HG1M12
Engineering Mathematics 2

Chapter 4
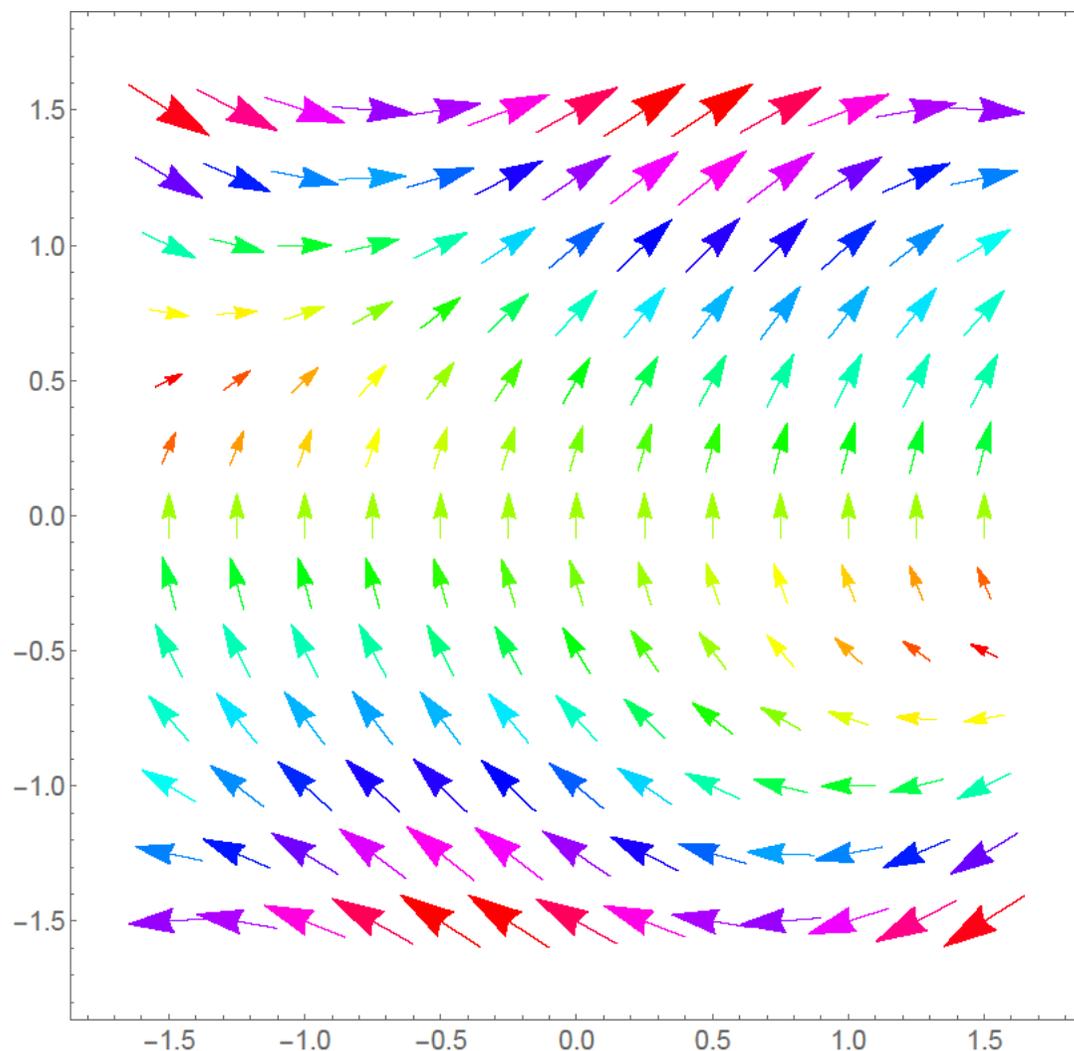## (Vector Fields)

Lecture #19

2D vector field

$$\boldsymbol{F}(x, y) = y\boldsymbol{i} + \sin(1 + xy)\,\boldsymbol{j}$$

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Vector fields (see Chapter 3)

On the previous slide we have a typical representation of a 2D vector field, in this case

$$F(x, y) = y\mathbf{i} + \sin(1 + xy)\,\mathbf{j}$$

The **length** of the arrows is scaled with the magnitude of this vector, $|F(x, y)|$: long arrows correspond to a large magnitude, while short arrows are associated with small magnitude (based on this, we can say that in some regions of the xy-plane the vector field is *strong* or *weak*, depending on the size of the corresponding arrows).

The direction of the arrows is given by the slope of the vector field:

$$\text{Slope at } (x, y) = \frac{\sin(1 + xy)}{y}$$

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Vector fields

$\mathbf{x}_0$
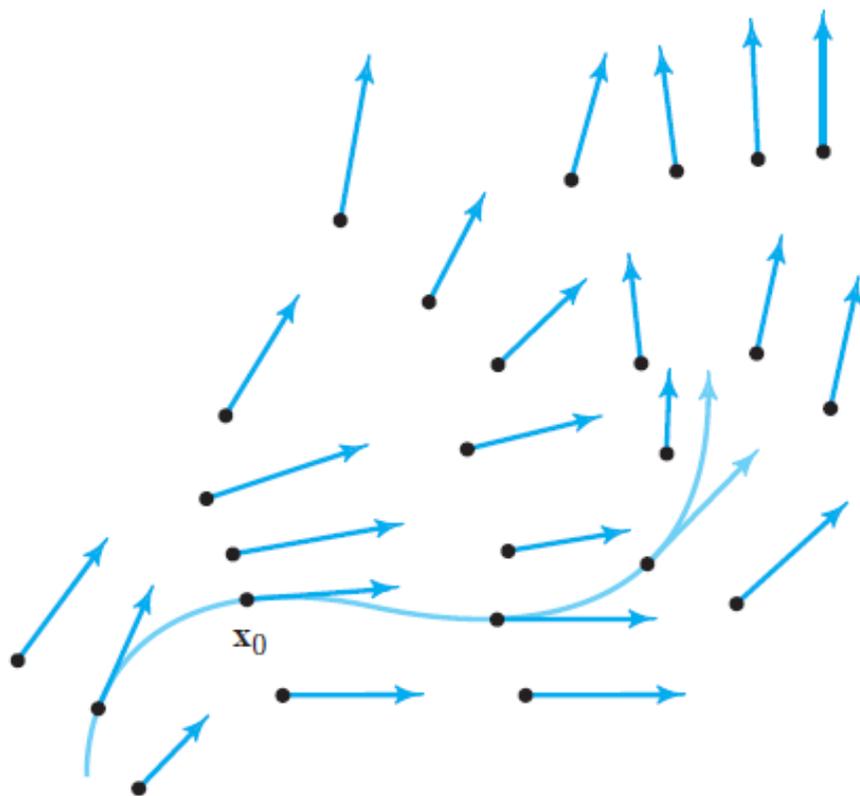
A **field line** of a given vector field is defined as a curve which, at every point through which it passes, has the *same direction as the vector field.*

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Vector fields

For a 2D vector field $\boldsymbol{F}(x,y) = \big(f_1(x,y), f_2(x,y)\big)$ the **field lines** are defined by the differential equation

$$\frac{dy}{dx} = \frac{f_2(x,y)}{f_1(x,y)}$$

OBS. The solutions of the ODE $\quad \dfrac{dy}{dx} = f(x,y)$

are the field lines of the vector field $\boldsymbol{F}(x,y) = \big(1, f(x,y)\big)$

see **CHAPTER 3**

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Vector fields

We could define a field line for a 3D vector field as well, but in this module we will not do that.

It is useful to think of these 2D vector fields as velocity fields for the **steady** flow of some liquid in a 2D pipe ("steady" means that at each point inside the pipe the velocity of the fluid passing through that point does **not** change with time).

In this context, a field line is the path followed by a small particle suspended in the fluid. In fluid mechanics the field lines of the velocity vector for a steady flow are known as **streamlines** or **flow lines**.

Velocity vector   Flow line
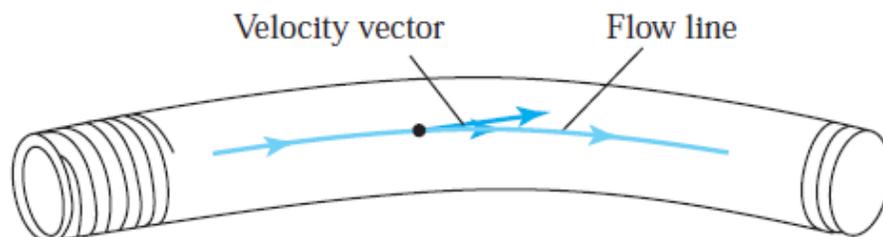
E.G.: The velocity vector of a fluid is tangent to a field line passing through that point

DIVERGENCE of a vector field

Given a vector field $A(x) = A(x, y, z)$ with components

$$A(x) = A_1(x)i + A_2(x)j + A_3(x)k,$$

the divergence of A is a scalar field, defined as

$$\text{div } A = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

This can be written in terms of the symbol ∇,

$$\text{div } A = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (A_1, A_2, A_3) = \nabla \cdot A$$

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Example 4.5: If $\mathbf{A} = (2xy, xy^2z, \ln z)$, with $z > 0$, find $\nabla \cdot \mathbf{A}$.

$$\frac{\partial A_1}{\partial x} = 2y, \quad \frac{\partial A_2}{\partial y} = 2xyz, \quad \frac{\partial A_3}{\partial z} = \frac{1}{z}$$

$$\implies \quad \nabla \cdot \mathbf{A} = 2y + 2xyz + \frac{1}{z}$$

Example 4.6: If $\mathbf{F} = x^2 y \mathbf{i} + xz \mathbf{j} + xyz \mathbf{k}$, then find $\nabla \cdot \mathbf{F}$

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xyz) = 2xy + 0 + xy = 3xy$$

Heuristics

The value of the divergence of a vector field at a particular point gives a measure of the "net mass flow" or "flux density" of the vector field IN or OUT of that point.

Imagine that the vector field F represents velocity of a fluid.

If $\nabla \cdot F$ is zero at a point, then the rate at which the fluid is flowing INTO that point is equal to the rate at which the fluid is flowing OUT.
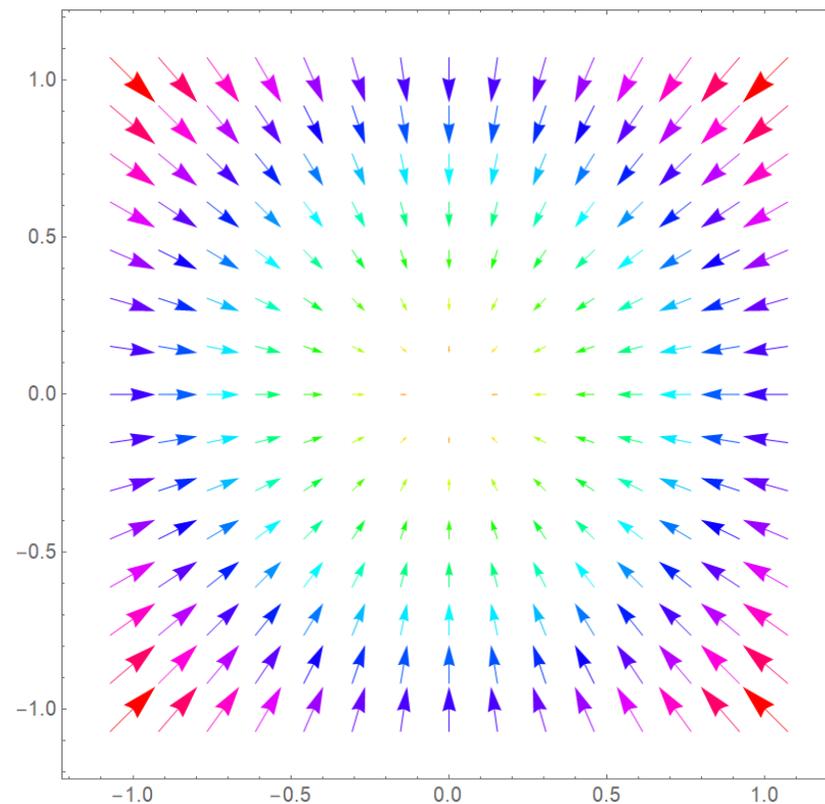
Positive divergence at a point signifies more fluid is flowing out than in, while **negative divergence** signifies just the opposite.

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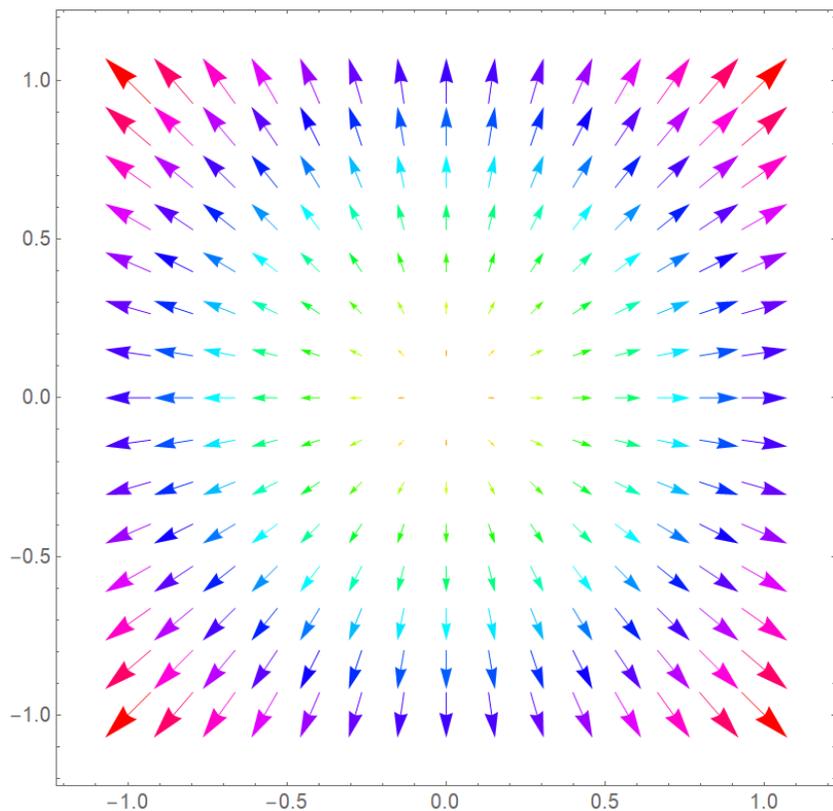
Heuristics (cont'd)

$$\boldsymbol{F}(x, y) = x\boldsymbol{i} + y\boldsymbol{j}$$

$$\boldsymbol{F}(x, y) = -x\boldsymbol{i} - y\boldsymbol{j}$$

There is greater flow AWAY from each point
than into it; that is, **F** is "diverging" at every point

Heuristics (cont'd)

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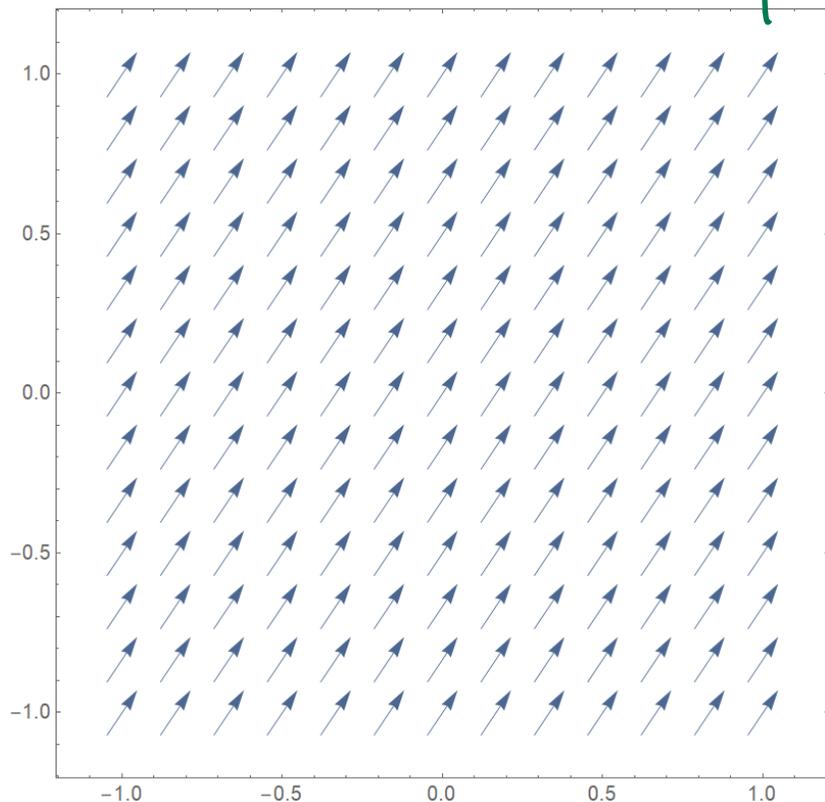
$$\boldsymbol{F}(x,y) = \boldsymbol{i} + 2\boldsymbol{j}$$

$$\boldsymbol{F}(x,y) = y\boldsymbol{i} - x\boldsymbol{j}$$

$$\nabla \cdot \boldsymbol{F}(x,y) = 0$$

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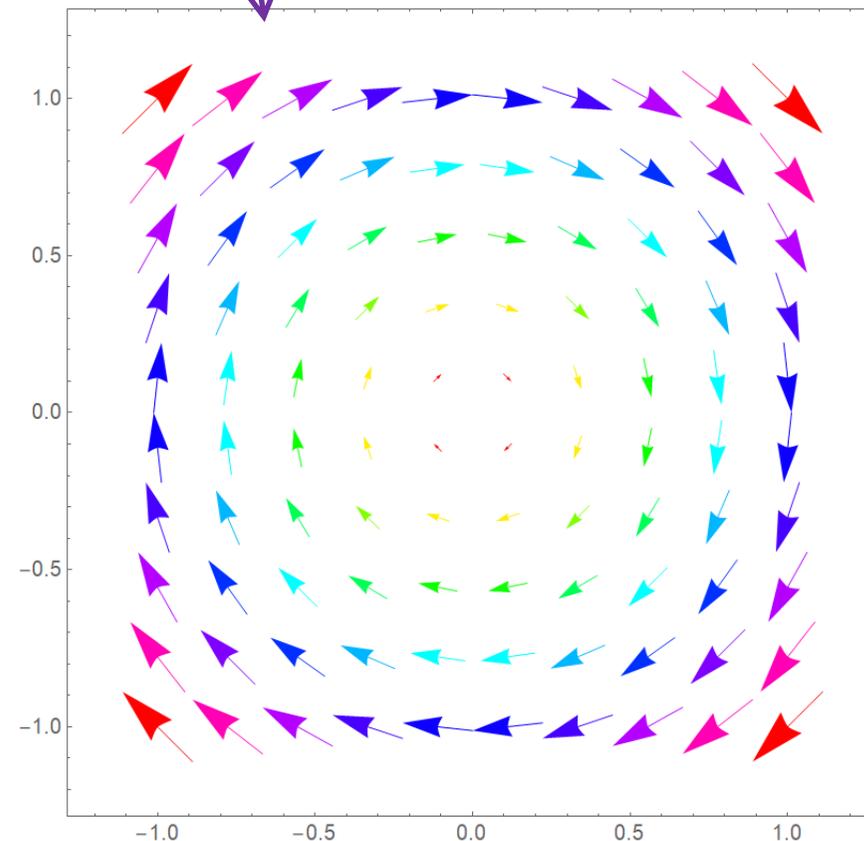
Physical interpretation (2D case)

In 2D the divergence of a velocity field measures **local changes in area** (while in 3D it describes local change of volumes). We are interested in the former.

We consider an imaginary small fluid parcel in the shape of a rectangle with sides Δx and Δy, as indicated below:

$(x, y + \Delta y)$ $(x + \Delta x, y + \Delta y)$

(x, y) $(x + \Delta x, y)$
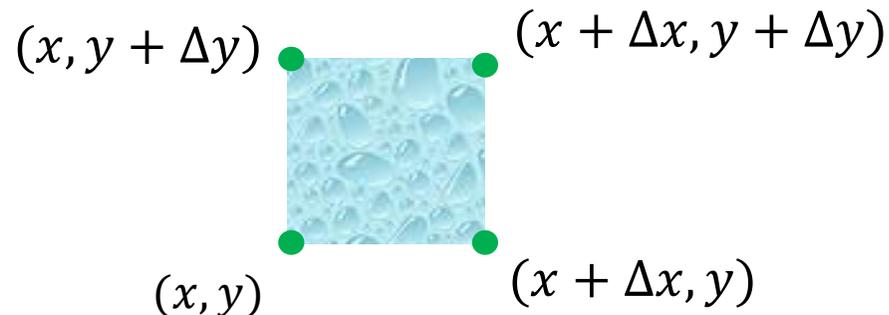
Note that **different parts** of this fluid parcel will move with **different speeds** because the components of the velocity field depend on position (i.e., x and y).

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Let's consider steady 2D fluid flow, with a given velocity field

$$\boldsymbol{v}(x, y) = u(x, y)\boldsymbol{i} + v(x, y)\boldsymbol{j}$$

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Nottingham
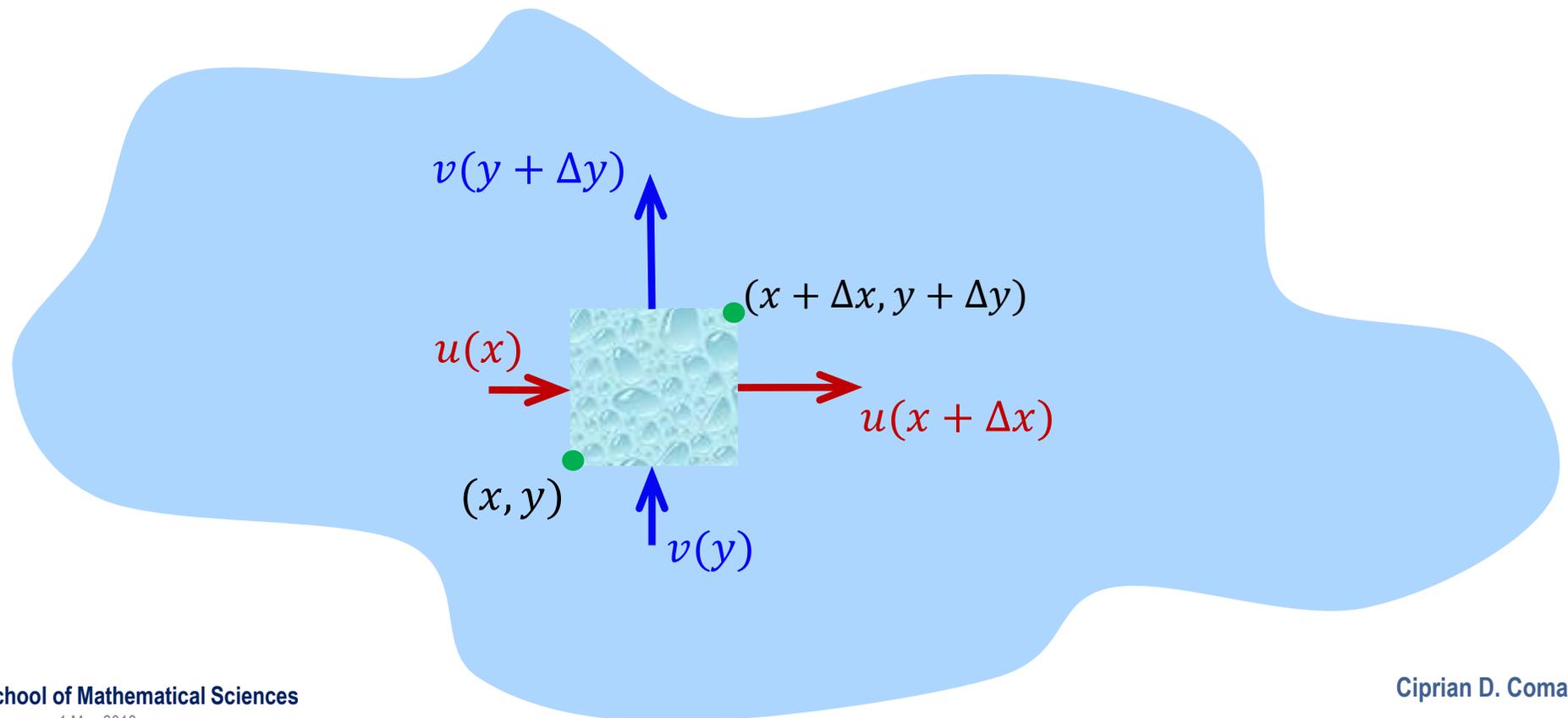
Fluid parcel area: $A = \Delta x \Delta y$

$$\frac{dA}{dt} = \frac{d}{dt}(\Delta x \Delta y) = (\Delta x)\frac{d}{dt}(\Delta y) + (\Delta y)\frac{d}{dt}(\Delta x)$$

$$\underbrace{\phantom{(\Delta x)\frac{d}{dt}(\Delta y)}}_{v(y + \Delta y) - v(y)} \qquad \underbrace{\phantom{(\Delta y)\frac{d}{dt}(\Delta x)}}_{u(x + \Delta x) - u(x)}$$

Divide this equation by the initial area:

$$\frac{1}{A}\frac{dA}{dt} = \frac{v(y + \Delta y) - v(y)}{\Delta y} + \frac{u(x + \Delta x) - u(x)}{\Delta x}$$

fractional change in area

Take the limit as $\Delta x, \Delta y \to 0$:

$$\lim_{A \to 0} \frac{1}{A}\frac{dA}{dt} = \frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} = \nabla \cdot \boldsymbol{v}$$

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NUMERICAL EXAMPLES:

The next few slides show the changes undergone by a small circular area of fluid under different types of flows (everything is in 2D – think of the surface of a lake or a calm river).

For some of these flows the divergence is zero, in which case it turns out that the deformed 2D fluid region preserves its area, although the shape might be drastically altered. It is also possible that both the area and the shape of the initial 2D fluid region remain unchanged (e.g., in the case of a bulk rotation type flow).

Other flows have a positive divergence, and thus the initial area keeps increasing.

In each set of plots there are 4 snapshots of the vector field and the changes undergone by the small circular area of fluid: these are taken at 4 different instants of time (t=0.1 is the initial time, the subsequent values of 't' are in increasing order).

$$\boldsymbol{v}(x, y) = x\boldsymbol{i}$$
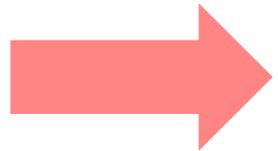
$$\nabla \cdot \boldsymbol{v}(x, y) = 1 > 0$$

area

$A_1 = 3.17$

$t_1 = 0.1$

time
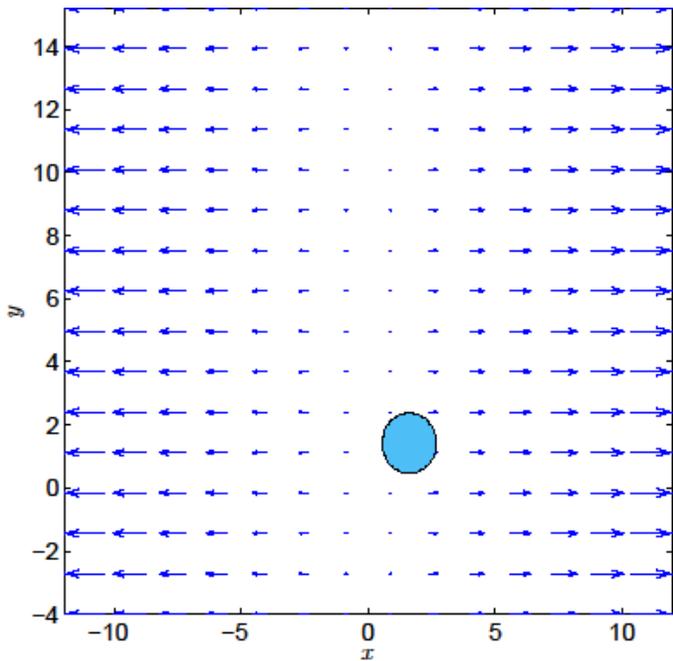
$A_2 = 4.73$

$t_2 = 0.5$
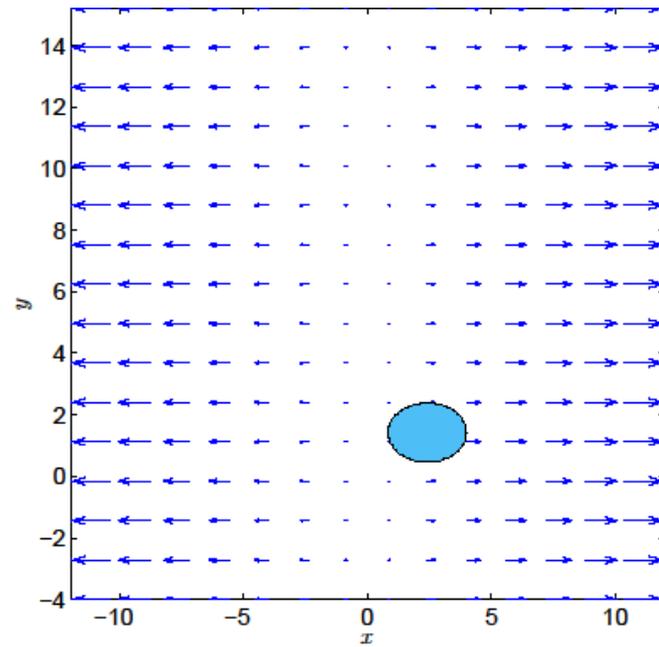
$A_3 = 7.1$

$t_3 = 0.9$
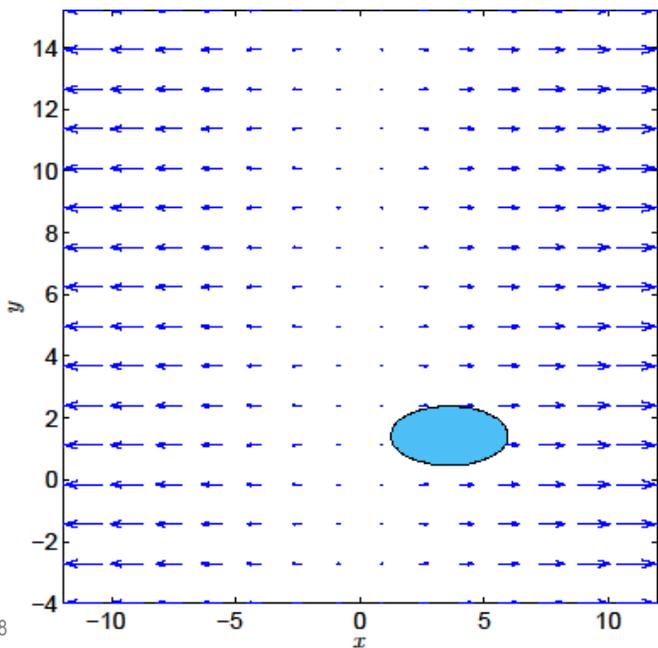
$A_4 = 10.54$

$t_4 = 1.3$
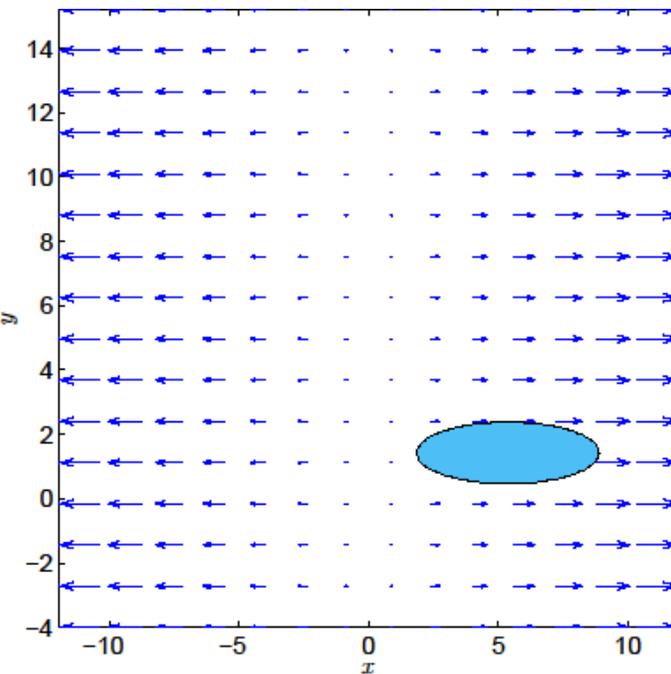
$$\boldsymbol{v}(x, y) = \frac{x}{\sqrt{x^2 + y^2}} \boldsymbol{i} + \frac{y}{\sqrt{x^2 + y^2}} \boldsymbol{j}$$
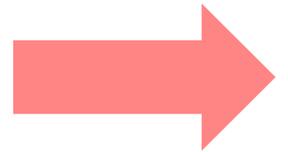
$$\nabla \cdot \boldsymbol{v}(x, y) \equiv \frac{1}{\sqrt{x^2 + y^2}} > 0$$
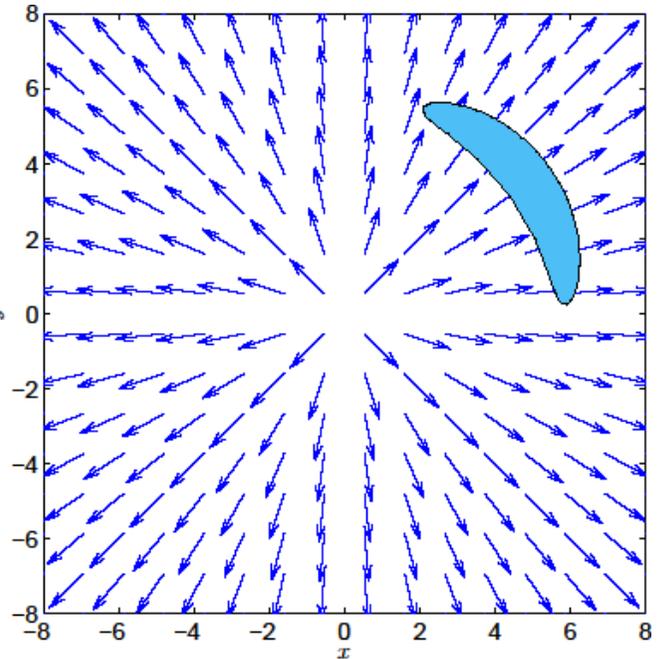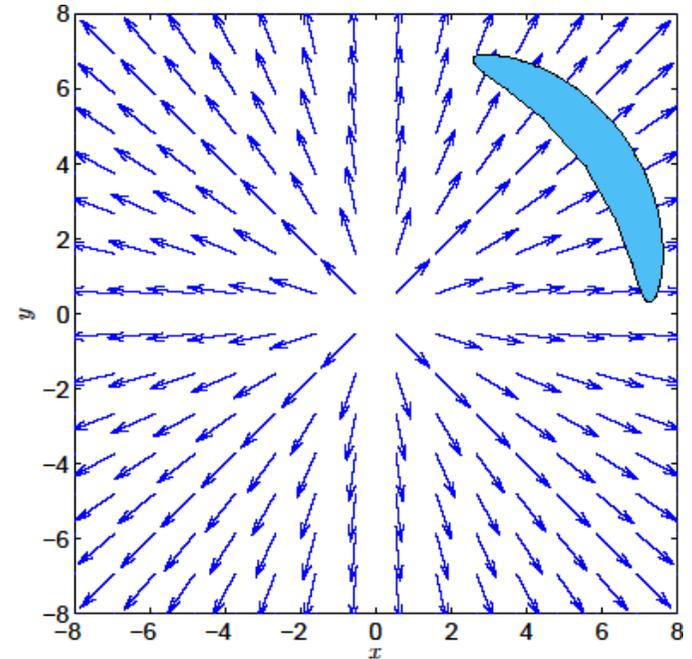
$A_1 = 1.39$

$t_1 = 0.1$
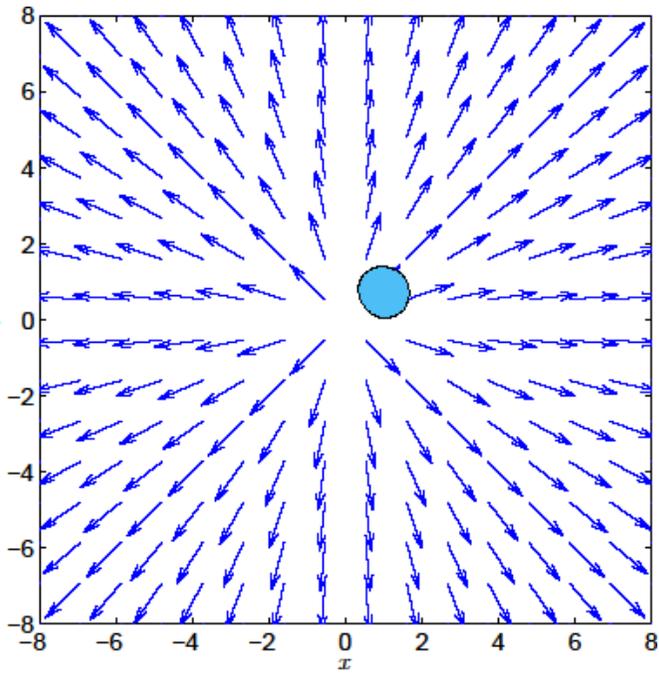
$A_1 = 4.17$

$t_1 = 2.5$
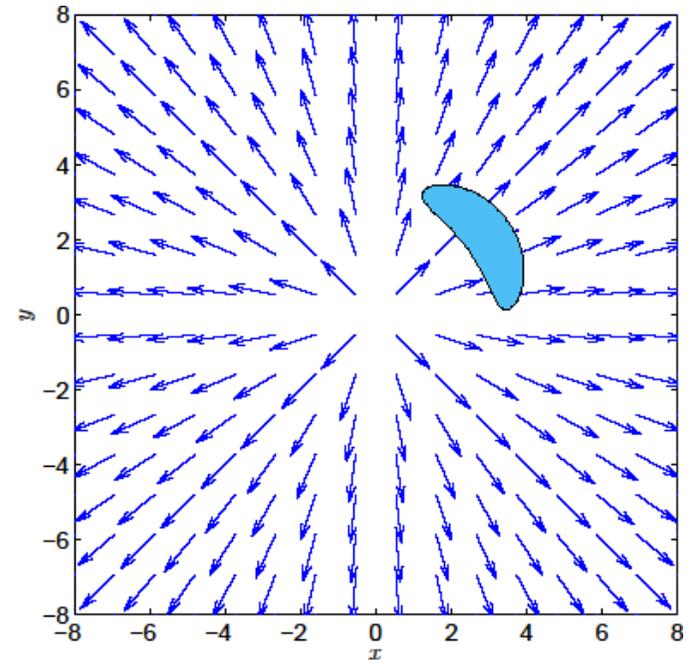
$A_1 = 6.98$

$t_1 = 4.9$

$A_1 = 8.64$

$t_1 = 6.3$

$$\boldsymbol{v}(x, y) = \frac{-y}{\sqrt{x^2 + y^2}}\boldsymbol{i} + \frac{x}{\sqrt{x^2 + y^2}}\boldsymbol{j}$$
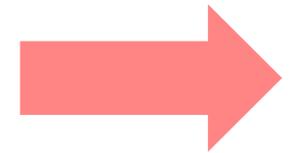
$$\nabla \cdot \boldsymbol{v}\,(x, y) \equiv 0$$
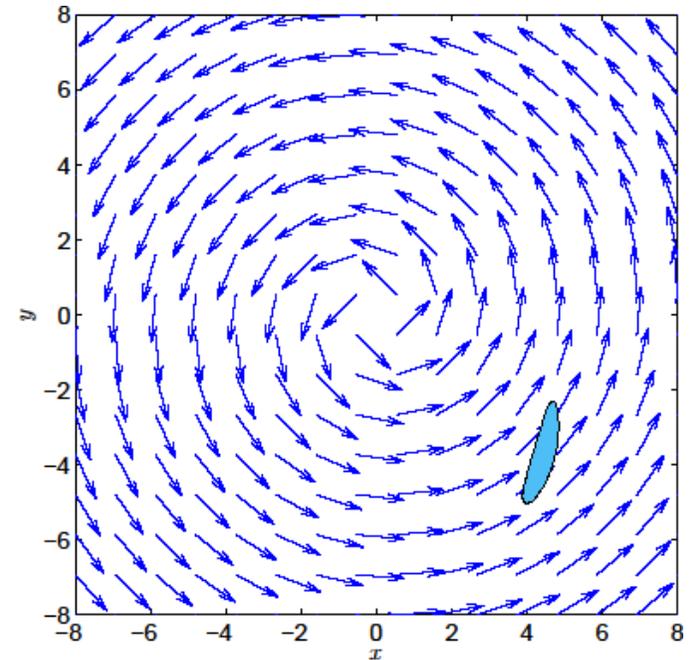
$A_1 = 1.27$

$t_1 = 0.1$
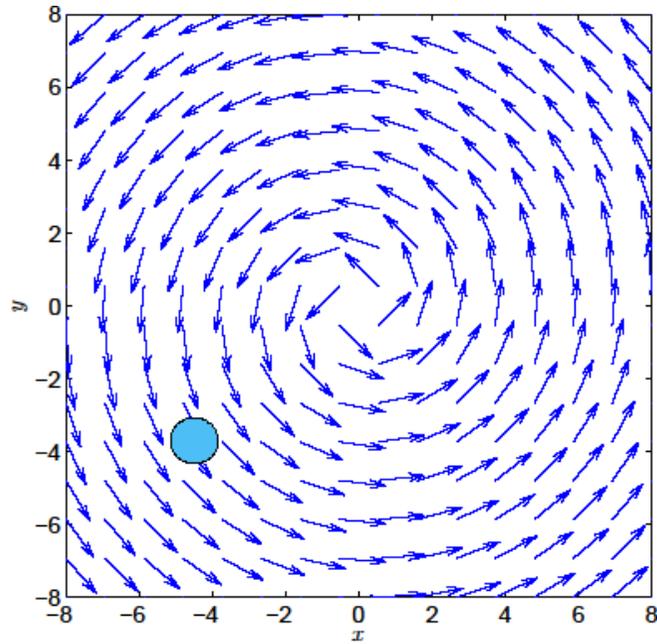
$A_1 = 1.27$

$t_1 = 2.5$
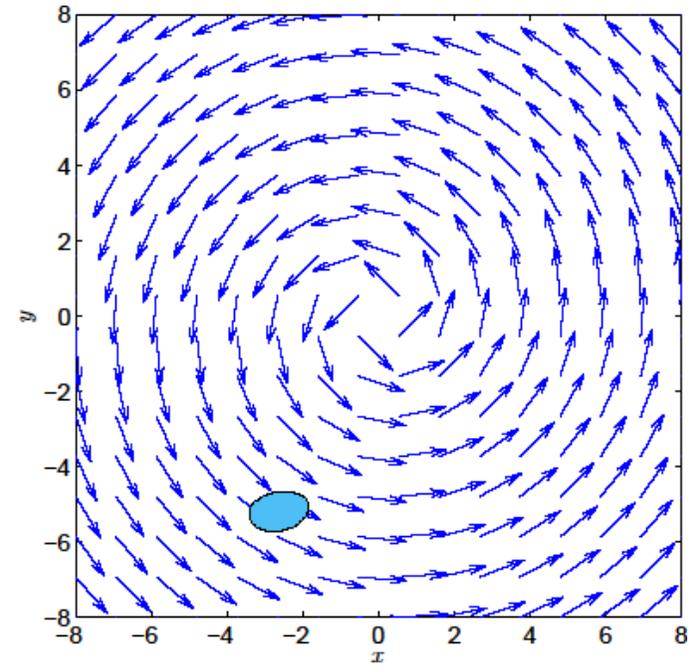
$A_1 = 1.27$

$t_1 = 5.9$
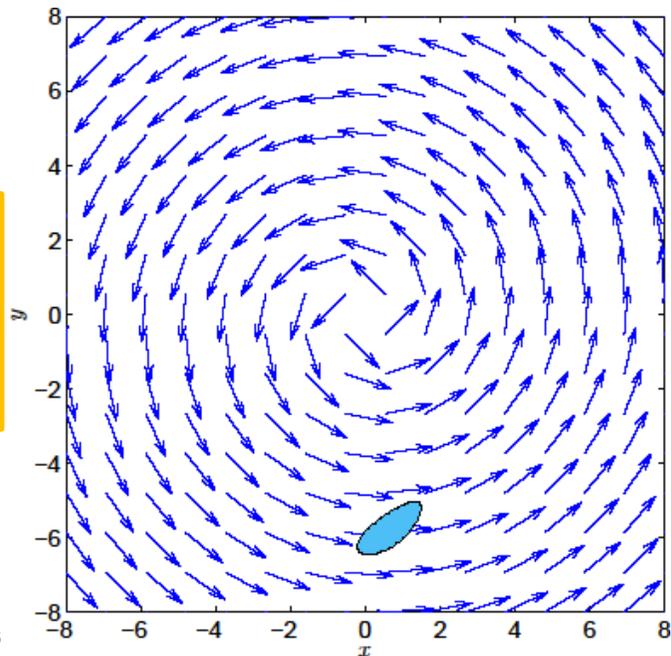
$A_1 = 1.27$

$t_1 = 10.3$

$$\boldsymbol{v}(x, y) = -y\boldsymbol{i} + x\boldsymbol{j}$$
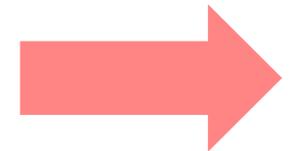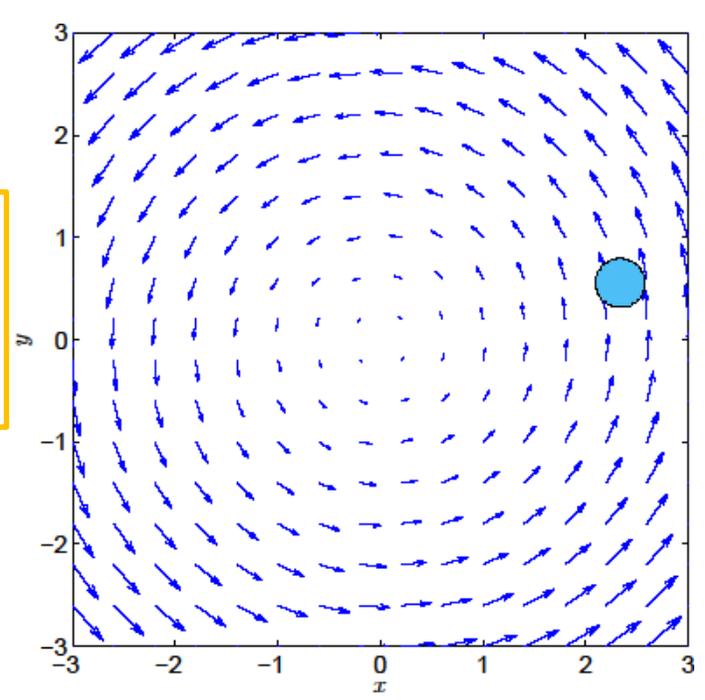
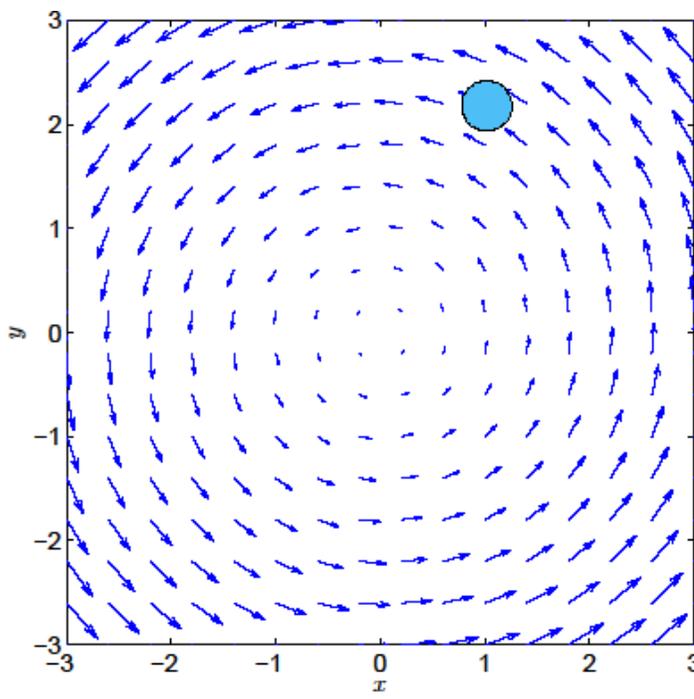$$\nabla \cdot \boldsymbol{v}(x, y) \equiv 0$$

$A_1 = 0.18$

$t_1 = 0.1$
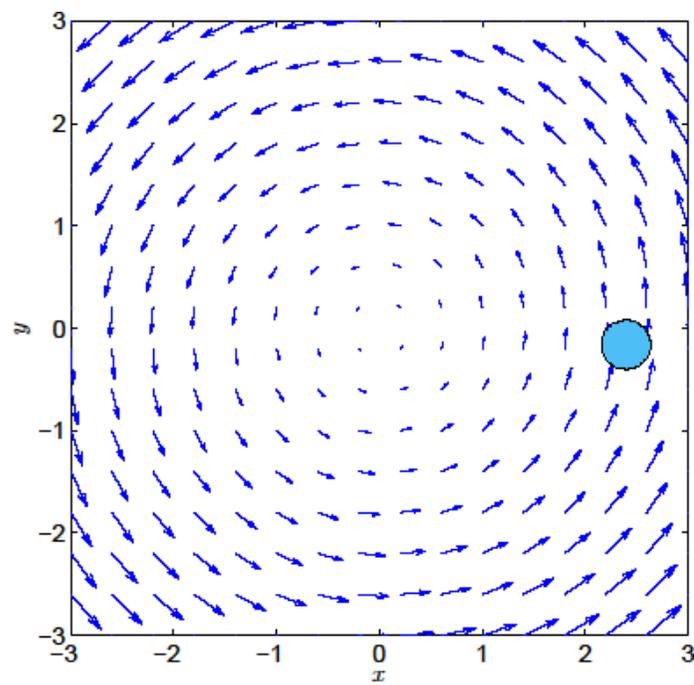
$A_1 = 0.18$

$t_1 = 1.1$

$A_1 = 0.18$

$t_1 = 2.0$

$A_1 = 0.18$

$t_1 = 3.0$