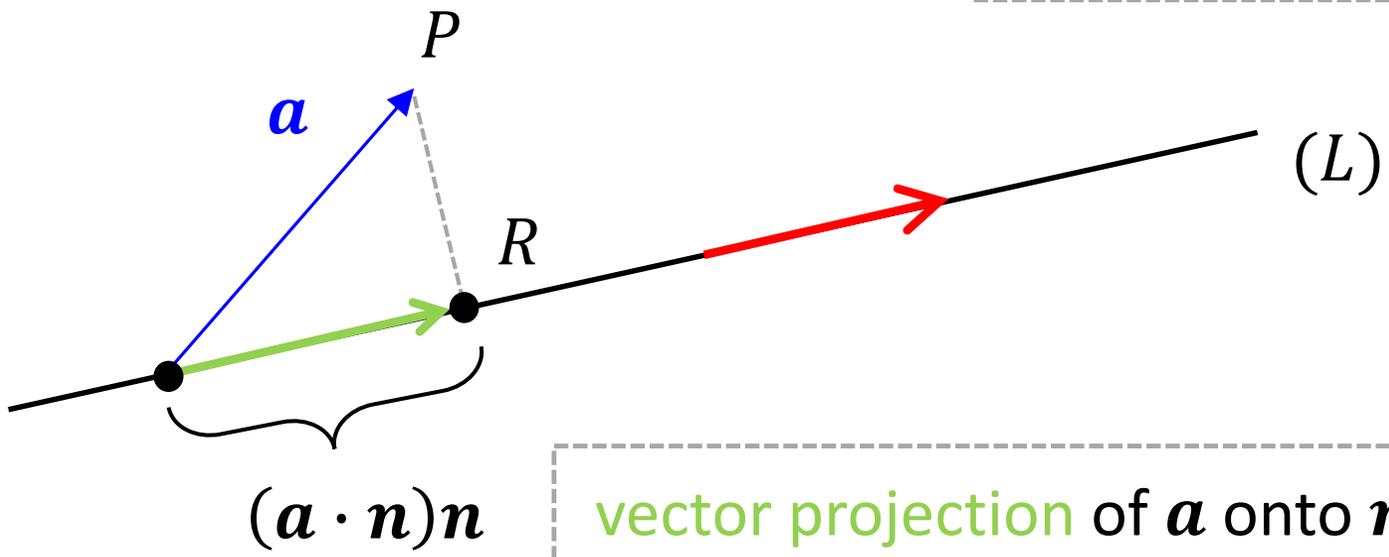
$$\boxed{\text{acute angle between planes}} = \pi - \frac{2\pi}{3} = \boxed{\frac{\pi}{3}}$$

$$|\mathbf{n}| = 1$$

$$\mathbf{a} \neq \mathbf{0}$$



component of a vector \mathbf{a}
in the direction of the unit vector \mathbf{n} \rightarrow SCALAR



vector projection of \mathbf{a} onto \mathbf{n} \rightarrow VECTOR

EXAMPLE 3: If $\mathbf{a} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and

$$\mathbf{n} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{2}}$$

then find: (i) the component of \mathbf{a} in the direction of \mathbf{n} ; (ii) the vector projection of \mathbf{a} onto \mathbf{n}

SCALAR

$$\begin{aligned} & \underline{\mathbf{a}} \cdot \underline{\mathbf{n}} \\ &= (1, 1, 3) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \\ &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2} \end{aligned}$$

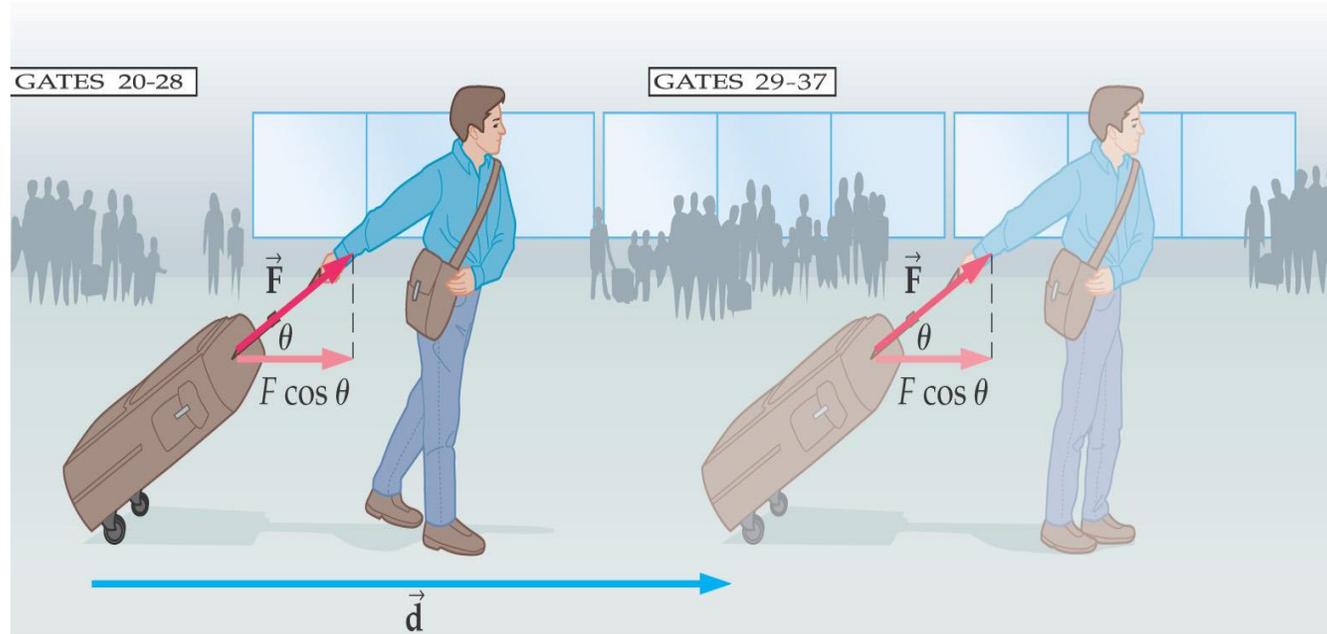
$$\underline{\mathbf{a}} \cdot \underline{\mathbf{n}} = \sqrt{2}$$

vector projection

$$\begin{aligned} & (\underline{\mathbf{a}} \cdot \underline{\mathbf{n}}) \underline{\mathbf{n}} \\ &= \sqrt{2} \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \\ &= (1, 1, 0) \\ &= \underline{\mathbf{i}} + \underline{\mathbf{j}} \end{aligned}$$

WORK DONE BY A FORCE:

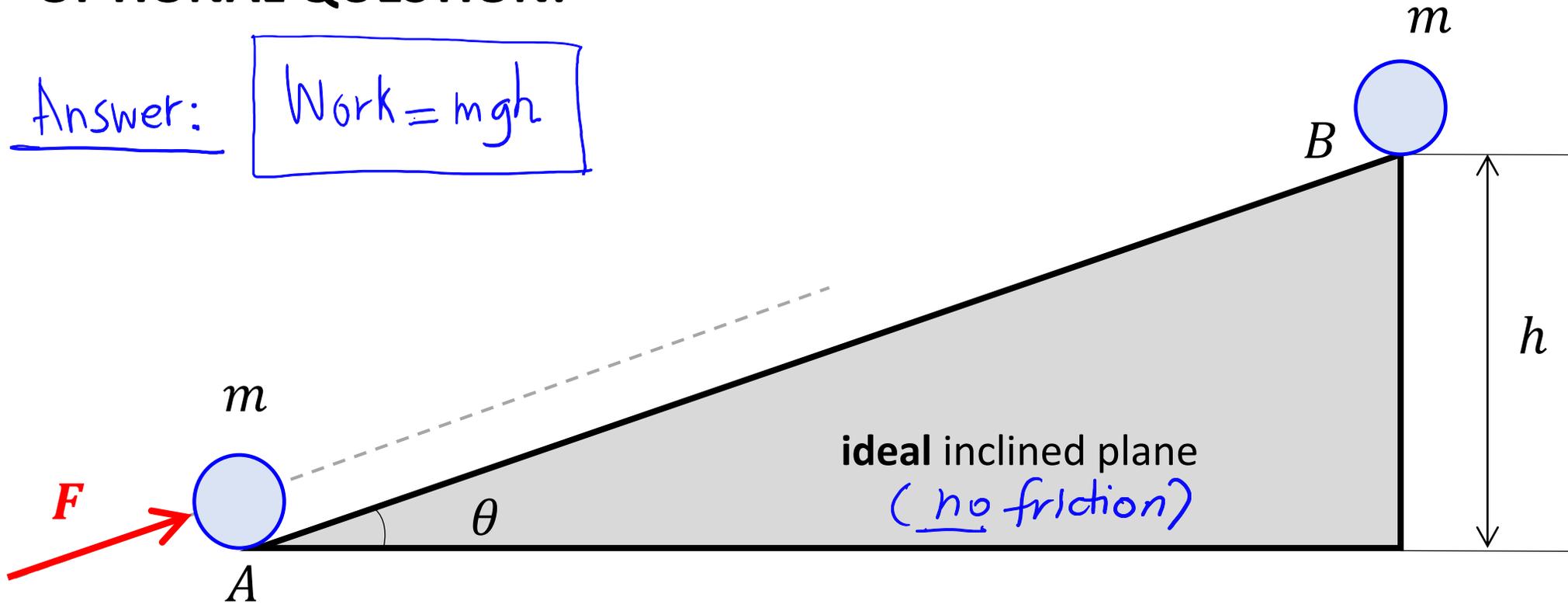
$$W = \mathbf{F} \cdot \mathbf{d} = |\mathbf{F}| |\mathbf{d}| \cos \theta$$
$$= (|\mathbf{F}| \cos \theta) |\mathbf{d}|$$



The **work** done on a body by a constant force is defined as the product of the magnitudes of the displacement and the vector projection of the force in the direction of the displacement.

OPTIONAL QUESTION:

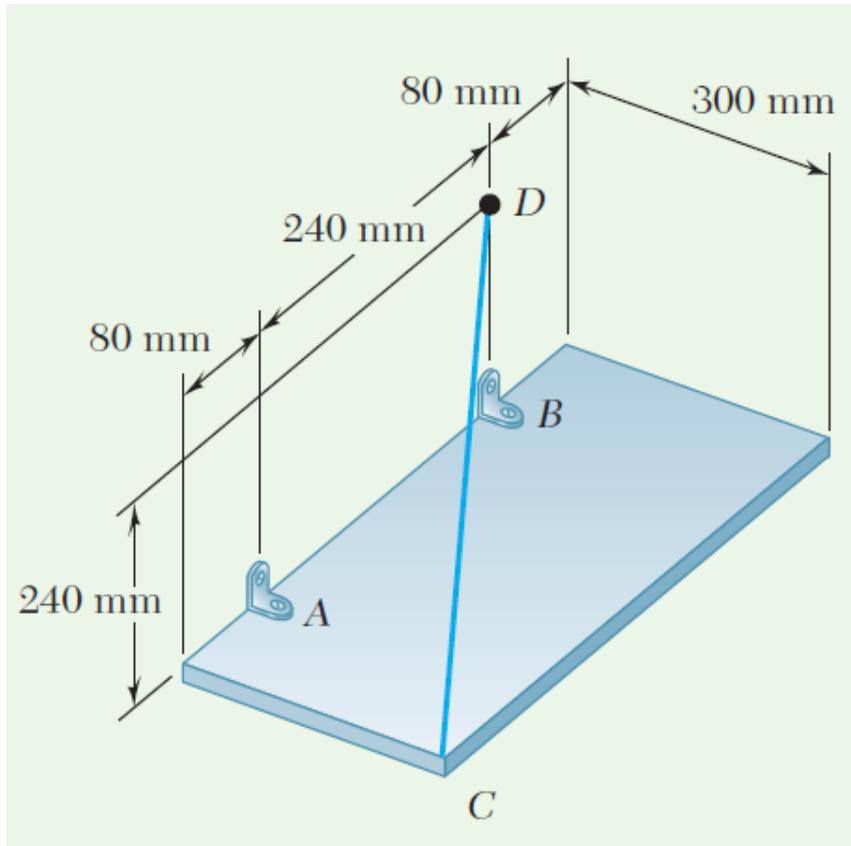
Answer: $W = mgh$



A constant force F acting parallel to an inclined plane moves a mass m from A to B , without accelerating it. **What work is done by the force F ?**

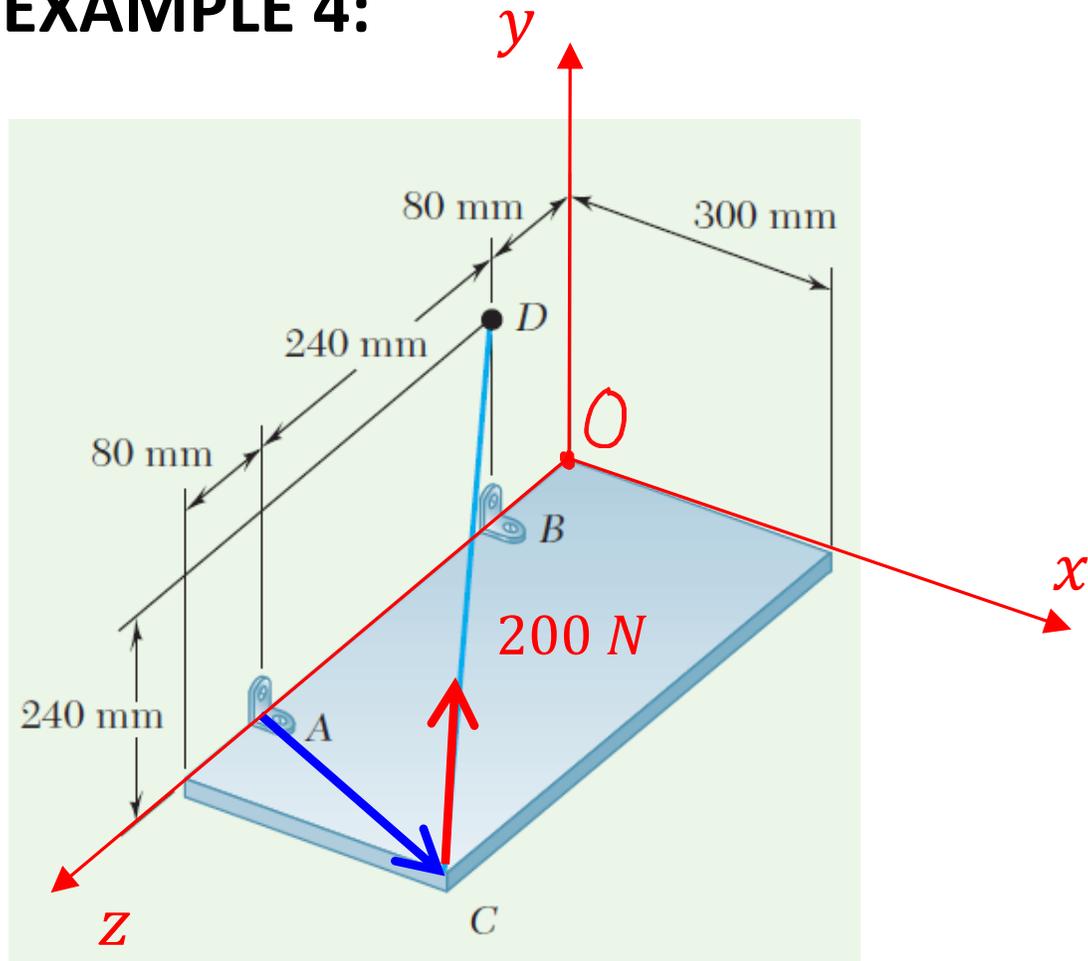
(use the definition of W in terms of the scalar product to get your answer)

EXAMPLE 4:



A rectangular plate is supported by brackets at A and B and by a wire CD . If the tension in the wire is 200 N , determine the **moment about A** of the force exerted by the wire on point C .

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$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

$$\mathbf{r} = \overrightarrow{AC} = p.v.(C) - p.v.(A)$$

$$= (0.3, 0, 0.08) = 0.3\mathbf{i} + 0.08\mathbf{k}$$

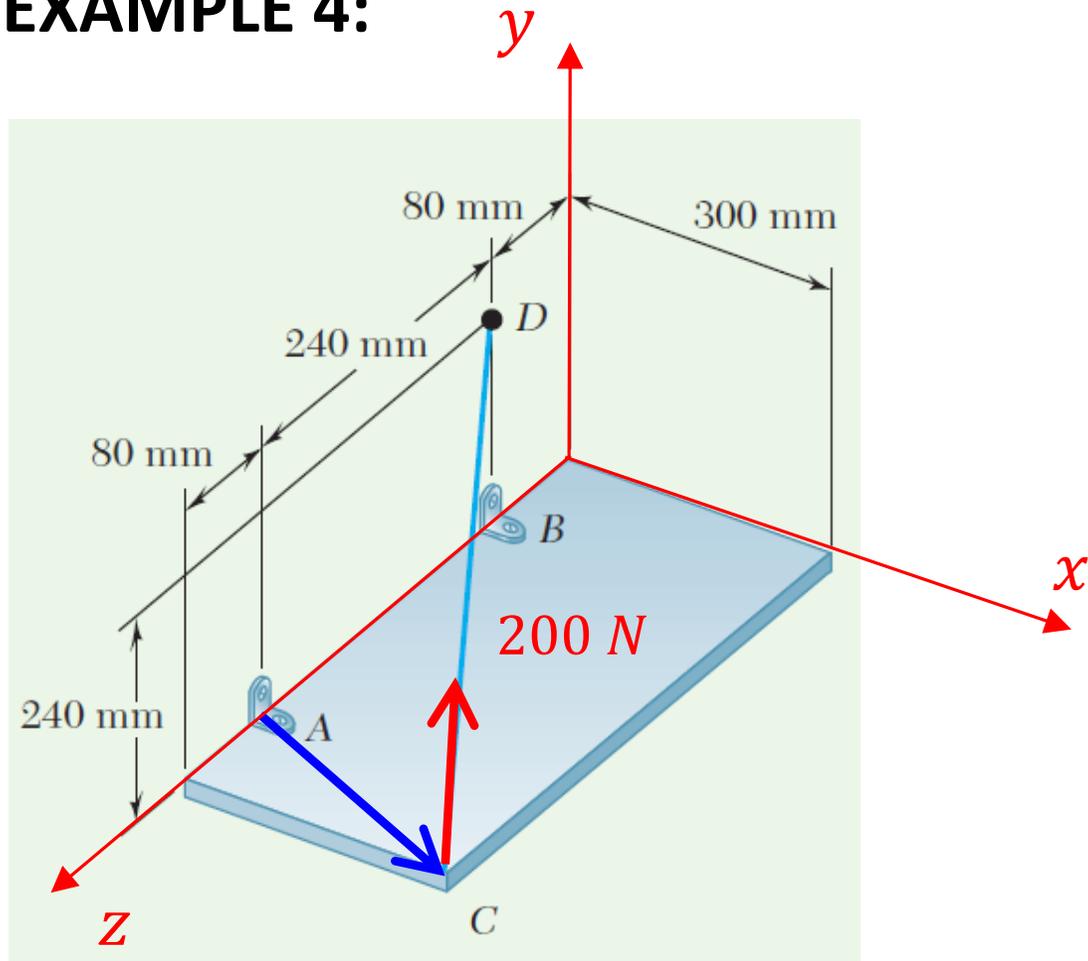
$$\mathbf{F} = ? \quad (|\underline{F}| = 200\text{ N, given})$$

$$A(0, 0, 0.32)$$

(metres)

$$C(0.3, 0, 0.4)$$

EXAMPLE 4:



A rectangular plate is supported by brackets at A and B and by a wire CD . If the tension in the wire is 200 N , determine the **moment about A** of the force exerted by the wire on point C .

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

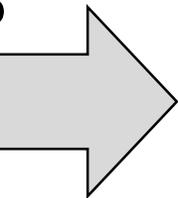
$$\mathbf{F} = 200 \underbrace{\frac{\overrightarrow{CD}}{|\overrightarrow{CD}|}}_{\text{UNIT VECTOR}}$$

$$C(0.3, 0, 0)$$

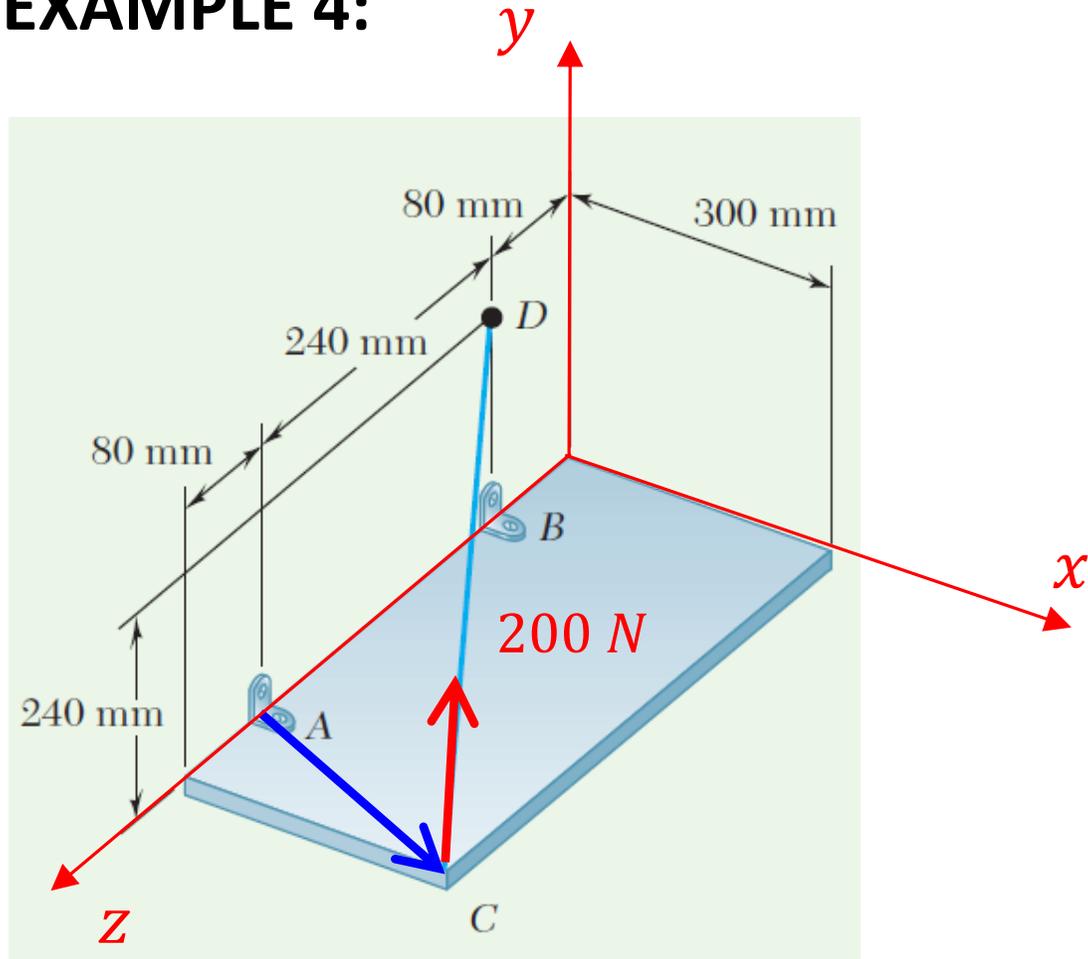
$$D(0, 0.24, 0.08)$$

$$\begin{aligned} \overrightarrow{CD} &= p.v.(D) - p.v.(C) \\ &= (-0.3, 0.24, -0.08) \end{aligned}$$

$$|\overrightarrow{CD}| = 0.5$$



EXAMPLE 4:



A rectangular plate is supported by brackets at A and B and by a wire CD . If the tension in the wire is 200 N , determine the **moment about A** of the force exerted by the wire on point C .

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

$$\mathbf{F} = 200 \frac{\overrightarrow{CD}}{|\overrightarrow{CD}|} = -120\mathbf{i} + 96\mathbf{j} - 128\mathbf{k} \quad (\text{N})$$

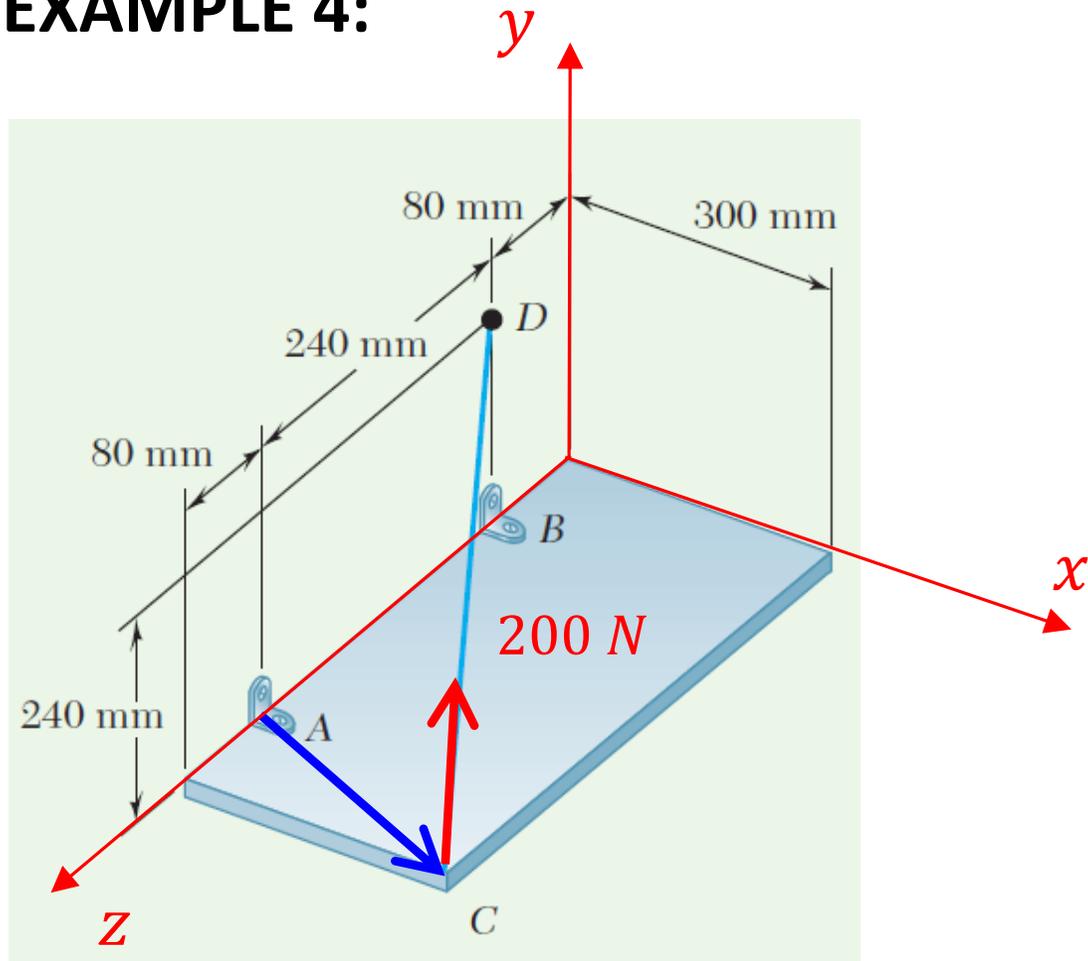
$$\mathbf{r} = (0.3, 0, 0.08)$$

(found earlier)

$$\overrightarrow{CD} = (-0.3, 0.24, -0.32)$$

$$|\overrightarrow{CD}| = 0.5$$

EXAMPLE 4:



A rectangular plate is supported by brackets at A and B and by a wire CD . If the tension in the wire is 200 N, determine the **moment about A** of the force exerted by the wire on point C .

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.3 & 0 & 0.08 \\ -120 & 96 & -128 \end{vmatrix}$$

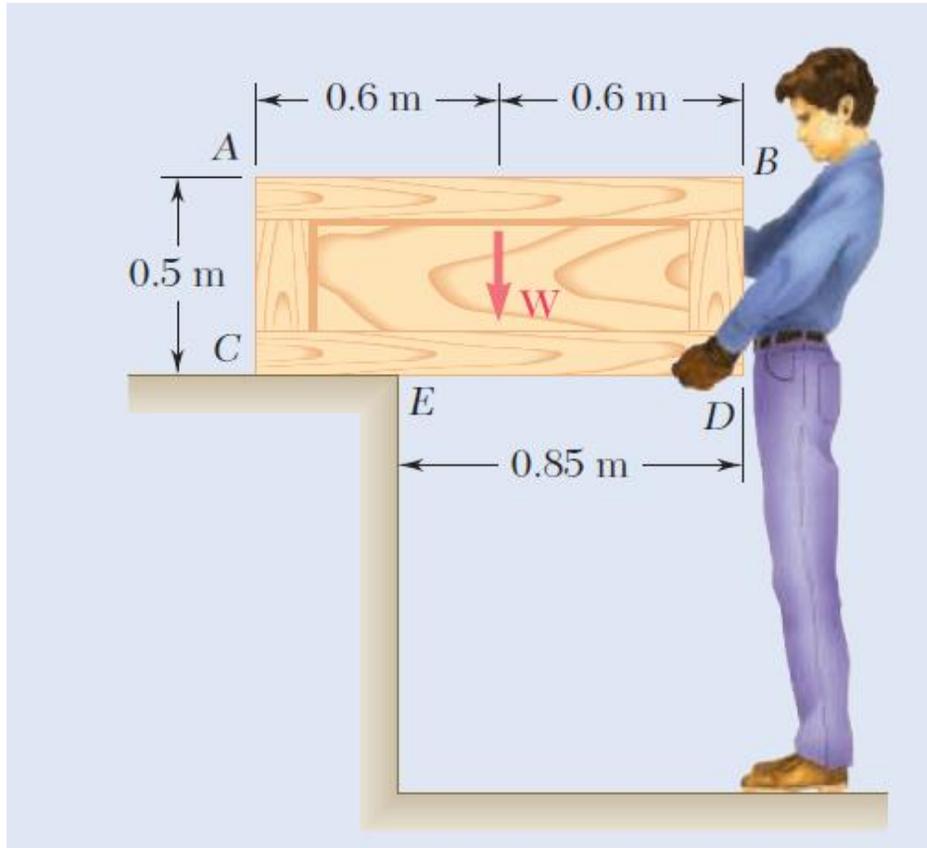
$$\mathbf{F} = (-120, 96, -128)$$

$$\mathbf{r} = (0.3, 0, 0.08)$$

$$= -7.68 \mathbf{i} + 28.8 \mathbf{j} + 28.8 \mathbf{k}$$

(final answer)

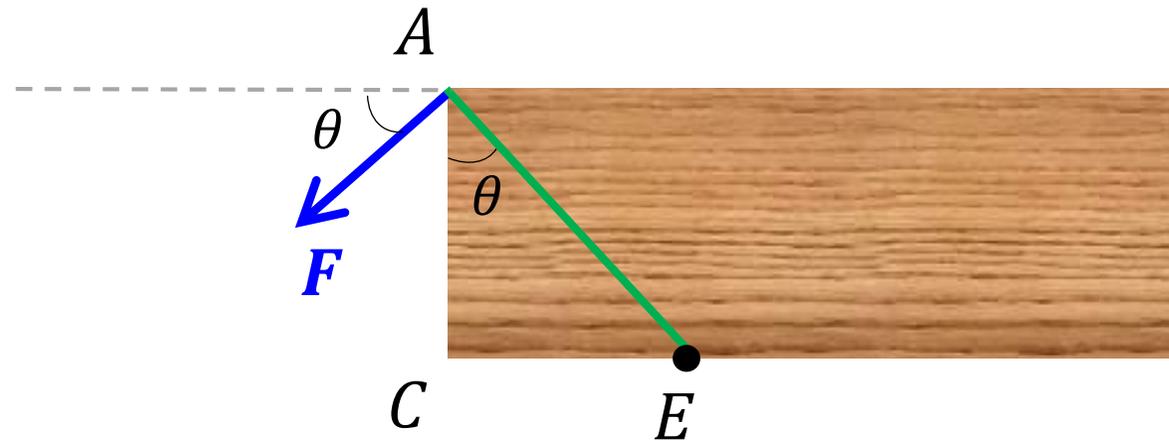
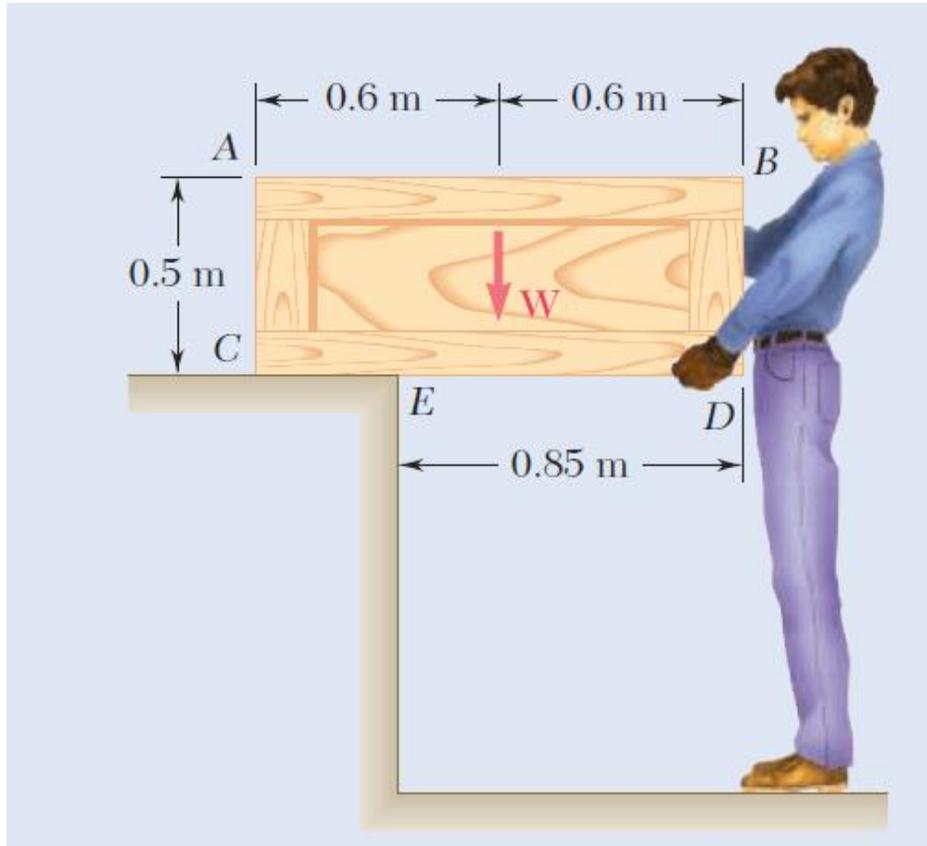
EXAMPLE 5:



A crate of mass 80 Kg is held in the position shown. Determine:

- The moment produced by the weight W of the crate about E .
- The smallest force applied at A that creates a moment of equal magnitude and opposite sense about E .

EXAMPLE 5:



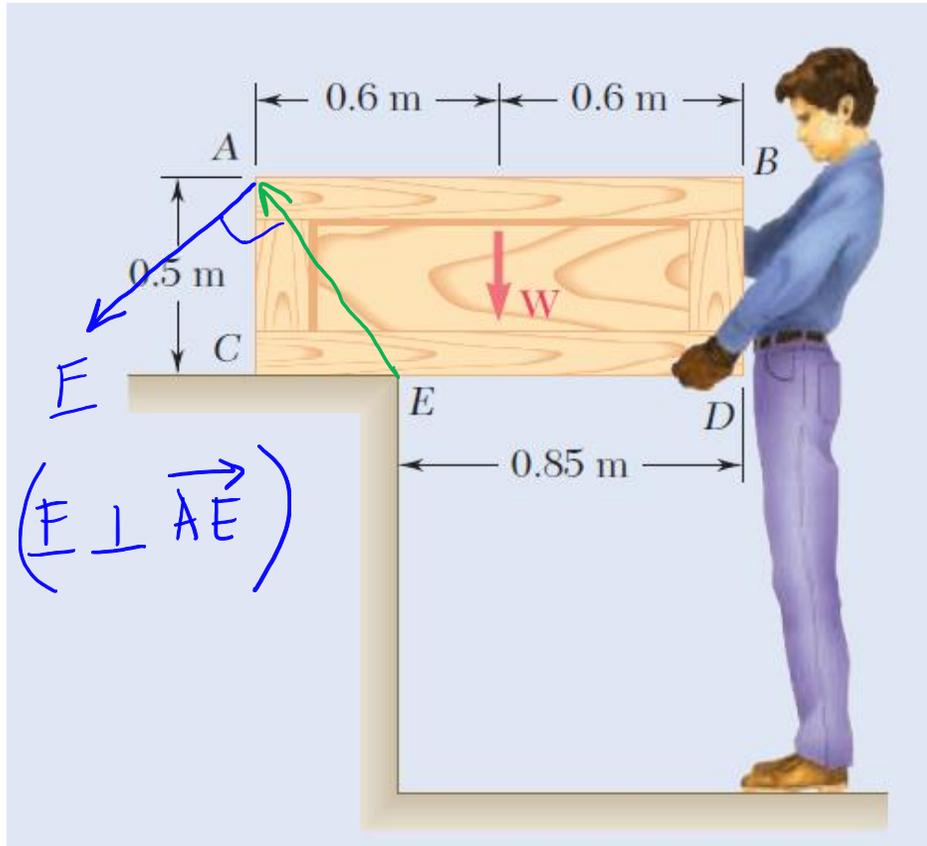
For F to be minimum, it must be perpendicular to the line joining points A and E. Then, with F directed as shown, we have

$$|M_E| = |\overrightarrow{AE}| |F|$$

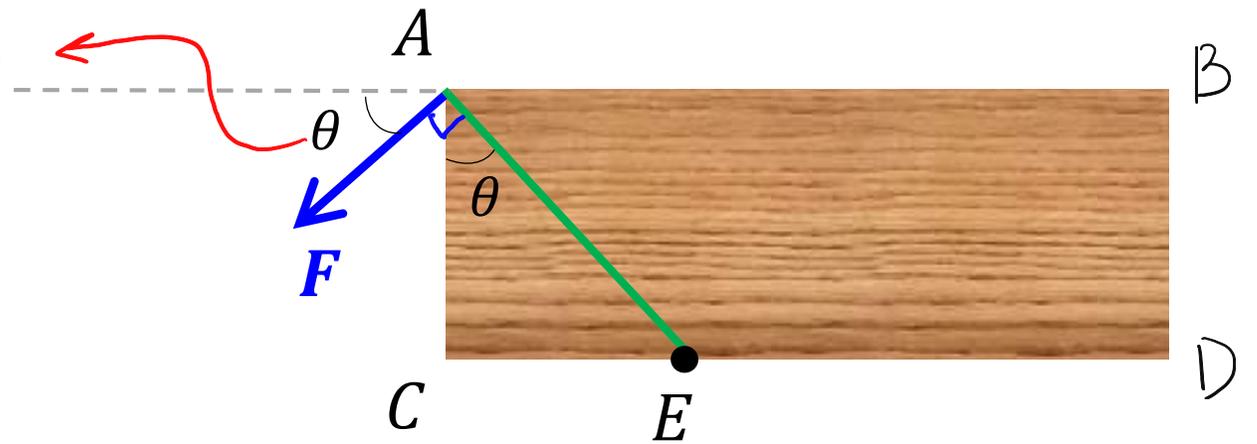
$$\begin{aligned} |M_E| &= |\overrightarrow{EH}| |W| = (0.25 \text{ m})(784.8 \text{ N}) \\ &= 196.2 \text{ N} \cdot \text{m} \end{aligned}$$

EXAMPLE 5:

this angle gives the orientation of \underline{F}



$$|M_E| = 196.2 \text{ N} \cdot \text{m}$$



$$|M_E| = |\overline{AE}| |F|$$

$$\sqrt{(0.35 \text{ m})^2 + (0.5 \text{ m})^2} = 0.61033 \text{ m}$$

Hence $196.2 \text{ N} \cdot \text{m} = (0.61033 \text{ m}) |F|$

$$\rightarrow |F| = 321 \text{ N} \text{ (magnitude)}$$

$$\tan \theta = \frac{CE}{AC} = \frac{0.35 \text{ m}}{0.5 \text{ m}} \Rightarrow \theta = 35^\circ \text{ (orientation)}$$