HG1M12
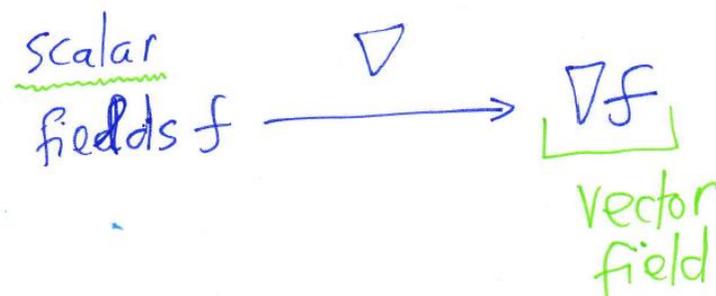## Engineering Mathematics 2

Chapter 4
(Vector Fields)

Lecture #18

Nabla operator:

$$\nabla = \underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}$$

↑
symbolic
vector

scalar fields f $\xrightarrow{\nabla}$ $\boxed{\nabla f}$
vector field

$$\nabla f = \left(\underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z} \right) f$$

$$= \underline{i}\frac{\partial f}{\partial x} + \underline{j}\frac{\partial f}{\partial y} + \underline{k}\frac{\partial f}{\partial z}$$

scalar field: $f(x,y,z)$

$$f = f(x, y, z) \qquad \text{OR} \qquad f = f(x, y)$$

$$\underline{v} = \text{unit vector} \qquad \left(|\underline{v}| = 1 \right)$$

$$\underline{a} = (a_1, a_2, a_3) \qquad \text{OR} \qquad \underline{a} = (a_1, a_2)$$

Rate of change
of f at $\underline{a}$
in the direction $\underline{v}$:

$$\boxed{D_{\underline{v}} f(\underline{a}) = \nabla f(\underline{a}) \cdot \underline{v}}$$

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Let $f(x,y,z) = x^2 e^{-yz}$.

Compute the rate of change of f in the direction of the unit vector

$$\underline{v} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

at the point $(1,0,0)$.

Solution: We are asked to calculate

$$D_{\underline{v}} f (1,0,0)$$

$$D_{\underline{v}} f = (\nabla f) \cdot \underline{v}$$

$$\nabla f (x,y,z) = \left(2x e^{-yz}, -xz^2 e^{-yz}, -x^2 y e^{-yz} \right)$$

$$\Downarrow$$

$$\nabla f (1,0,0) = (2,0,0)$$

Hence $D_{\underline{v}} f (1,0,0) = (2,0,0) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) = \frac{2}{\sqrt{3}}$

required answer

Assume that $\nabla f(\boldsymbol{a}) \neq \boldsymbol{0}$.

Then $\nabla f(\boldsymbol{a})$ points in the direction along which f increases **the fastest**.

Justification:

If $\boldsymbol{v}$ is a <u>unit</u> vector, the rate of change of f in direction $\boldsymbol{v}$ is

$$D_{\boldsymbol{v}}f(\boldsymbol{a}) = \nabla f(\boldsymbol{a}) \cdot \boldsymbol{v} = |\nabla f(\boldsymbol{a})||\boldsymbol{v}|\cos\theta$$

$$= |\nabla f(\boldsymbol{a})|\cos\theta$$

This expression is maximum when $\theta = 0$; that is, when $\boldsymbol{v}$ and $\nabla f(\boldsymbol{a})$ are parallel

(minimum when $\theta = \pi$)

Maximum rate of change = $|\nabla f|$

Important properties (ctd.)

In other words, if one wishes to move in a direction in which f will **increase most quickly**, one should proceed in the direction

$$\frac{\nabla f(\boldsymbol{a})}{|\nabla f(\boldsymbol{a})|}$$

In a similar way, if one wishes to move in a direction in which f **decreases fastest**, one should proceed in the direction
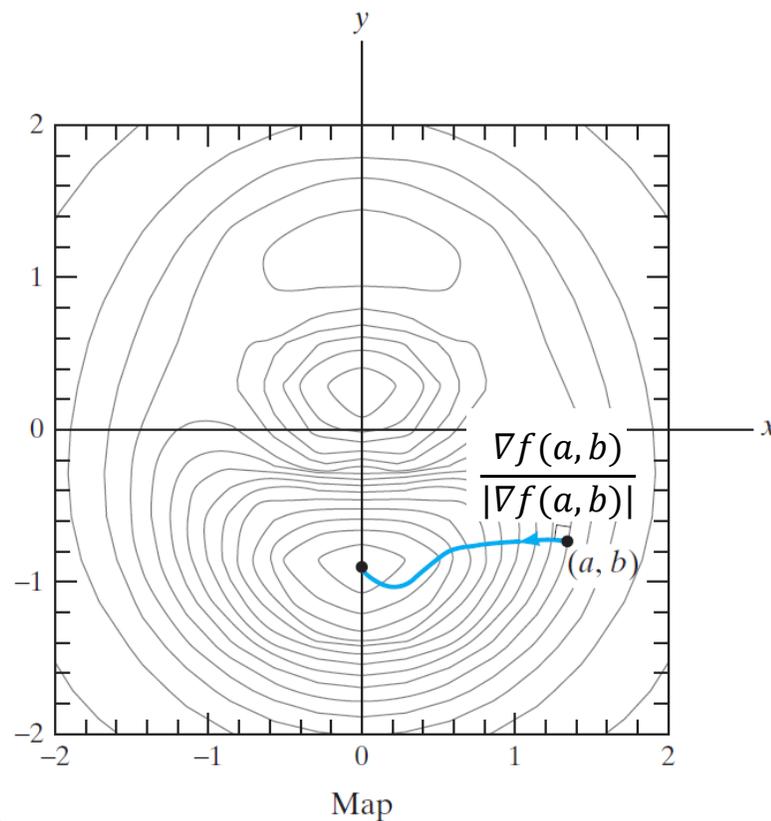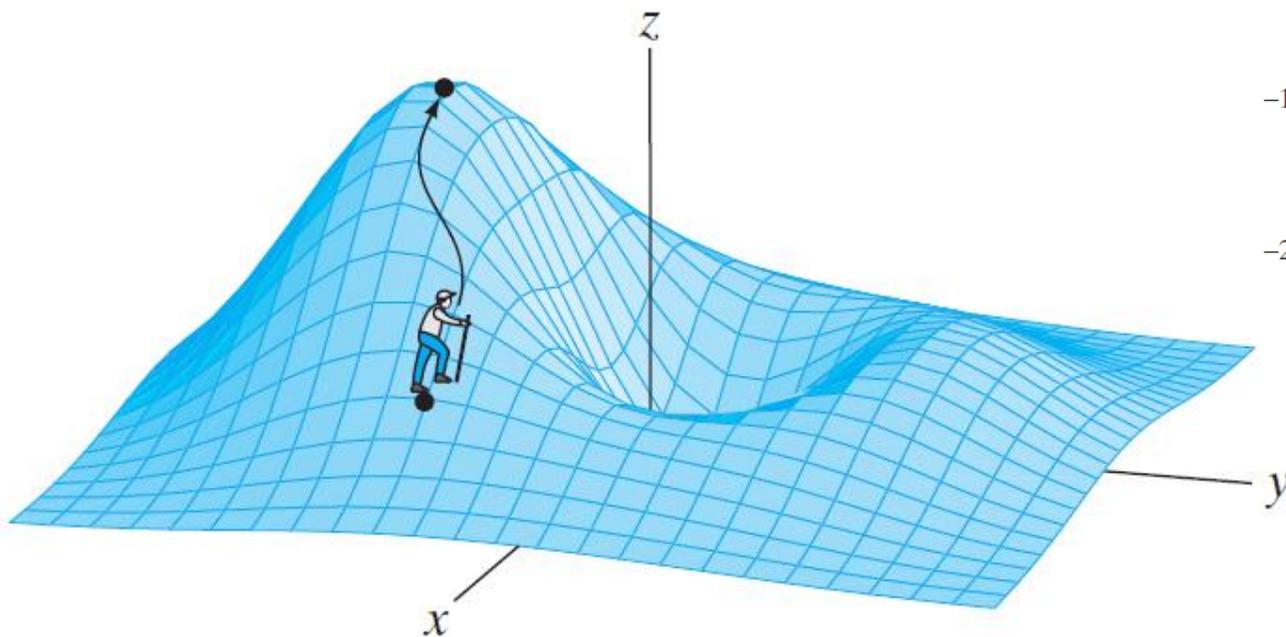
$$-\frac{\nabla f(\boldsymbol{a})}{|\nabla f(\boldsymbol{a})|}$$

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Application: steepest ascent/descent

$$z = f(x,y) \qquad \boldsymbol{a} = (a,b)$$

(**2D** example)

$$\frac{\nabla f(\boldsymbol{a})}{|\nabla f(\boldsymbol{a})|}$$

$$\frac{\nabla f(a,b)}{|\nabla f(a,b)|}$$

(a,b)

Map

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> If x_0 is a point in the level set $f(x) = C$,
> then the vector $\nabla f(x_0)$ is perpendicular to this level set.

The 2D case (i.e., $x = (x, y)$)-- particular example: $f(x, y) = x^4 - 3xy + 2y^2$
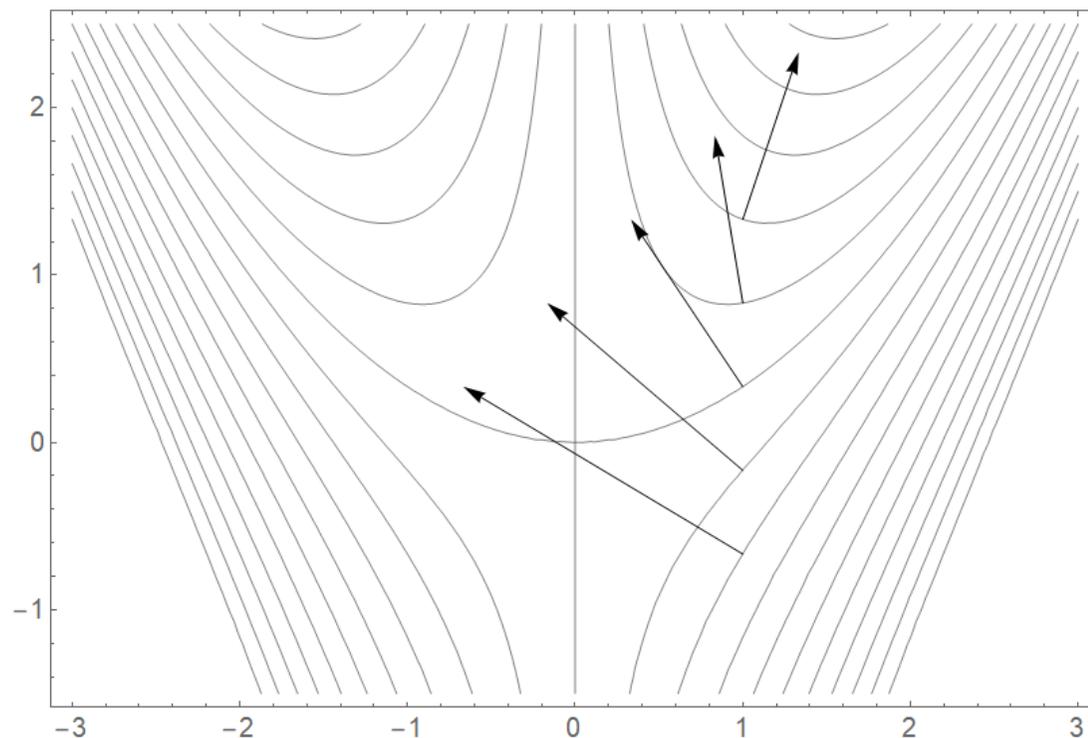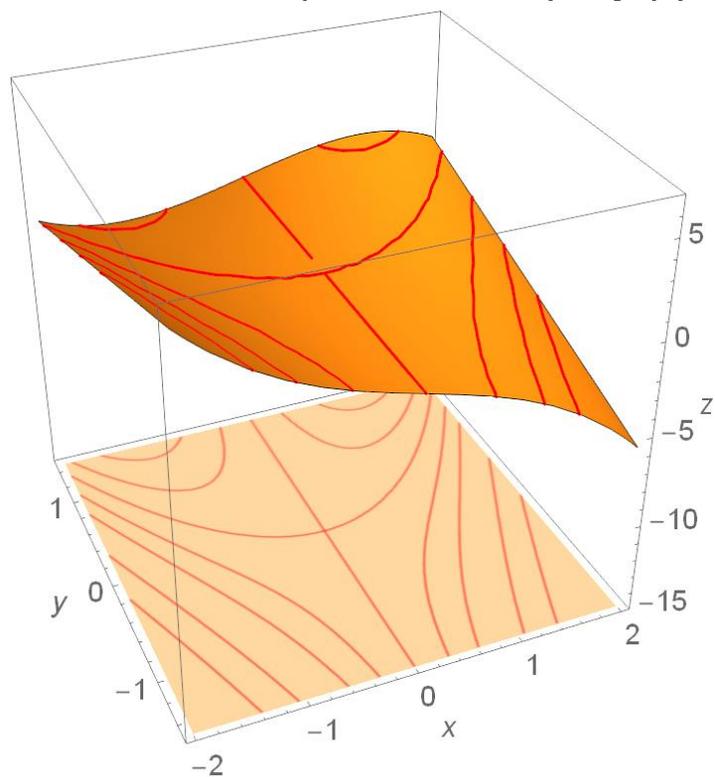
Important properties (II)

Justification Case 1: $f = f(x, y)$

The level set $f(x, y) = C$ represents a curve situated in the xy-plane, so we can think of this as $(x(t), y(t)) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j}$, with $t \in \mathbb{R}$.

Since $f(x(t), y(t)) = C$ $\implies$ $\dfrac{d}{dt} f(x(t), y(t)) = 0$

Using the rule for **total derivatives**: $\dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt} = 0$

$\implies$ $\underbrace{\left(\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}\right)}_{\nabla f} \cdot \underbrace{\left(\dfrac{dx}{dt}, \dfrac{dy}{dt}\right)}_{\text{"velocity" vector}} = 0$ $\implies$ $\nabla f \perp \underbrace{(\text{"velocity" vector})}_{\text{tangent to the curve}}$ $\implies$ $\boxed{\begin{array}{c} \nabla f \text{ is} \\ \text{orthogonal} \\ \text{to the level} \\ \text{set} \end{array}}$

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Important properties (II)

Case 2: $f = f(x, y, z)$

This is slightly more complicated because the level sets $f(x, y, z) = C$ are 3D surfaces.
In this case $\nabla f(x_0)$ represents the normal at x_0 to those surfaces.

Although we do not prove this, the idea is the following: if v is any vector tangent to the level set at x_0,
then $\nabla f(x_0)$ is perpendicular to v, i.e. $\nabla f(x_0) \cdot v = 0$.

By a tangent vector to the level set at x_0, we mean that v
is the velocity vector of a curve that lies in
that level set and passes through x_0.

A generic example can be seen in the sketch
included on this slide.
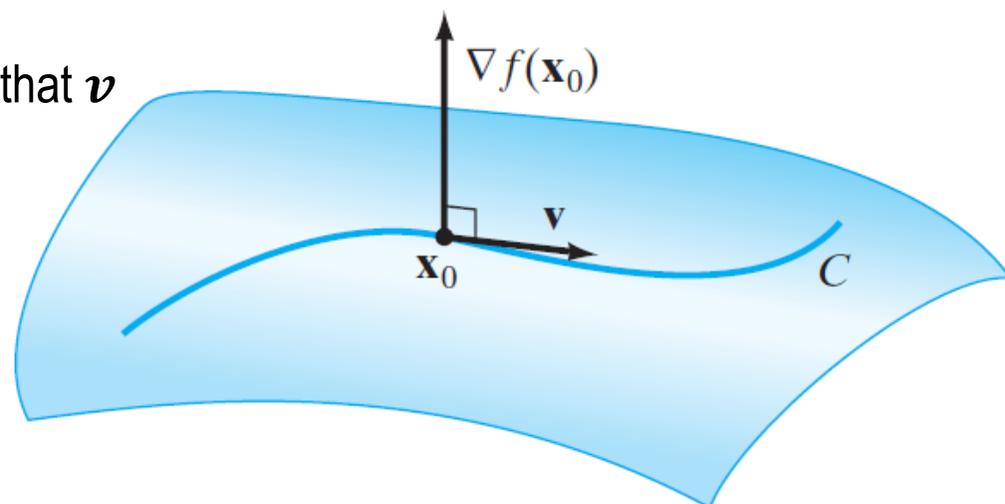
✖ The unit normal vector to a **plane curve** defined by an equation of the form

$f(x, y) = C$ is calculated by the formula $\quad \widehat{\boldsymbol{n}} = \dfrac{\nabla f}{|\nabla f|}$

✖ The unit normal vector to a **3D surface** defined by an equation of the form

$f(x, y, z) = C$ is calculated by the formula $\quad \widehat{\boldsymbol{n}} = \dfrac{\nabla f}{|\nabla f|}$

OBS.

in 2D:

$$\nabla f = \frac{\partial f}{\partial x} \boldsymbol{i} + \frac{\partial f}{\partial y} \boldsymbol{j}$$

in 3D:

$$\nabla f = \frac{\partial f}{\partial x} \boldsymbol{i} + \frac{\partial f}{\partial y} \boldsymbol{j} + \frac{\partial f}{\partial z} \boldsymbol{k}$$

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Worked examples

Example 4.1: In what direction from $(0,1)$ does the function
$$f(x,y) = x^2 - y^2$$
increases the fastest?

Solution:

$$\nabla f = 2x\boldsymbol{i} - 2y\boldsymbol{j} \qquad \Longrightarrow \qquad \nabla f(0,1) = -2\boldsymbol{j}$$

Answer: $\quad -\boldsymbol{j}$

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Example 4.2: Find the maximum rate of change of the scalar field
$$f(x, y, z) = x^2 + 2yz^2$$
at the point $(2,1,1)$. Find also the directional derivative in the direction of the vector $\boldsymbol{v} = (1,1,0)$.

Solution:

$$\nabla f = 2x\boldsymbol{i} - 2z^2\boldsymbol{j} + 4yz\boldsymbol{k} \implies \nabla f(2,1,1) = (4,2,4)$$

Maximum rate of change = $|\nabla f(2,1,1)| = \sqrt{4^2 + 2^2 + 4^2} = 6$

$$|\boldsymbol{v}| = \sqrt{2} \implies \hat{\boldsymbol{v}} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$ this is a **unit vector** in the given direction

The rate of change of f in the direction of $\boldsymbol{v}$: $\frac{1}{\sqrt{2}}(4, 2, 4) \cdot (1, 1, 0) = \frac{6}{\sqrt{2}}$.

Example 4.3: Find the unit normal to the surface
$$x^2 + y^2 - z = 0$$
at the point (1,1,2).

Solution: Define $f(x, y, z) = x^2 + y^2 - z$. Then the surface is $f(x, y, z) = 0$, a level surface of f.

The unit normal $\hat{\mathbf{n}}$ is given by
$$\hat{\mathbf{n}} = \frac{\nabla f}{|\nabla f|},$$

where
$$\nabla f = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}, \quad \text{and} \quad |\nabla f| = \sqrt{4x^2 + 4y^2 + 1}.$$

Hence
$$\hat{\mathbf{n}} = \frac{2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}}{\sqrt{4x^2 + 4y^2 + 1}} \implies \hat{\mathbf{n}} = \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$$

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Evaluated at $(1, 1, 2)$ we obtain

Example 4.4: Find the unit normal to the surface
$$x^3 y - yz^2 + z^5 - 9 = 0$$
at the point $(3, -1, 2)$.

Solution: Define $f(x, y, z) = x^3 y - yz^2 + z^5$. Then the surface is $f(x, y, z) = 9$, a particular level surface of f.

The unit normal $\widehat{\mathbf{n}}$ is given by $\qquad \widehat{\mathbf{n}} = \dfrac{\nabla f}{|\nabla f|},$

where

$$\nabla f = 3x^2 y \mathbf{i} + (x^3 - z^2)\mathbf{j} + (5z^4 - 2yz)\mathbf{k}.$$

Hence

$$\nabla f(3, -1, 2) = -27\mathbf{i} + 23\mathbf{j} + 84\mathbf{k} \quad \text{and} \quad |\nabla f(3, -1, 2)| = \sqrt{8314}.$$

Finally, the required unit normal is $\qquad \dfrac{1}{\sqrt{8314}}(-27, 23, 84)$

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