



# **HG1M12**

## **Engineering Mathematics 2**

### **Chapter 4** **(Vector Fields)**

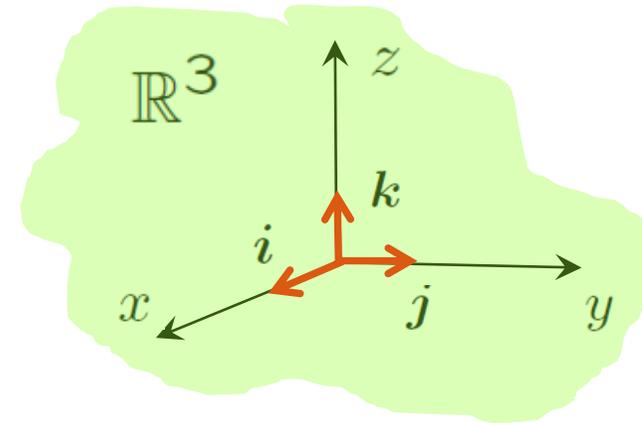
Lecture #20



# CURL of a vector field

Let  $F = F(x, y, z)$  be a vector field:

$$F = F_1i + F_2j + F_3k \equiv (F_1, F_2, F_3)$$



The 'curl' of the above vector field is defined to be another vector field, denoted by  $\text{curl}F$ , and defined as:

$$\text{curl}F = \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) i + \left( \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) j + \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) k$$



# CURL of a vector field

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$



## Worked example

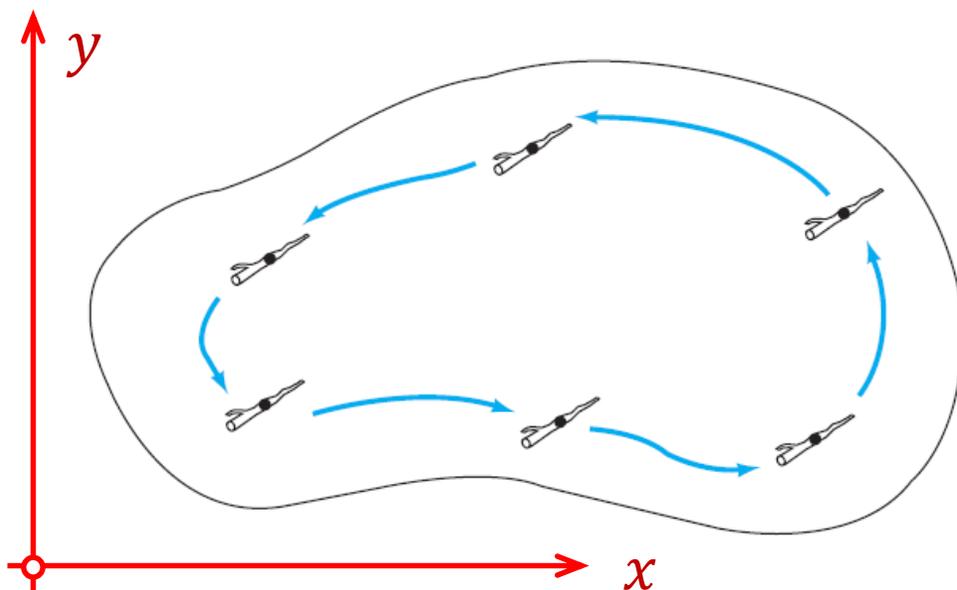
**Example 4.7:** If  $F = (2xy, xy^2, \ln z)$ , find  $\nabla \times F$ .

$$\begin{aligned}\nabla \times F &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & xy^2z & \ln z \end{vmatrix} = \mathbf{i}(-xy^2) + \mathbf{j}(0) + \mathbf{k}(y^2z - 2x) \\ &= -xy^2\mathbf{i} + (y^2z - 2x)\mathbf{k}\end{aligned}$$

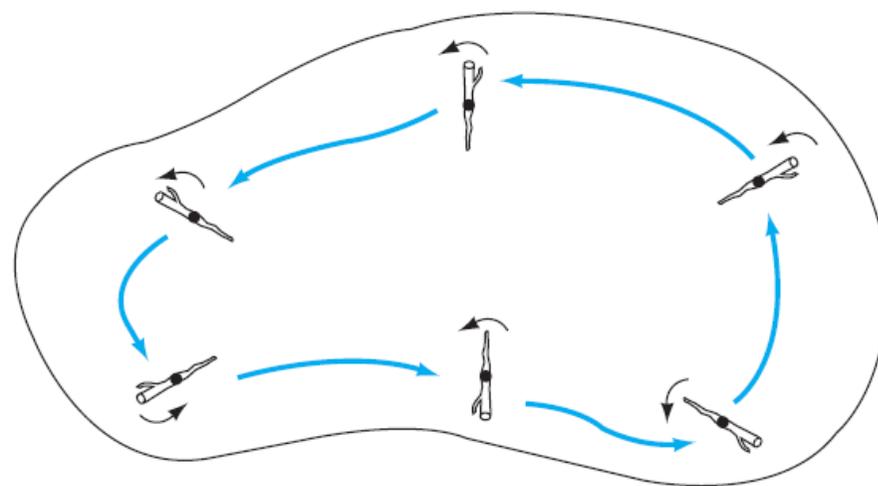


# Physical interpretation of **CURL**

A twig in a pond where water moves with velocity given by a vector field  $\mathbf{V}(x, y)$ :



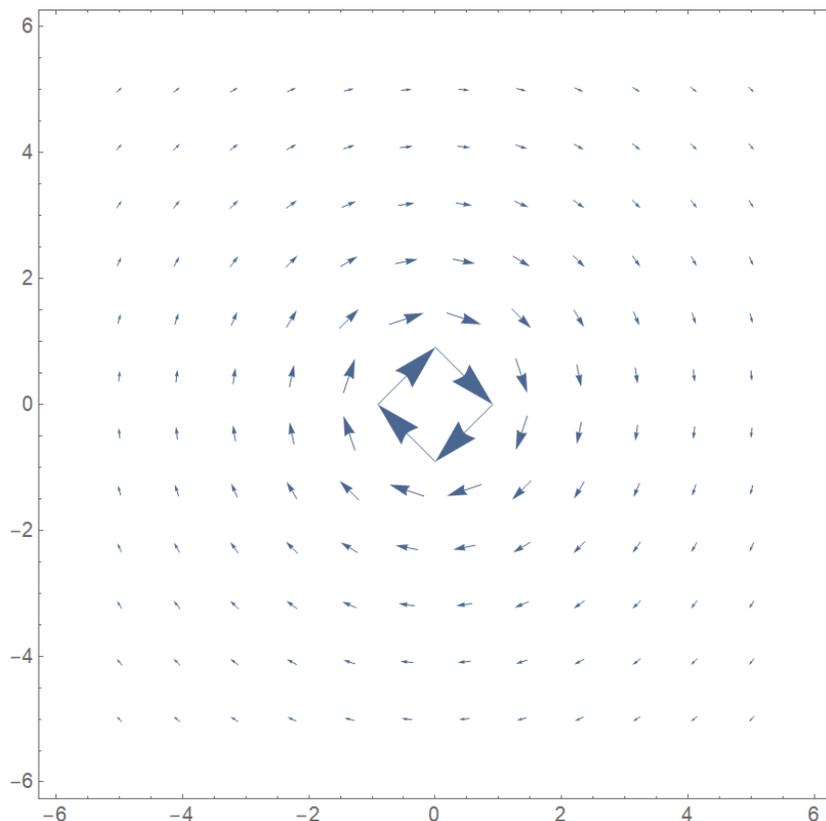
The twig does not rotate as it travels, so  $\nabla \times \mathbf{V} = \mathbf{0}$



In this case the twig **spins around** as it travels around the pond. Hence  $\nabla \times \mathbf{V} \neq \mathbf{0}$

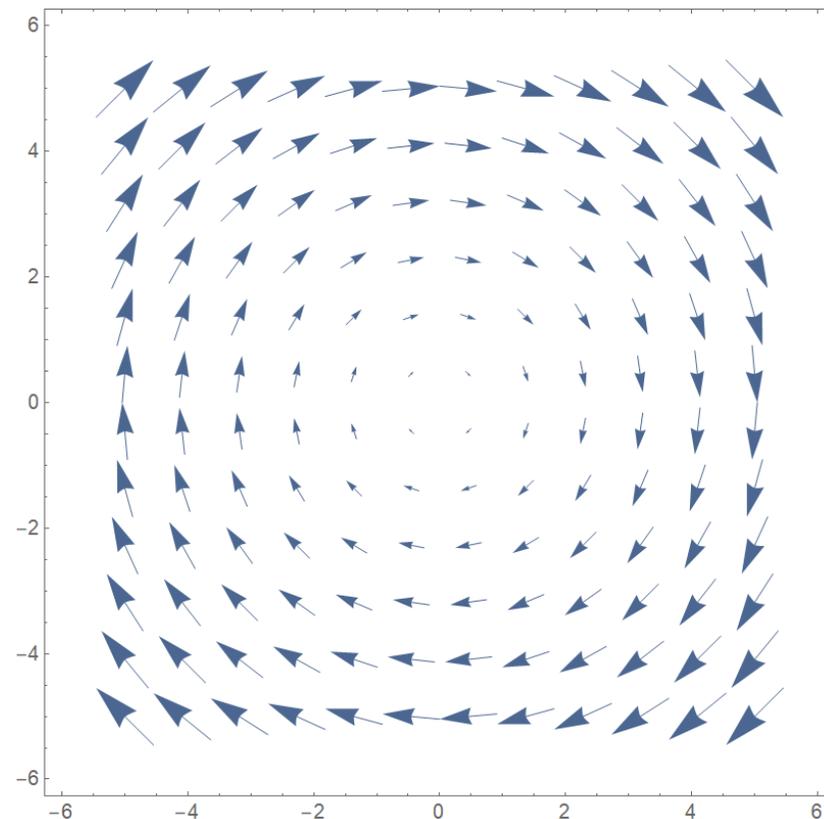


# Interpretation...



$$\mathbf{V}(x, y) = \frac{y}{x^2 + y^2} \mathbf{i} - \frac{x}{x^2 + y^2} \mathbf{j}$$

$$\nabla \times \mathbf{V} = \mathbf{0}$$

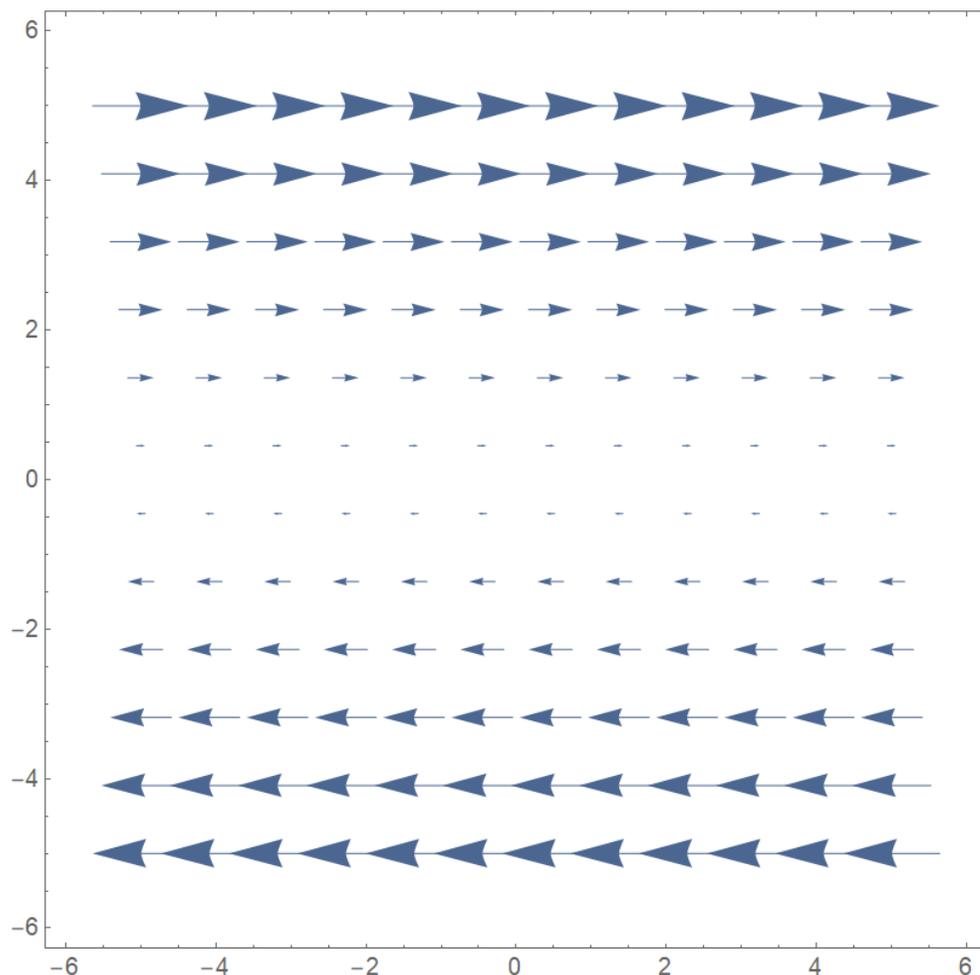


$$\mathbf{V}(x, y) = y \mathbf{i} - x \mathbf{j}$$

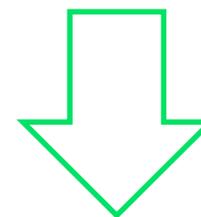
$$\nabla \times \mathbf{V} = -2 \mathbf{k}$$



# Interpretation...



$$\mathbf{V}(x, y) = y\mathbf{i}$$



$$\nabla \times \mathbf{V} = -\mathbf{k}$$

**Summary:** The curl has nothing to do with bulk rotation. It measures the tendency of the vector field to swirl around (it measure local spin at a point).



# CURL and ROTATIONS

Consider a solid rigid body  $\mathcal{B}$  rotating about an axis  $L$ .

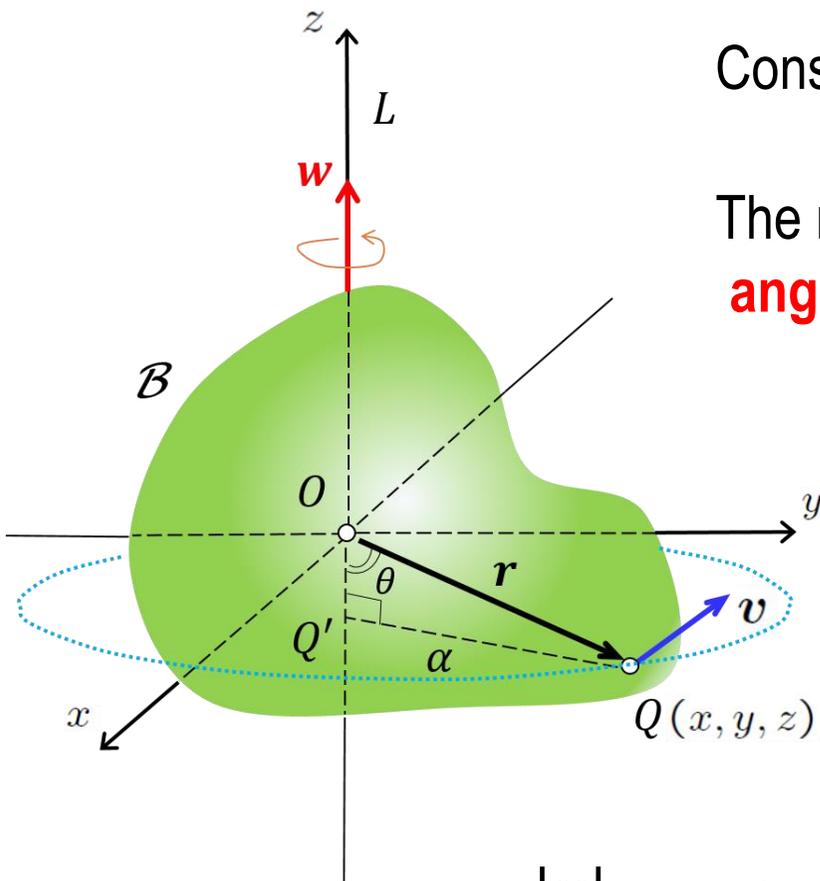
The rotational motion of the body is described by the **angular velocity vector  $w$** :

$$|w| = \omega \quad \text{and} \quad w = \omega k$$

(the length of this vector is taken to be the **angular speed** of the body  $\mathcal{B}$ ; that is, the speed of any point of  $\mathcal{B}$  divided by its distance from  $L$ .)

$$\text{From } \triangle OQQ': \quad \alpha = |r| \sin \theta$$

$$|v| = \omega \alpha = \omega |r| \sin \theta = \underbrace{|\omega| |r| \sin \theta}_{|w \times r|}$$





# CURL and ROTATIONS

Since  $\mathbf{v}$  and  $\mathbf{w} \times \mathbf{r}$  have the same magnitude and the same direction, they must be equal to each other,

$$\mathbf{v} = \mathbf{w} \times \mathbf{r}$$

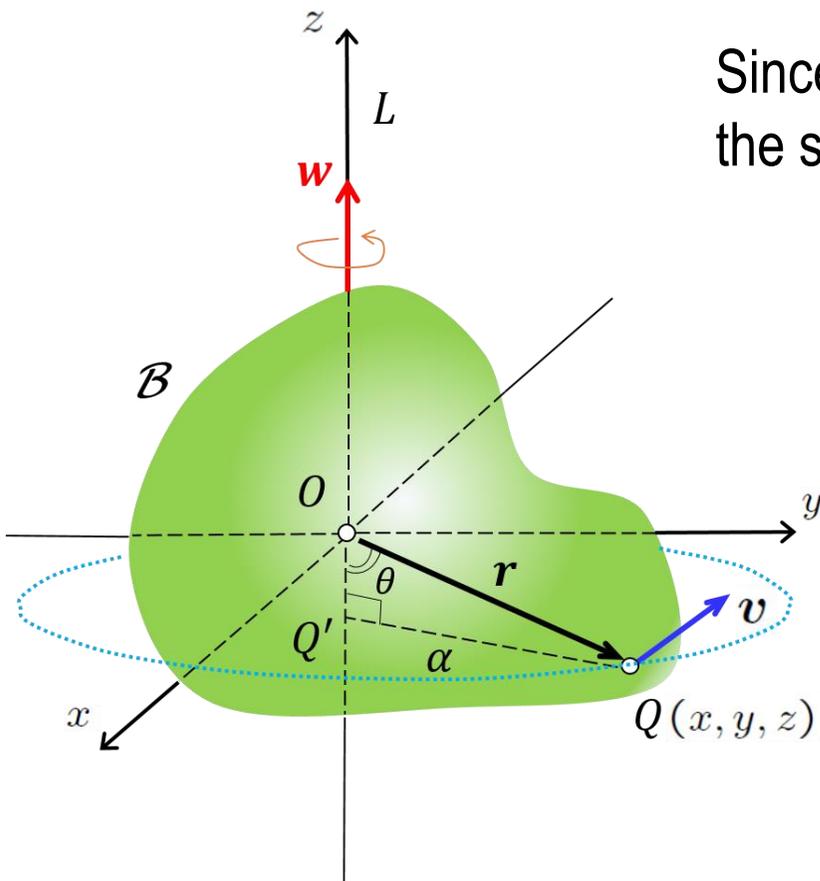
$$\mathbf{r} = (x, y, z) \quad \text{and} \quad \mathbf{w} = (0, 0, \omega)$$



$$\mathbf{v} = -\omega y \mathbf{i} + \omega x \mathbf{j}$$

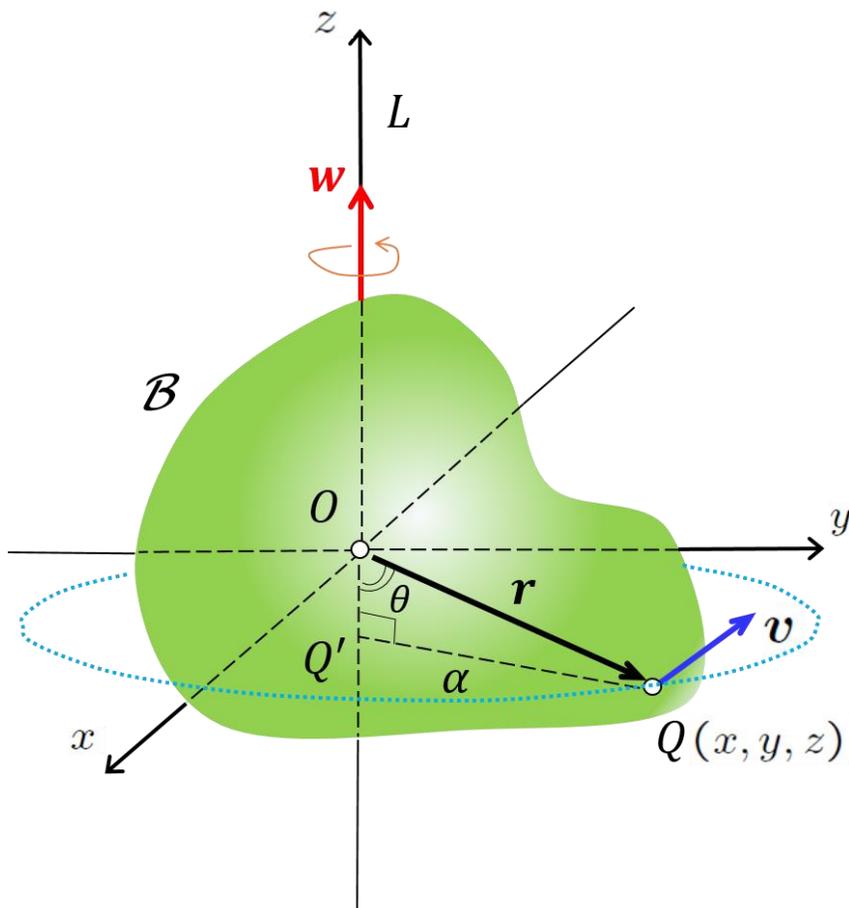


$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = 2\omega \mathbf{k} = 2\mathbf{w}$$





# CURL and ROTATIONS



In conclusion, we have showed that

$$\nabla \times \mathbf{v} = 2\mathbf{w}$$

For the rotation of a rigid body the 'curl' of the (linear) velocity field is a vector directed along the axis of rotation, with magnitude twice the angular speed.



# Combinations of GRAD, DIV & CURL

Grad, div and curl can be combined together in several ways. Suppose  $\phi$  is a scalar field, then

$$(i) \operatorname{div}(\operatorname{grad} \phi) = \nabla \cdot (\nabla \phi) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \nabla^2 \phi$$

To show this, let  $\mathbf{A} = \nabla \phi$  then

$$\mathbf{A} = (A_1, A_2, A_3) = \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right)$$

and

$$\begin{aligned} \nabla \cdot (\nabla \phi) = \nabla \cdot \mathbf{A} &= \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \\ &= \frac{\partial}{\partial x} \left( \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial \phi}{\partial y} \right) + \frac{\partial}{\partial z} \left( \frac{\partial \phi}{\partial z} \right) \\ &= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}. \end{aligned}$$

$\nabla^2 \phi$  is called the Laplacian of  $\phi$ , and sometimes written as  $\Delta$ .



# Combinations of GRAD, DIV & CURL

$$(ii) \text{ curl}(\text{grad } \phi) = \nabla \times (\nabla \phi) = \mathbf{0}$$

as

$$\begin{aligned} \nabla \times \nabla \phi &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix} \\ &= \mathbf{i} \left( \frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y} \right) - \mathbf{j} \left( \frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x} \right) \\ &\quad + \mathbf{k} \left( \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x} \right) \\ &= \mathbf{0} \end{aligned}$$

$$(iii) \text{ div}(\text{curl } \mathbf{A}) = \nabla \cdot (\nabla \times \mathbf{A}) = 0.$$



# Rules for $\nabla$

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$f, g =$  scalar fields

$A, B =$  vector fields

$$\nabla \cdot (fA) = (\nabla f) \cdot A + f(\nabla \cdot A)$$

$$\nabla \times (fA) = (\nabla f) \times A + f(\nabla \times A)$$

$$\nabla \cdot (A \times B) = (\nabla \times A) \cdot B - (\nabla \times B) \cdot A$$

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$$\nabla \times (f\nabla g) = (\nabla f) \times (\nabla g)$$

$$\nabla \times (A \times B) = A(\nabla \cdot B) - B(\nabla \cdot A) + (B \cdot \nabla)A - (A \cdot \nabla)B$$

$$\nabla(\nabla \cdot A) - \nabla \times (\nabla \times A) = \nabla^2 A$$



# Irrotational vector fields

We have seen that if  $\mathbf{A} = \nabla\phi$ , then  $\nabla \times \mathbf{A} = \mathbf{0}$

If  $\mathbf{A}$  is any vector field such that  $\nabla \times \mathbf{A} = \mathbf{0}$ , then it is called **irrotational**.

Clearly, the gradient of a scalar field is always an irrotational vector field.

**The converse is also true**, i.e. if the vector field  $\mathbf{A}$  is irrotational, then it can be expressed in terms of a scalar field  $\phi$  (called a **scalar potential function**) by

$$\mathbf{A} = \nabla\phi$$

This can be used to simplify the solution to a variety of engineering problems, i.e. to find a scalar field  $\phi$  rather than three components of the vector field  $\mathbf{A}$ .



# Solution strategy....

...for finding the scalar potential function.

$$\mathbf{A} = A_1(x, y, z)\mathbf{i} + A_2(x, y, z)\mathbf{j} + A_3(x, y, z)\mathbf{k}$$

← this is given and  
satisfies  $\nabla \times \mathbf{A} = \mathbf{0}$

$$\mathbf{A} = \nabla\phi \Rightarrow \begin{cases} \frac{\partial\phi}{\partial x} = A_1(x, y, z) \\ \frac{\partial\phi}{\partial y} = A_2(x, y, z) \\ \frac{\partial\phi}{\partial z} = A_3(x, y, z) \end{cases}$$

- integrate the first eqn.; the result will involve an arbitrary function depending on  $y$  &  $z$
- Use this expression to integrate the second eqn.; the dependence on  $y$  of the previous arbitrary function is fixed; you are still left with an arbitrary function of  $z$
- This arbitrary function of  $z$  is found by integrating the last equation
- The final result will always depend on an arbitrary constant



## Worked examples (irrotational vector fields)

### Example 4.8:

Take  $\mathbf{A} = (x, y, -z)$

then

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & -z \end{vmatrix} = \mathbf{0}.$$

For  $\mathbf{A} = \nabla\phi$  we need  $\frac{\partial\phi}{\partial x} = x$ ,  $\frac{\partial\phi}{\partial y} = y$ ,  $\frac{\partial\phi}{\partial z} = -z$ .

Let us integrate the derivative with respect to  $x$ , while keeping  $y$  and  $z$  fixed as constants.

$$\phi(x, y, z) = \frac{x^2}{2} + f_1(y, z).$$



## Worked examples (ctd.)

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Now let's look at the derivative with respect to  $y$ .

$$\frac{\partial \phi}{\partial y} = y, \quad \text{ie} \quad \frac{\partial f_1}{\partial y} = y.$$

Integrating this with respect to  $y$ , and keeping  $z$  as a fixed constant, we get

$$f_1(y, z) = \frac{y^2}{2} + f_2(z).$$

Time to look at the derivative with respect to  $z$ .

$$\frac{\partial \phi}{\partial z} = -z, \quad \text{ie} \quad \frac{df_2}{dz} = -z.$$

Integrating this gives

$$f_2(z) = -\frac{z^2}{2} + C, \quad f_1(y, z) = \frac{y^2}{2} - \frac{z^2}{2} + C,$$

and

$$\phi(x, y, z) = \frac{x^2}{2} + \frac{y^2}{2} - \frac{z^2}{2} + C.$$



## Worked examples (irrotational vector fields)

### Example 4.9:

$$\mathbf{A} = (y^2 z^3, 2xyz^3, 3xy^2 z^2)$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z^3 & 2xyz^3 & 3xy^2 z^2 \end{vmatrix} = \mathbf{0}.$$

For  $\mathbf{A} = \nabla \phi$  we need  $\frac{\partial \phi}{\partial x} = y^2 z^3$ ,  $\frac{\partial \phi}{\partial y} = 2xyz^3$ ,  $\frac{\partial \phi}{\partial z} = 3xy^2 z^2$ .

Again, let's first consider the derivative with respect to  $x$ . Integrate this with respect to  $x$ , and keep  $y$  and  $z$  as fixed constants.

$$\phi(x, y, z) = xy^2 z^3 + f_1(y, z).$$



## Worked examples (ctd.)

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Now consider the partial derivative with respect to  $y$ .

$$\frac{\partial \phi}{\partial y} = 2xyz^3 + \frac{\partial f_1}{\partial y} = 2xyz^3,$$

which means

$$\frac{\partial f_1}{\partial y} = 0, \quad \text{ie } f_1(y, z) = f_2(z),$$

and

$$\phi(x, y, z) = xy^2z^3 + f_2(z).$$

Finally, consider the derivative with respect to  $z$ .

$$\frac{\partial \phi}{\partial z} = 3xy^2z^2 + \frac{df_2}{dz} = 3xy^2z^2,$$

so

$$f_2(z) = C, \quad \text{ie } \phi(x, y, z) = xy^2z^3 + C.$$



## Worked examples (irrotational vector fields)

$$\mathbf{A} = (ye^x, e^x - z \sin y, \cos y)$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^x & e^x - z \sin y & \cos y \end{vmatrix} = \mathbf{0}.$$

For  $\mathbf{A} = \nabla\phi$  we need  $\frac{\partial\phi}{\partial x} = ye^x$ ,  $\frac{\partial\phi}{\partial y} = e^x - z \sin y$ ,  $\frac{\partial\phi}{\partial z} = \cos y$ .

As usual, integrate the derivative with respect to  $x$  first, keeping  $y$  and  $z$  fixed.

$$\phi(x, y, z) = ye^x + f_1(y, z).$$



## Worked examples (ctd.)

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Now consider the derivative with respect to  $y$ .

$$\frac{\partial \phi}{\partial y} = e^x + \frac{\partial f_1}{\partial y} = e^x - z \sin y,$$

so

$$\frac{\partial f_1}{\partial y} = -z \sin y,$$

which we can integrate (keeping  $z$  fixed),

$$f_1(y, z) = z \cos y + f_2(z), \quad \text{ie} \quad \phi(x, y, z) = ye^x + z \cos y + f_2(z).$$

Finally, we can consider the derivative with respect to  $z$ .

$$\frac{\partial \phi}{\partial z} = \cos y + \frac{df_2}{dz} = \cos y,$$

which means

$$\frac{df_2}{dz} = 0, \quad \text{ie} \quad f_2(z) = C.$$



## Worked examples (ctd.)

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Thus the final result is

$$\phi(x, y, z) = ye^x + z \cos y + C.$$



# Incompressible/Solenoidal vector fields

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If  $\mathbf{B}$  is any vector field such that  $\nabla \cdot \mathbf{B} = 0$ , then it is called **incompressible** (or **solenoidal**).

We have seen before that the curl of a vector field is incompressible. The converse statement is also true: if  $\mathbf{B}$  is a vector field that satisfies

$$\nabla \cdot \mathbf{B} = 0,$$

then there exists a vector field  $\mathbf{A}$  such that

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

The vector field  $\mathbf{A}$  is called the **vector potential** of  $\mathbf{B}$ .

Finding a vector potential is much harder than finding a scalar potential, and is beyond the scope of this module.