VECTORS
(lines & planes)

VECTORS

Geometrical point of view:

magnitude + direction

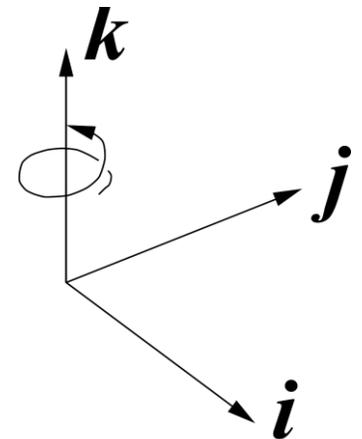$$\boldsymbol{a}$$
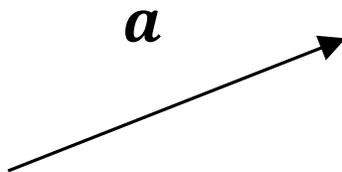
Algebraic point of view:

$$\boldsymbol{a} = (a_1, a_2, a_3)$$ ordered triple

$$= a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$$

$\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ = orthonormal vectors

VECTORS

Geometrical point of view:

Algebraic point of view:

Vector addition: the head-to-tail method

$$\boldsymbol{a} = (a_1, a_2, a_3)$$

$$\boldsymbol{b} = (b_1, b_2, b_3)$$
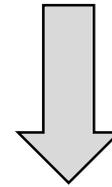
$$\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1, \ a_2 + b_2, \ a_3 + b_3)$$

VECTORS

Geometrical point of view:
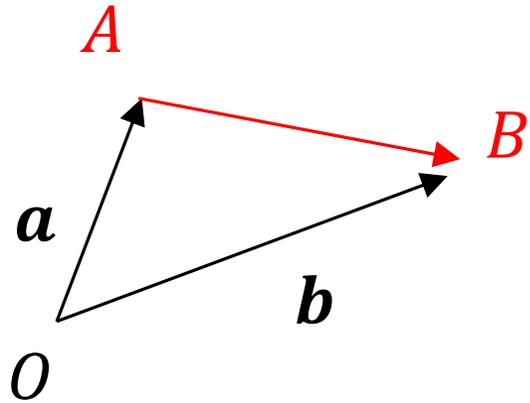
Algebraic point of view:

$$a = p.v.(A)$$

$$b = p.v.(B)$$

$$\overrightarrow{AB} = b - a$$
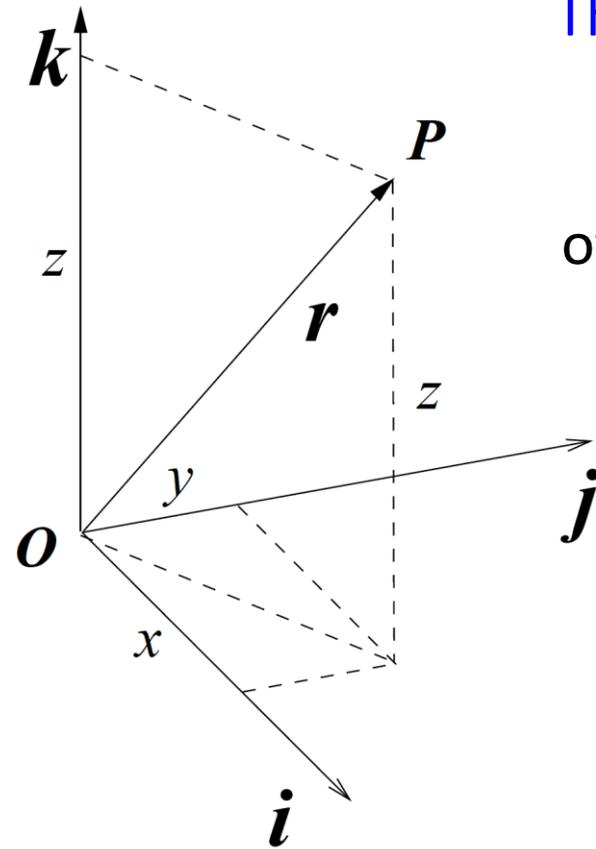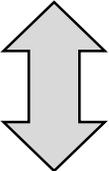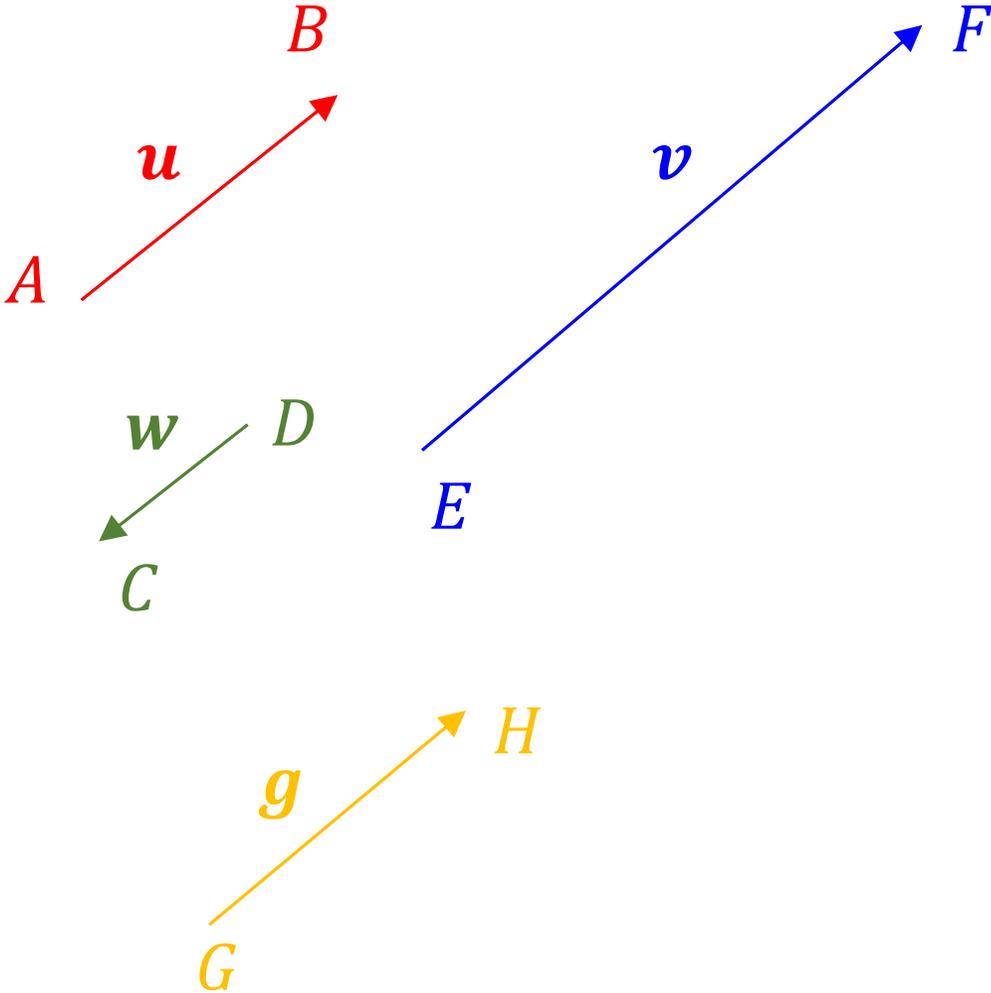
The **coordinates** of P
are
the **components**
of its position vector

PARALLEL VECTORS: **vectors that have the same direction**

Two vectors are parallel if they are **scalar** multiples of one another:

E.G. $\boldsymbol{u} = \alpha\boldsymbol{v}$

FOR SOME **SCALAR** α

$\boldsymbol{w} = \beta\boldsymbol{g}$

EQUATION OF A LINE

A straight **line** can be specified by

- two points it passes through, OR

- one point it passes through AND a given direction

OUR **OBJECTIVE:**

Find various ways to express the **equation of a line (in 3D)**

ASIDE: 2D stuff
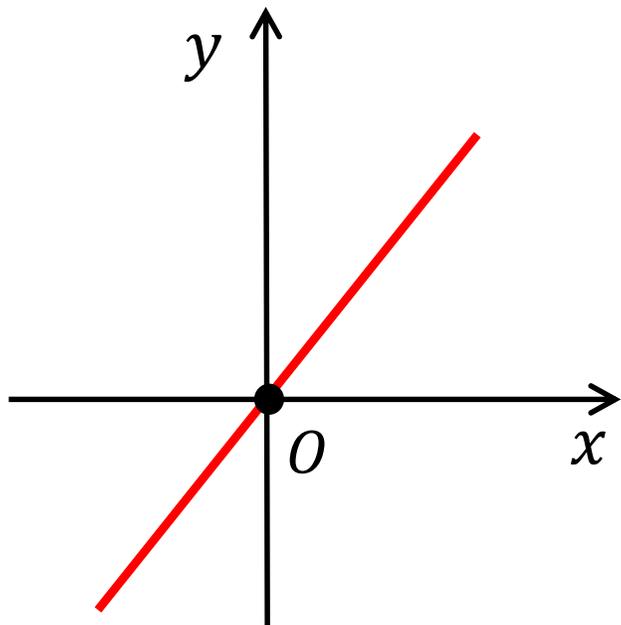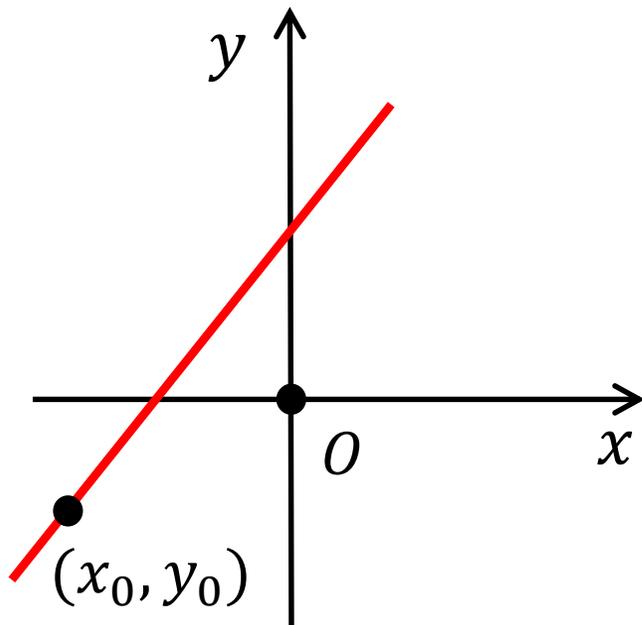
$$y = mx \qquad\qquad y - y_0 = m(x - x_0)$$

SLOPE

ASIDE: 2D stuff

$$\vec{AB} = \vec{CD} = \vec{EF}$$
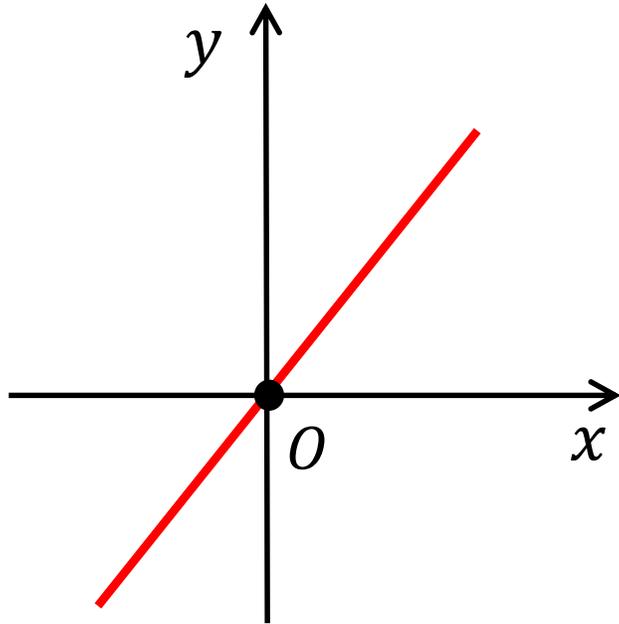
$y = mx$
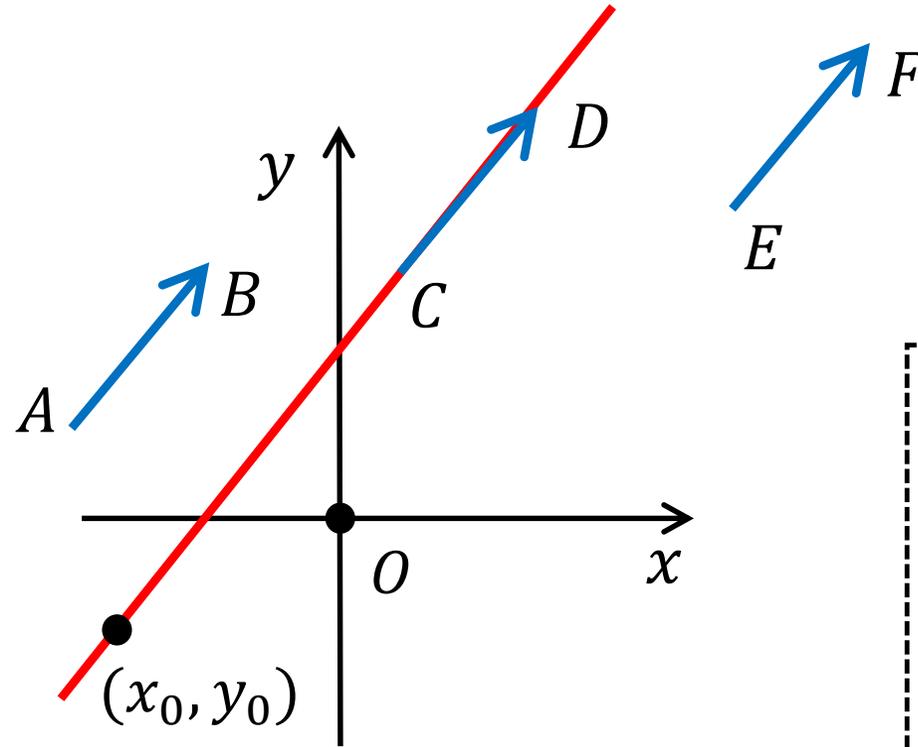
$y - y_0 = m(x - x_0)$

SLOPE

$\Updownarrow$
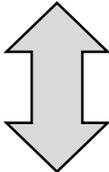
any vector parallel to the line
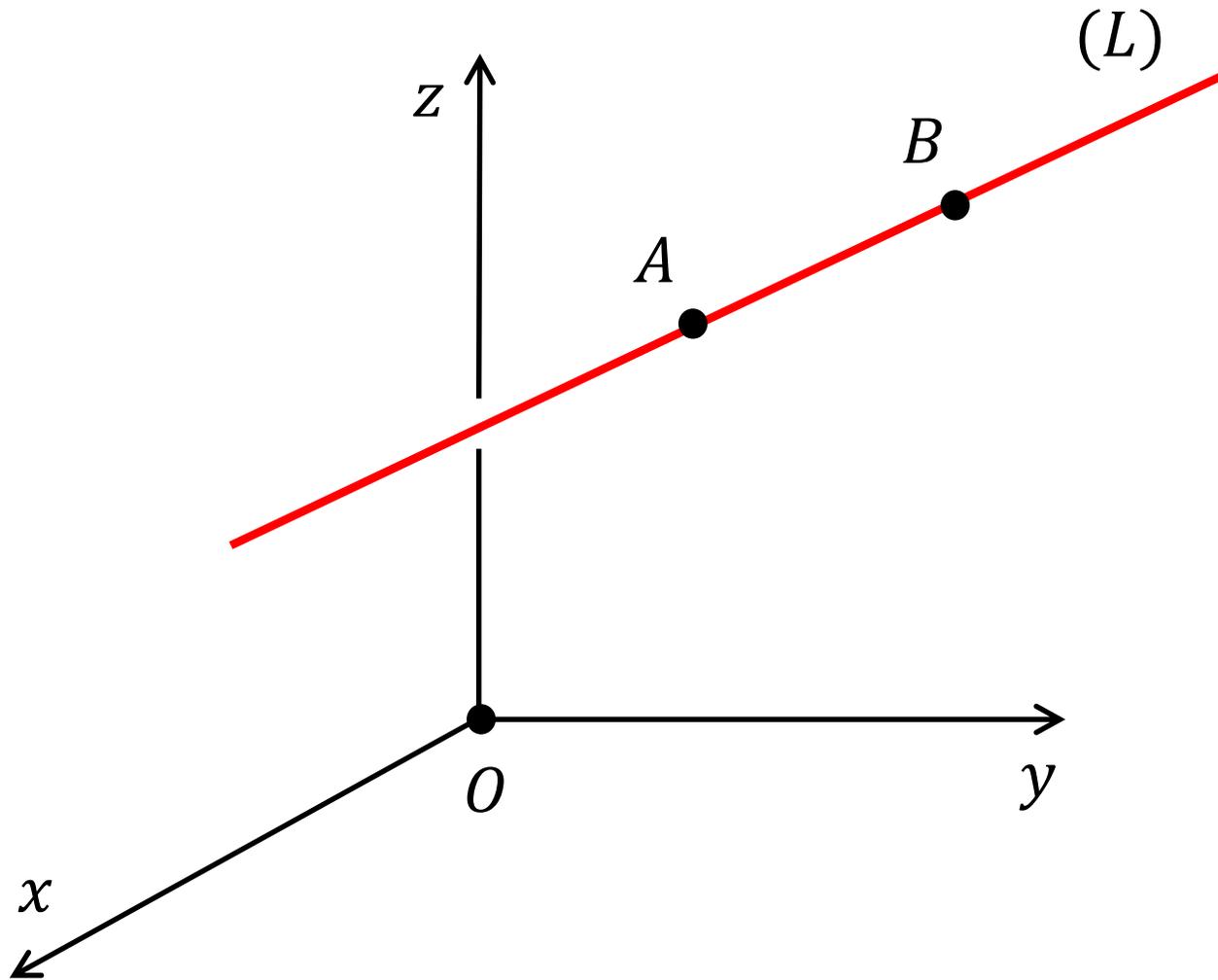
Line passing through two given points
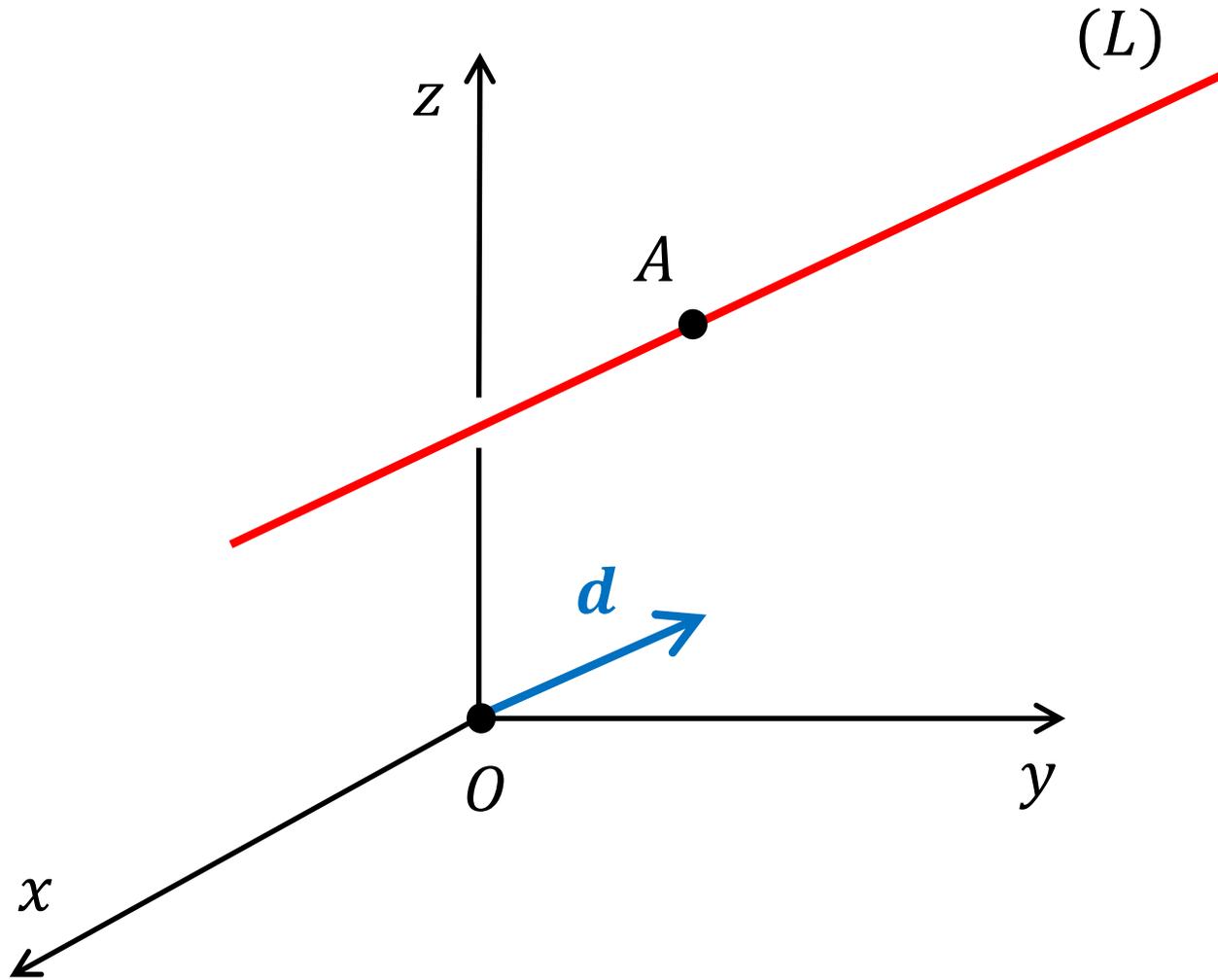
Line passing through a given point A and having the direction defined by the vector **d**

GIVEN POINT

z

(L)

$P(x, y, z)$

$A(x_0, y_0, z_0)$

$\boldsymbol{d} = (d_1, d_2, d_3)$

O

y

x

GIVEN VECTOR
("direction" vector)

z

(L)

$P(x, y, z)$

$A(x_0, y_0, z_0)$

r

r_0

$d = (d_1, d_2, d_3)$

O

y

x

$$\overrightarrow{AP} = \text{p.v.}(P) - \text{p.v.}(A)$$

$$= r - r_0$$
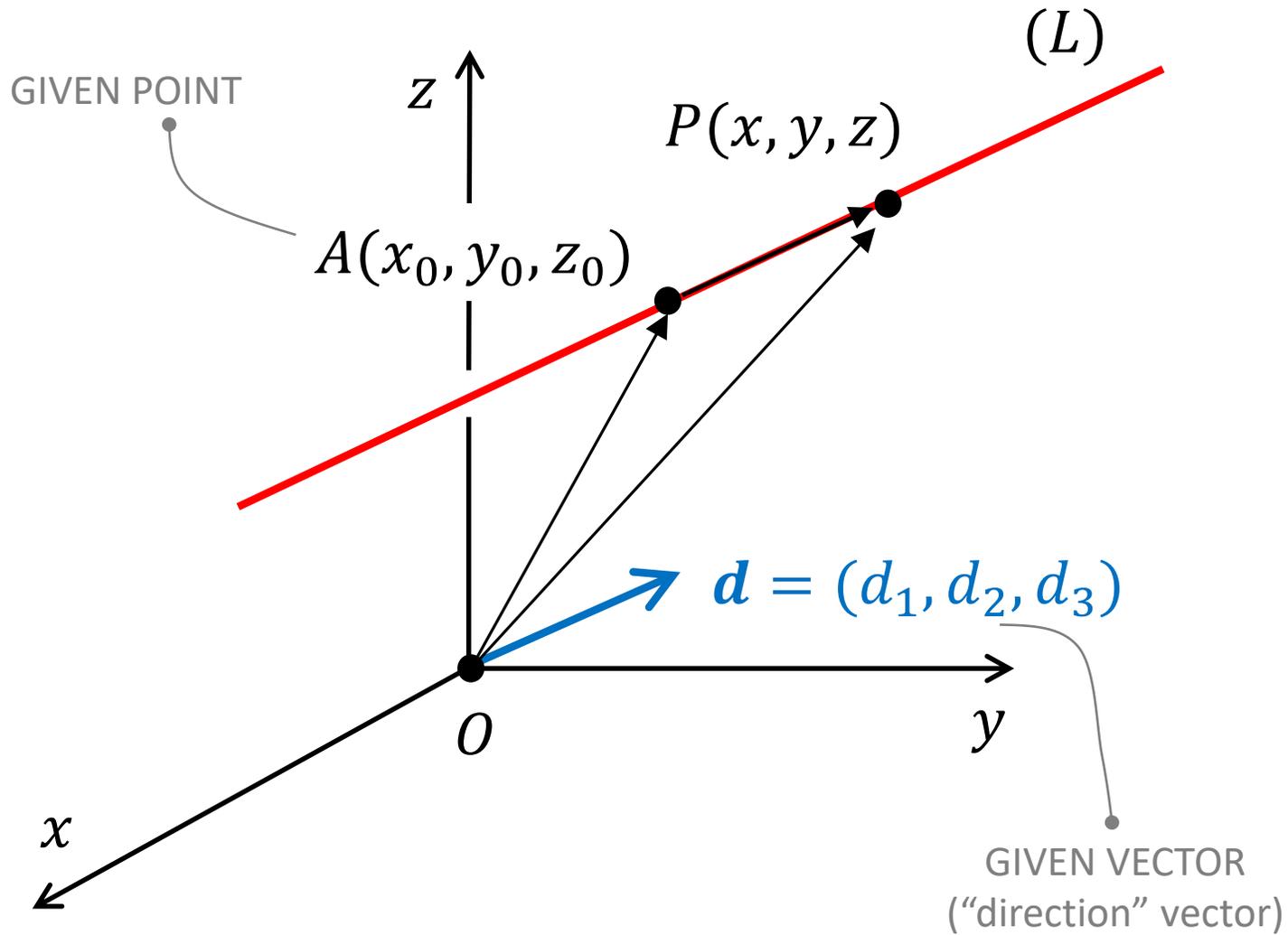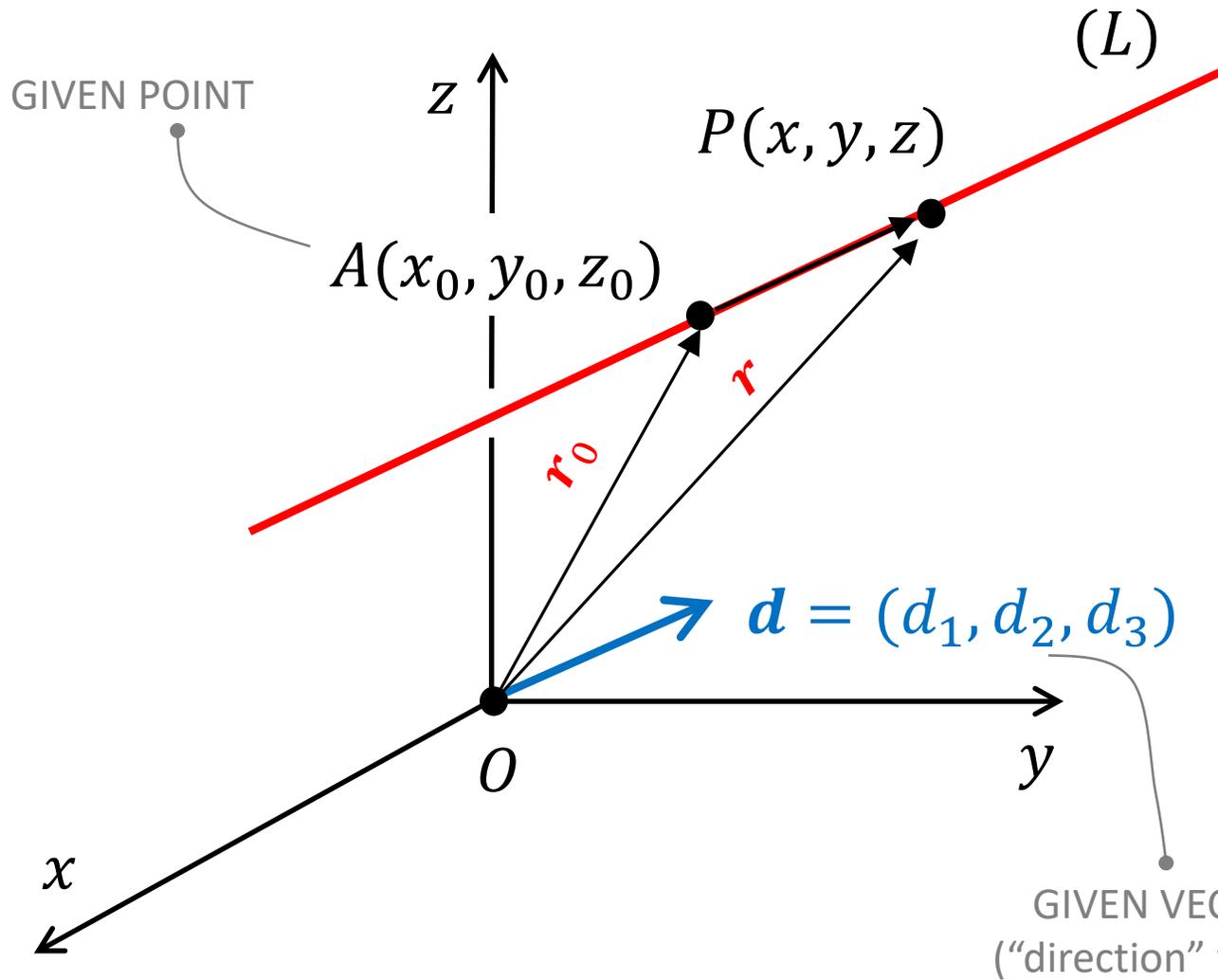
$$\overrightarrow{AP} \parallel d$$

HENCE

$$\overrightarrow{AP} = \lambda d \qquad (\lambda = \text{real number})$$

$$r - r_0 = \lambda d$$

$$r = r_0 + \lambda d$$

VECTOR FORM....

$$r = r_0 + \lambda d$$

$$r = (x, y, z)$$
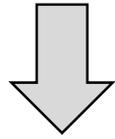
$$r_0 = (x_0, y_0, z_0)$$

$$d = (d_1, d_2, d_3)$$

$$(x, y, z) = (x_0, y_0, z_0) + \lambda(d_1, d_2, d_3)$$

$$(x, y, z) = (x_0 + \lambda d_1, y_0 + \lambda d_2, z_0 + \lambda d_3)$$

$$x = x_0 + \lambda d_1$$

$$y = y_0 + \lambda d_2$$

$$z = z_0 + \lambda d_3$$

PARAMETRIC FORM....

$$x = x_0 + \lambda d_1$$
$$y = y_0 + \lambda d_2$$
$$z = z_0 + \lambda d_3$$

OBS. Different values of the parameter λ generate different points on the line

$$x = x_0 + \lambda d_1$$
$$y = y_0 + \lambda d_2$$
$$z = z_0 + \lambda d_3$$

PARAMETRIC FORM....

$$\boldsymbol{r} = (x, y, z)$$

$$\boldsymbol{r}_0 = (x_0, y_0, z_0)$$

$$\boldsymbol{d} = (d_1, d_2, d_3)$$

$$\begin{cases} x - x_0 = \lambda d_1 \\ y - y_0 = \lambda d_2 \\ z - z_0 = \lambda d_3 \end{cases}$$

$$\Rightarrow$$
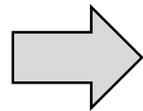
$$\begin{cases} \dfrac{x - x_0}{d_1} = \lambda \\[2mm] \dfrac{y - y_0}{d_2} = \lambda \\[2mm] \dfrac{z - z_0}{d_3} = \lambda \end{cases}$$

$$\Rightarrow$$
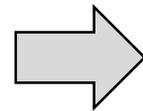
$$\frac{x - x_0}{d_1} = \frac{y - y_0}{d_2} = \frac{z - z_0}{d_3}$$

SYMMETRIC FORM....

Line passing through two given points

(L)

z

B

A

a

b

O

y

x
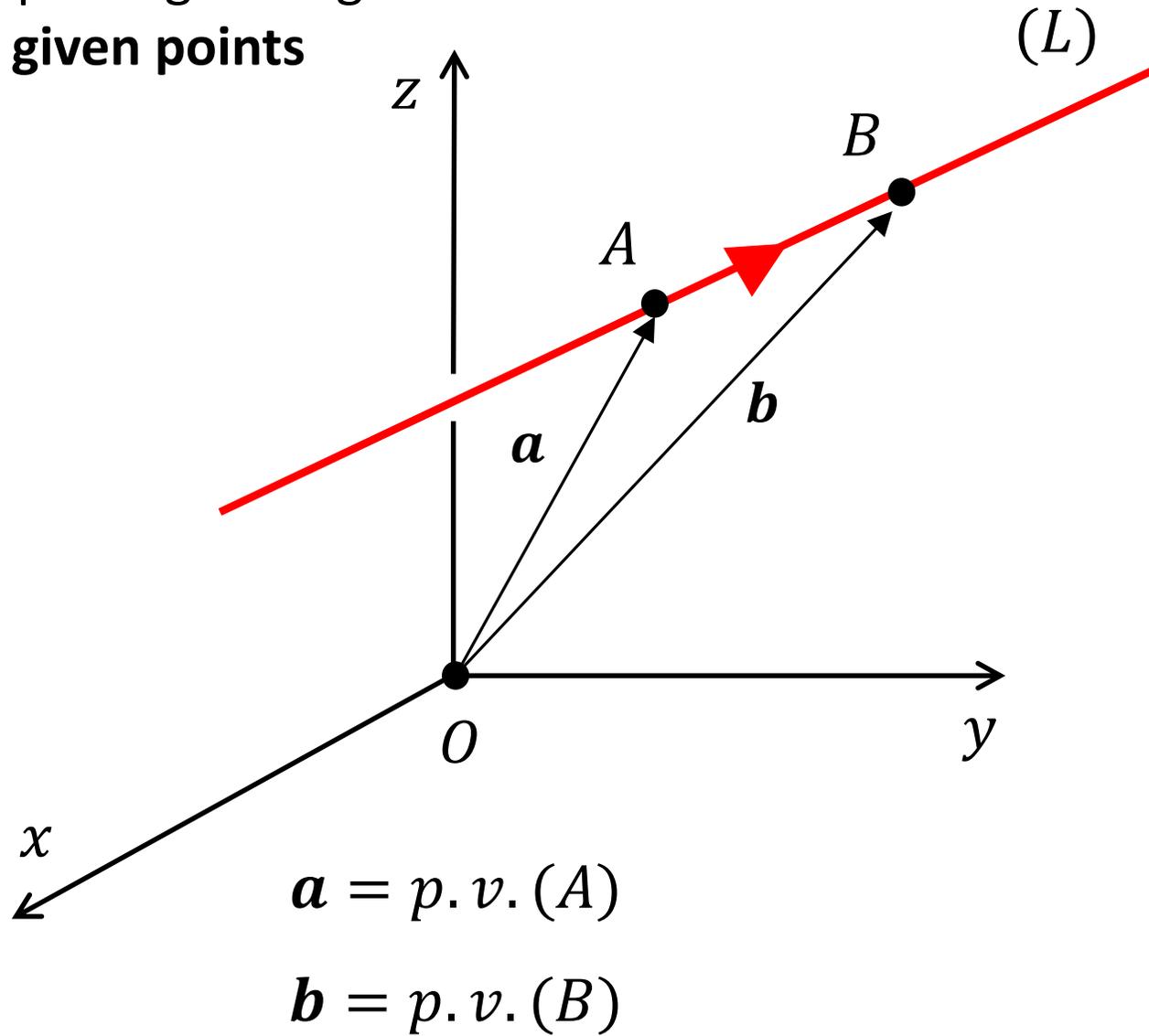
$a = p.v.(A)$

$b = p.v.(B)$

DIRECTION VECTOR:

$$\overrightarrow{AB} = b - a$$

(this is our d)

$$r = r_0 + \lambda(b - a)$$

Line passing through two given points

(L)

z

$B(x_B, y_B, z_B)$

$A(x_A, y_A, z_A)$

$\boldsymbol{a}$

$\boldsymbol{b}$

O

y

x
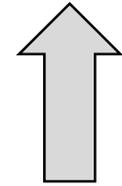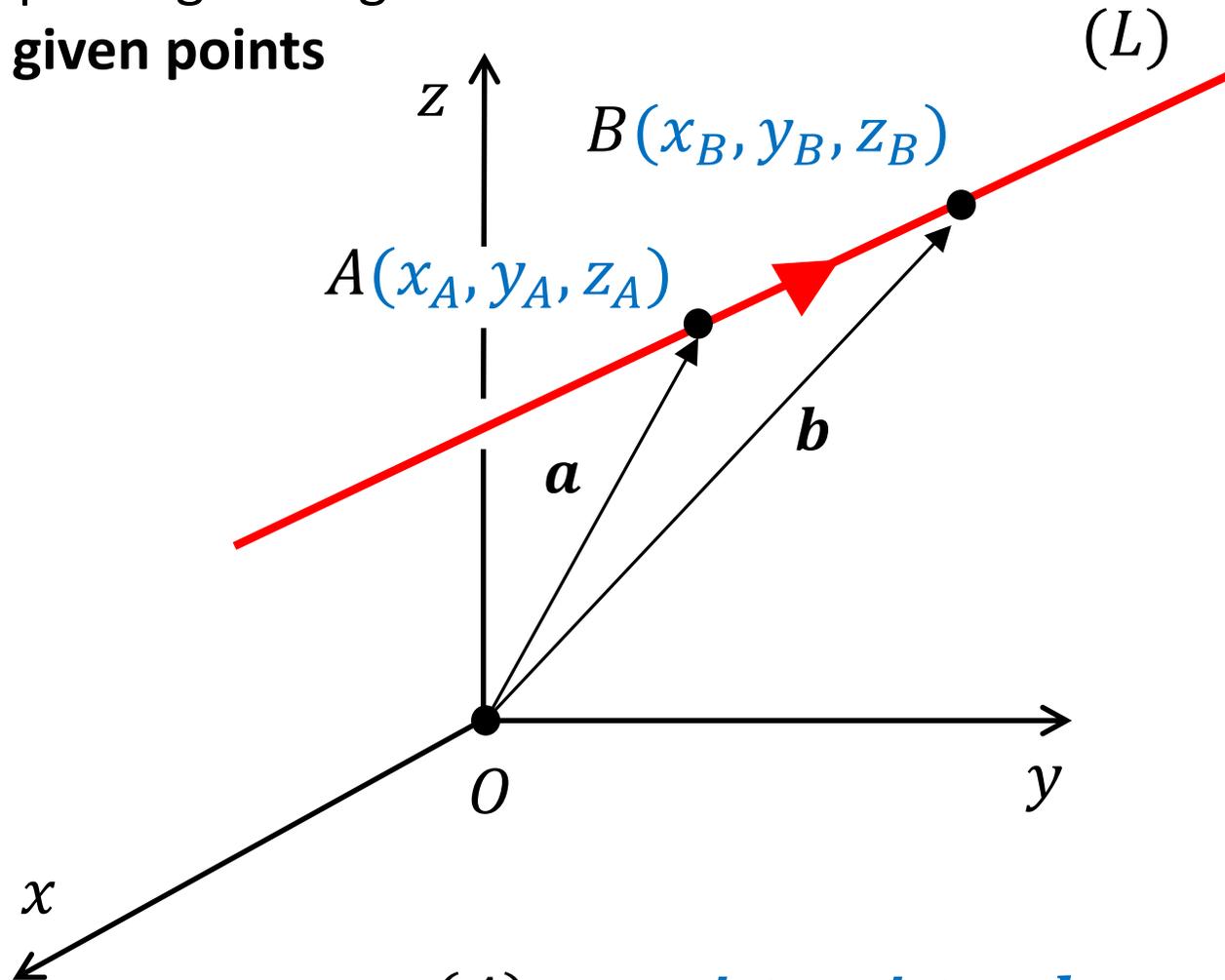
$\boldsymbol{a} = p.v.(A) = x_A\,\boldsymbol{i} + y_A\boldsymbol{j} + z_A\boldsymbol{k}$

$\boldsymbol{b} = p.v.(B) = x_B\,\boldsymbol{i} + y_B\boldsymbol{j} + z_B\boldsymbol{k}$

DIRECTION VECTOR:

$$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$$

(this is our $\boldsymbol{d}$)
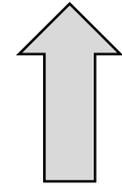
$$\boldsymbol{r} = \boldsymbol{r}_0 + \lambda(\boldsymbol{b} - \boldsymbol{a})$$

$(x_B - x_A,\ y_B - y_A,\ z_B - z_A)$

EXAMPLES:

1. Given the equation of a line $\dfrac{x+1}{2} = \dfrac{3-y}{6} = -\dfrac{z}{5}$

    a). Find the coordinates of a particular point on the line

    b). Is the vector *6i+18j+15k* <u>parallel</u> to the line?

$$\frac{X+1}{2} = \frac{-(y-3)}{6} = \frac{z}{-5} \Rightarrow \frac{X-(-1)}{2} = \frac{y-3}{-6} = \frac{z}{-5}$$

particular point : $(-1, 3, 0)$

dir. vector : $(2, -6, -5)$

$6\underline{i} + 18\underline{j} + 15\underline{K} = (6, 18, 15) = 3(2, 6, 5)$

<u>not</u> a scalar multiple of

Answer; <u>NO</u>

2. Consider the line $\dfrac{x-1}{3} = \dfrac{y-4}{1} = -\dfrac{z-2}{4}$

Find the coordinates of **three particular points** on the line.

Need the parametric form: $\dfrac{x-1}{3} = \dfrac{y-4}{1} = -\dfrac{z-2}{4} = \lambda$

$$\Rightarrow \begin{cases} x = 1+3\lambda \\ y = 4+\lambda \\ z = 2-4\lambda \end{cases} \Rightarrow$$

coordinates of generic point on the line $\longrightarrow (x,y,z) = (1+3\lambda, 4+\lambda, 2-4\lambda)$

$\lambda = 0:$ $(x,y,z) = (1,4,2)$

$\lambda = 1:$ $(x,y,z) = (4,5,-2)$

$\lambda = 2:$ $(x,y,z) = (7,6,-6)$

$\lambda = 3:$ $(x,y,z) = (10,7,-10)$

four specific points on the given line

$\vdots$

3. Find the **symmetric** form equation of the line passing through the points $A(-1, 3, 2)$ and $B(5, 4, -3)$

$$\text{dir. vector} = \overrightarrow{AB} = (5, 4, -3) - (-1, 3, 2) = \underbrace{(6, 1, -5)}_{\underline{d} = (d_1, d_2, d_3)} \quad \begin{array}{l}\text{direction}\\ \text{vector}\end{array}$$

$$\boxed{\frac{X - X_o}{d_1} = \frac{y - y_o}{d_2} = \frac{z - z_o}{d_3}}$$

If $(x_o, y_o, z_o) = \underbrace{(-1, 3, 2)}_{A} \Rightarrow \begin{array}{l}\text{eqn.}\\ \text{line}\end{array} \Rightarrow \frac{X+1}{6} = \frac{y-3}{1} = \frac{z-2}{-5}$

We can also choose $(x_o, y_o, z_o) = \underbrace{(5, 4, -3)}_{B} \Rightarrow \begin{array}{l}\text{eqn.}\\ \text{line}\end{array} \Rightarrow \frac{X-5}{6} = \frac{y-4}{1} = \frac{z+3}{-5}$

$\left(\text{Both eqns. are acceptable answers}\right)$

4. Find the **parametric** equation of the line passing through $A(2, -3, 7)$ and parallel to the line
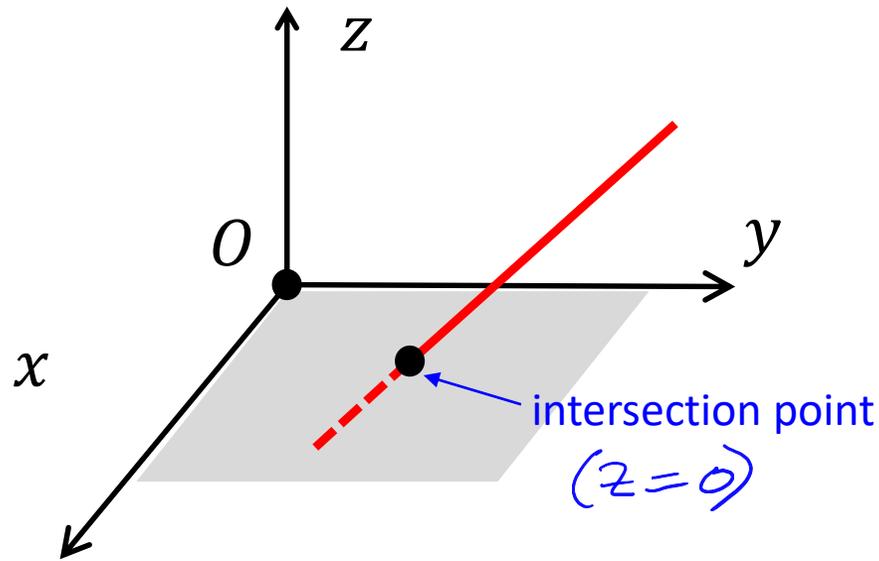
$$\frac{x-2}{2} = \frac{y-5}{5} = \frac{6-z}{7}$$

(x_0, y_0, z_0)

has the same dir. vector:

$(2, 5, -7)$

$$\begin{cases} x = x_0 + \lambda d_1 \\ y = y_0 + \lambda d_2 \\ z = z_0 + \lambda d_3 \end{cases} \Rightarrow \begin{cases} x = 2 + 2\lambda \\ y = -3 + 5\lambda \\ z = 7 - 7\lambda \end{cases}$$

↖ required answer

5. Find the equation of the line joining the points $A(2, 3, 4)$ and $B(1, 4, -3)$ and then determine the coordinates of its intersection point with the xy-coordinate plane.

intersection point
$(z = 0)$

This is "harder" → get parametric eqn.
Solution strategy ⟨
and then
find intersection point
$(z = 0)$

$\overrightarrow{AB} = (1, 4, -3) - (2, 3, 4) = (-1, 1, -7)$

$\begin{cases} x = 2 - \lambda \\ y = 3 + \lambda \\ z = 4 - 7\lambda \end{cases}$ ← parametric eqns. of red line

$4 - 7\lambda = 0 \implies \boxed{\lambda = \dfrac{4}{7}}$

Intersection point: $\left(2 - \dfrac{4}{7}, 3 + \dfrac{4}{7}, 0\right) \implies \boxed{\left(\dfrac{10}{7}, \dfrac{25}{7}, 0\right)}$ ← final answer

A class test type question:

The two lines

$$(\mathcal{L}_1): \quad \frac{x-1}{3} = y-2 = \frac{z+3}{-2},$$

→ $\underline{\text{direction}}$ $\underline{\text{vector}}$ $= \underline{d}_1 = (3, 1, -2)$

$$(\mathcal{L}_2): \quad \frac{x}{5} = -\frac{y+7}{9} = \frac{z-5}{3}$$

→ $\underline{\text{direction}}$ $\underline{\text{vector}}$ $= \underline{d}_2 = (5, -9, 3)$

are

→ $\underline{d}_1$ & $\underline{d}_2$ $\underline{\text{NOT}}$ scalar multiples of each other

(a) parallel;

(b) ~~right-handed~~ → NO SUCH THING!

YES (c) orthogonal;

(d) intersecting; → $\underline{d}_1 \cdot \underline{d}_2 = ?$

(e) none of the above.

$\underline{d}_1 \cdot \underline{d}_2 = (3, 1, -2) \cdot (5, -9, 3)$

$= (3)(5) + (1)(-9) + (-2)(3)$

$= 15 - 9 - 6 = 0 \Rightarrow \underline{d}_1 \perp \underline{d}_2$
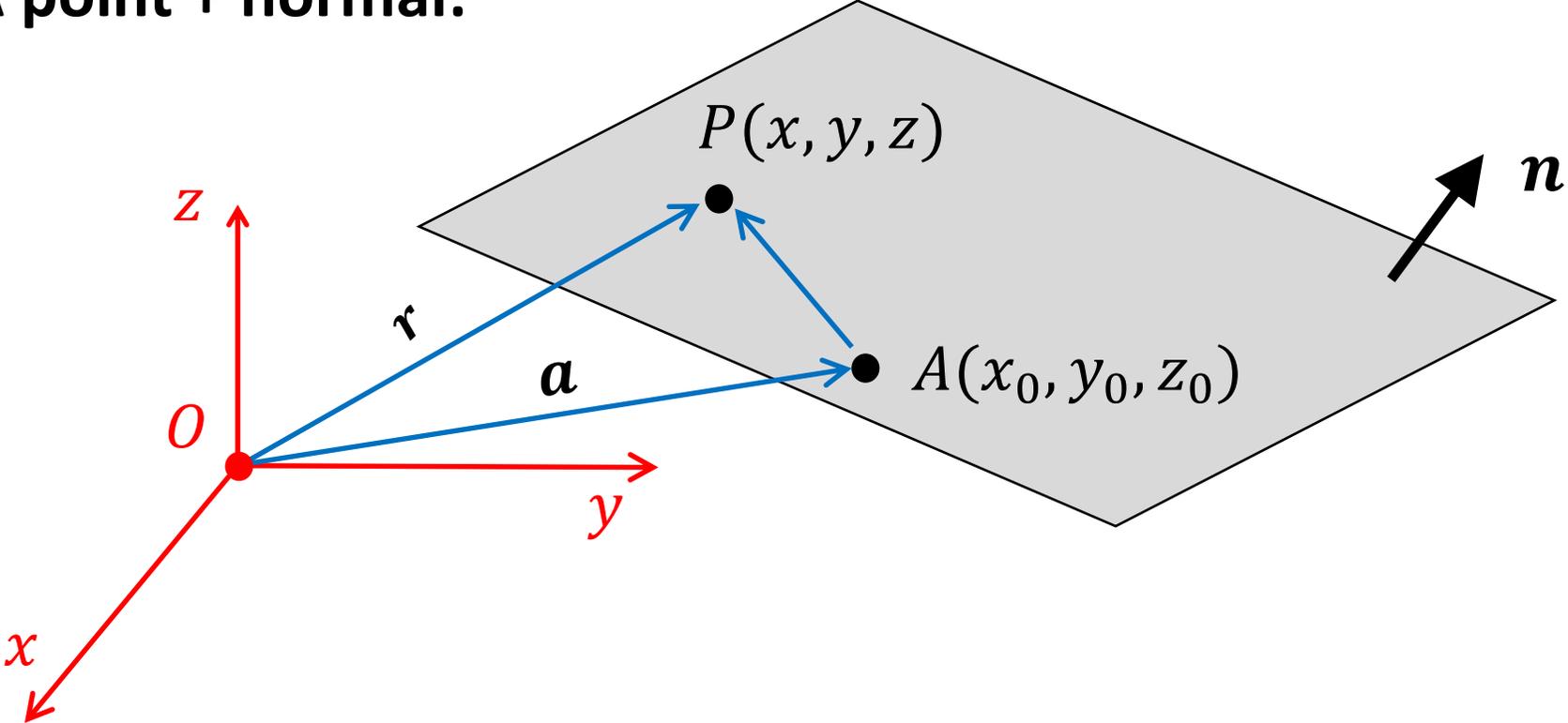
(i.e. lines orthogonal)
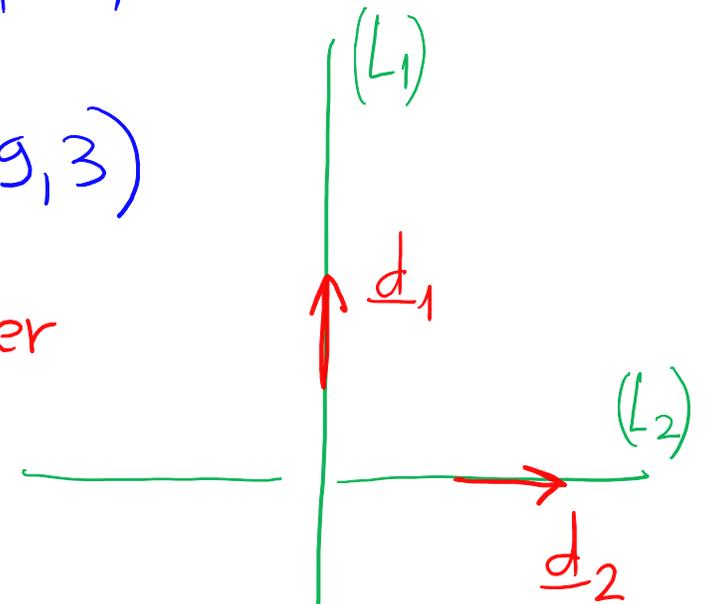
EQUATION OF A PLANE

A **plane** can be defined either by

- Specifying **three points** it contains
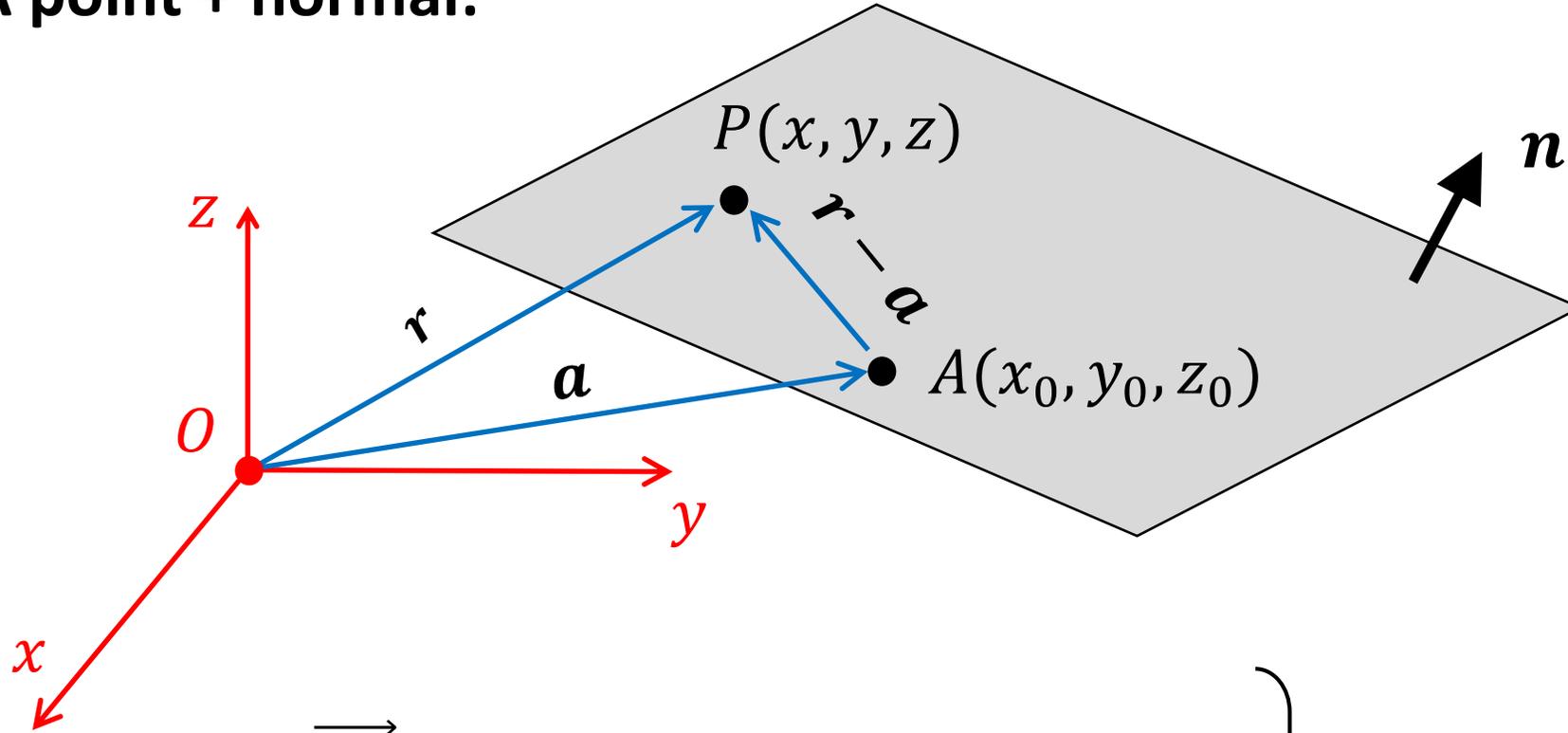
(provided these points do not lie on a straight line)

- Specifying **a point** on the plane AND **two directions** in the plane

- Specifying **a point** on the plane AND the **normal vector** to the plane

A point + normal:

A point + normal:

$$\overrightarrow{AP} = \mathrm{p.\,v.\,(P)} - \mathrm{p.\,v.\,(A)} = \boldsymbol{r} - \boldsymbol{a}$$

$$\boldsymbol{n} \perp \overrightarrow{AP} \implies \boldsymbol{n} \cdot \overrightarrow{AP} = 0$$
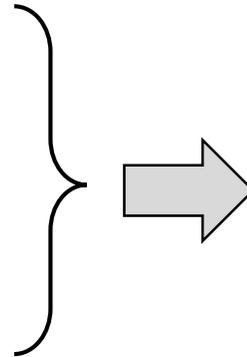
$$\boxed{\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{a}) = 0}$$

VECTOR EQUATION....

A point + normal:
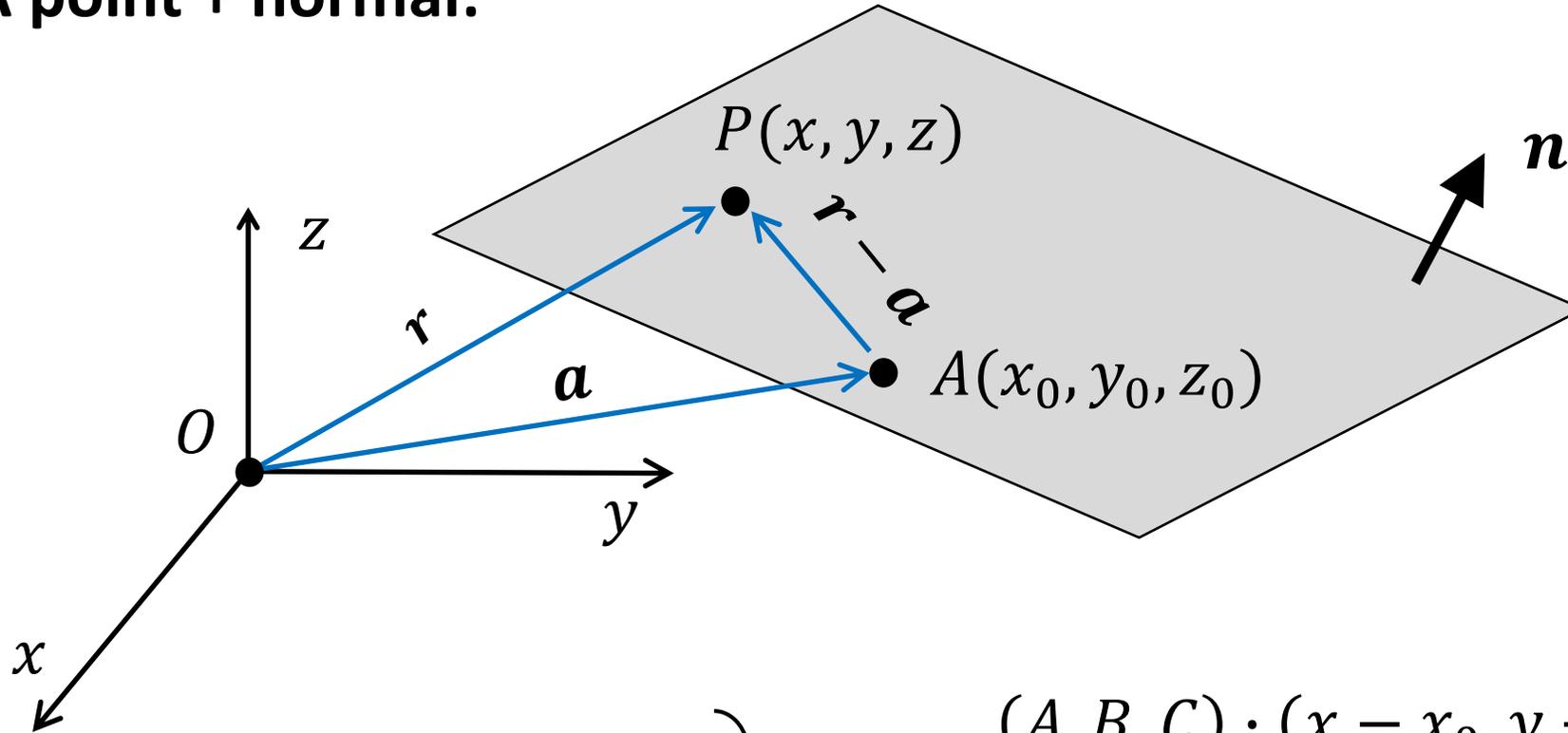
$$\boxed{\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{a}) = 0}$$

VECTOR EQUATION….

$$\left.\begin{array}{l} \boldsymbol{n} = (A, B, C) \\ \\ \boldsymbol{r} = (x, y, z) \\ \\ \boldsymbol{a} = (x_0, y_0, z_0) \end{array}\right\}$$

$$(A, B, C) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

$$\boxed{A(x - x_0) + B(y - y_0) + C(z - z_0) = 0}$$

OR $\quad\boxed{Ax + By + Cz = D}$

CARTESIAN EQUATION….

Three points:

$$\boldsymbol{n} = (\boldsymbol{c} - \boldsymbol{b}) \times (\boldsymbol{c} - \boldsymbol{a})$$
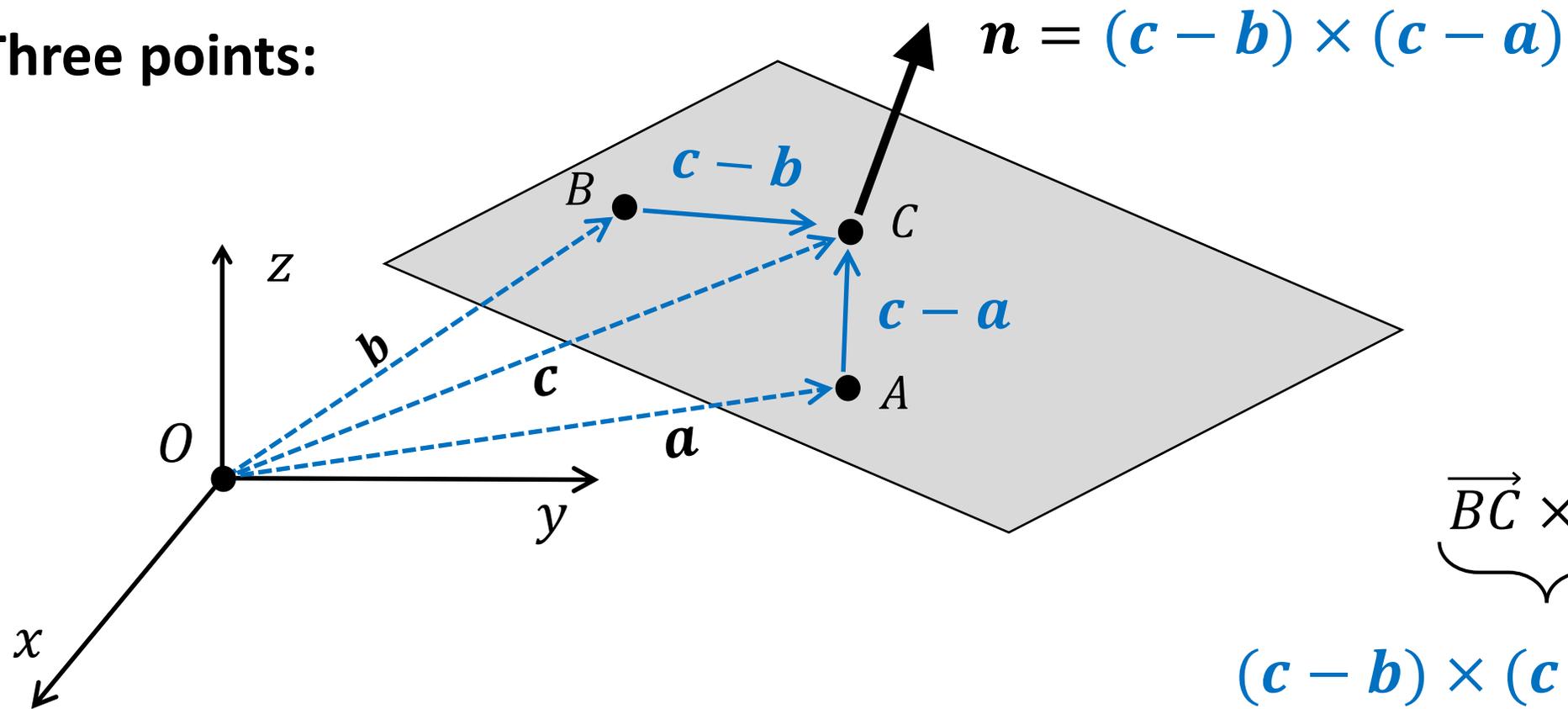
$$\overrightarrow{BC} = \boldsymbol{c} - \boldsymbol{b}$$

$$\overrightarrow{AC} = \boldsymbol{c} - \boldsymbol{a}$$

$$\underbrace{\overrightarrow{BC} \times \overrightarrow{AC}}_{} \perp PLANE$$

$$(\boldsymbol{c} - \boldsymbol{b}) \times (\boldsymbol{c} - \boldsymbol{a})$$

the three points: A, B, C

p.v. of the three points: $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$

$$\boxed{(\boldsymbol{r} - \boldsymbol{a}) \cdot \big((\boldsymbol{c} - \boldsymbol{b}) \times (\boldsymbol{c} - \boldsymbol{a})\big) = 0}$$

EXAMPLES:

1. Let (d) be the line $\dfrac{x-2}{3} = \dfrac{y+1}{4} = \dfrac{z}{1}$

Find the equation of the plane which is perpendicular to (d) and intersects it at (2, -1, 0).

dir. vector → ← given line

(2,-1,0)
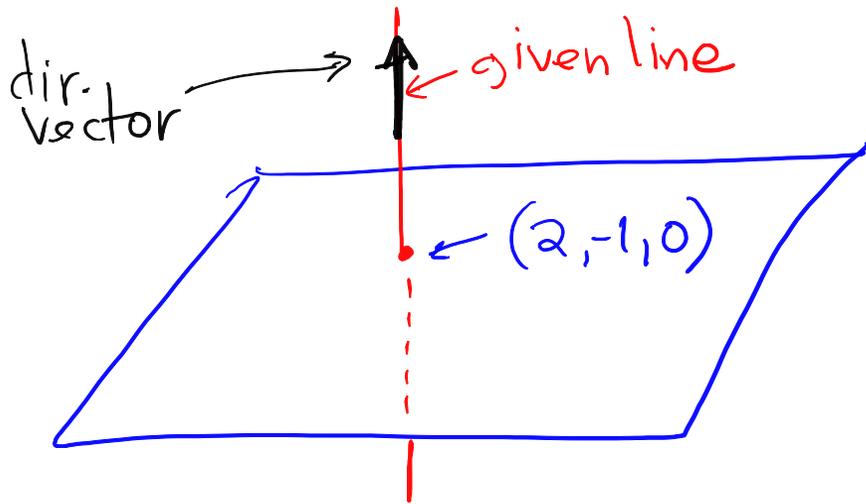
$\begin{pmatrix} \text{direction} \\ \text{vector of} \\ \text{line} \end{pmatrix} = \begin{pmatrix} \text{normal} \\ \text{to the plane} \end{pmatrix} \Rightarrow \underline{n} = (3,4,1)$

Eqn. plane: $Ax + By + Cz = D \Rightarrow 3x + 4y + z = D$

3   4   1

To find D use that $(2,-1,0)$ lies in the plane: $3(2) + 4(-1) + (0) = D \Rightarrow \boxed{D=2}$

Finally, eqn. plane $\Rightarrow$ $\boxed{3x + 4y + z = 2}$