



# **HG1M12**

## **Engineering Mathematics 2**

### **Chapter 4** **(Vector Fields)**

LECTURE #17



# Terminology (I)

**scalar field** = a scalar quantity that depends on position (either in 2D or 3D)

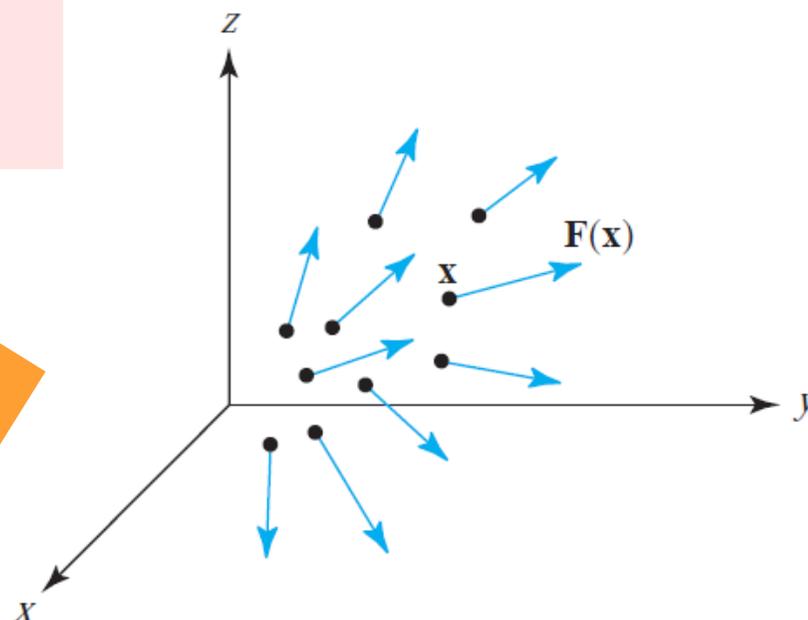
**E.G.:**      temperature:       $T = T(x, y)$       or       $T = T(x, y, z)$

                 pressure:       $p = p(x, y)$       or       $p = p(x, y, z)$

**vector field** = a vector quantity that depends on position (in 2D or 3D)

$$F: \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

A vector field  $F$  assigns a vector  $F(\mathbf{x})$  to each point  $\mathbf{x}$  of its domain.



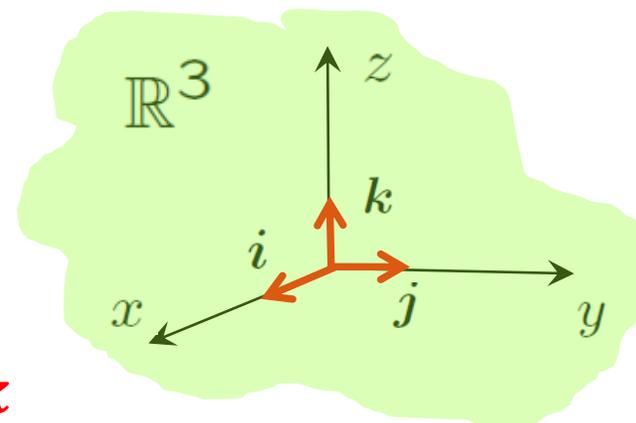


# Terminology (II)

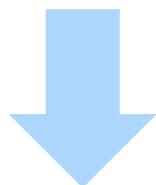
$$\mathbf{F}(\mathbf{x}) = F_1(\mathbf{x})\mathbf{i} + F_2(\mathbf{x})\mathbf{j} + F_3(\mathbf{x})\mathbf{k}$$

$$= (F_1(\mathbf{x}), F_2(\mathbf{x}), F_3(\mathbf{x}))$$

$\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$   
(a position vector)



$F_1, F_2, F_3$  are the components of the vector field  $\mathbf{F}(\mathbf{x})$



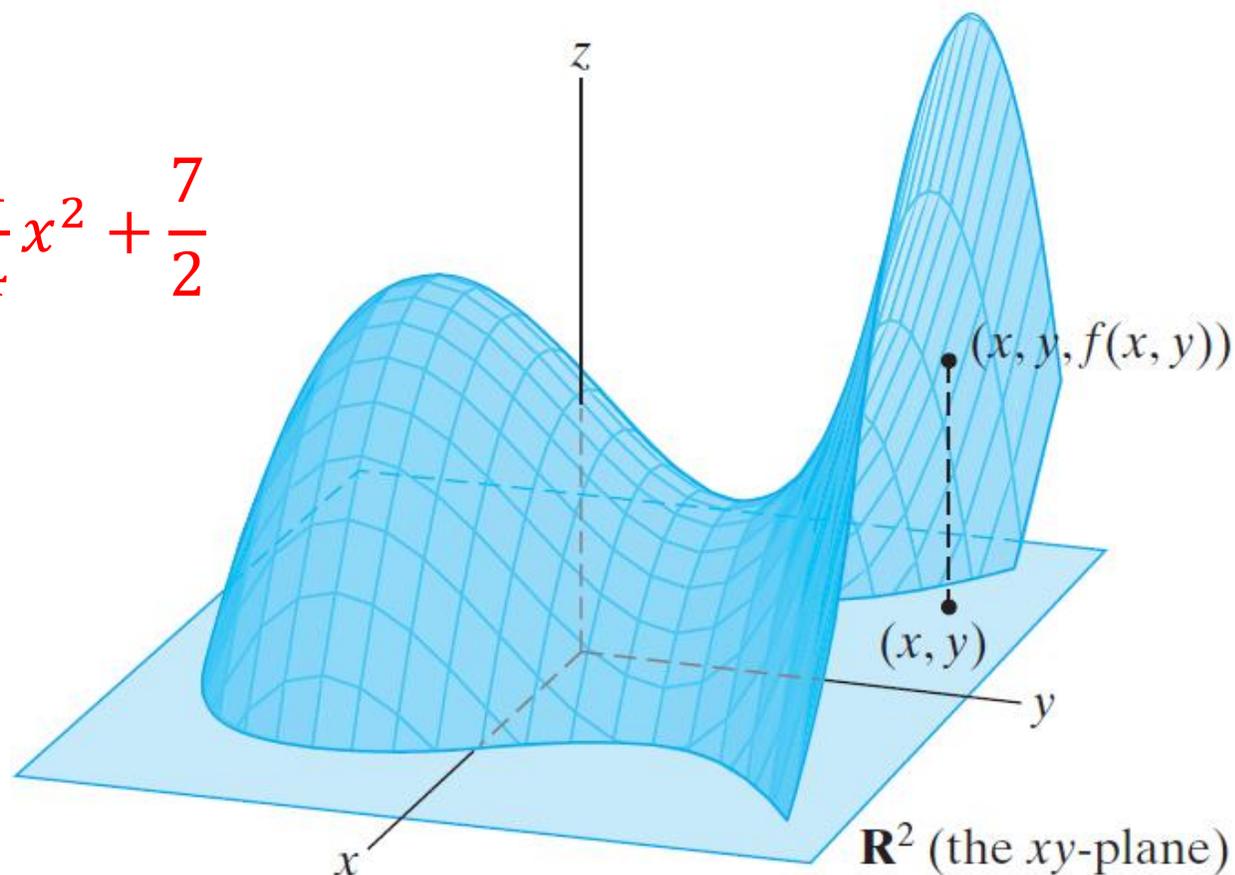
these are obviously scalar fields



# Graph of a function $z = f(x, y)$

$$f: \mathbf{R}^2 \rightarrow \mathbf{R}$$

$$f(x, y) = \frac{1}{12}y^3 - y - \frac{1}{4}x^2 + \frac{7}{2}$$





# Visual representation for scalar fields

$f(x, y, z) =$  a **scalar** field

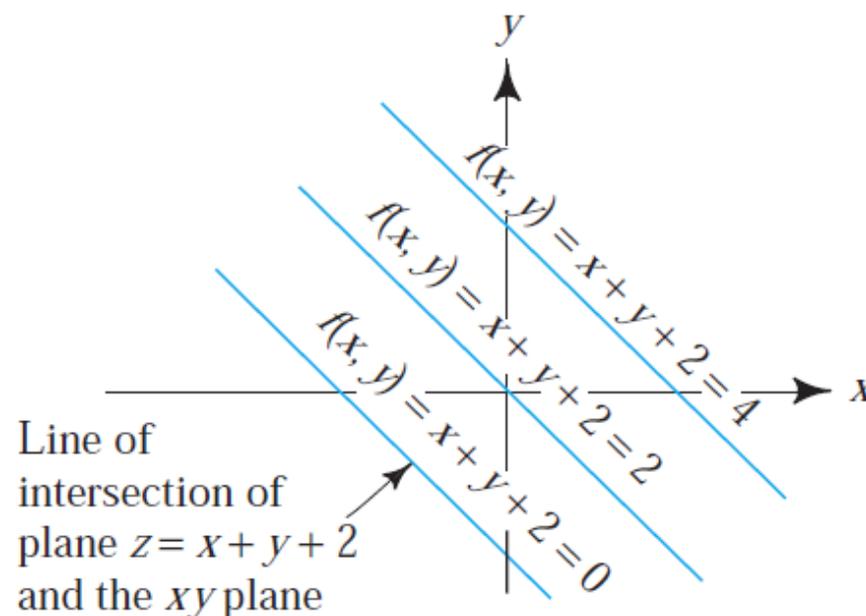
A **level set** for this field is the set of all points  $(x, y, z)$  for which  $f(x, y, z) = C$ , where  $C$  is a constant.

In the case of a **2D scalar field** these are known as **level curves** or **level contours**.

**E.G.:**

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$f(x, y) = x + y + 2$$

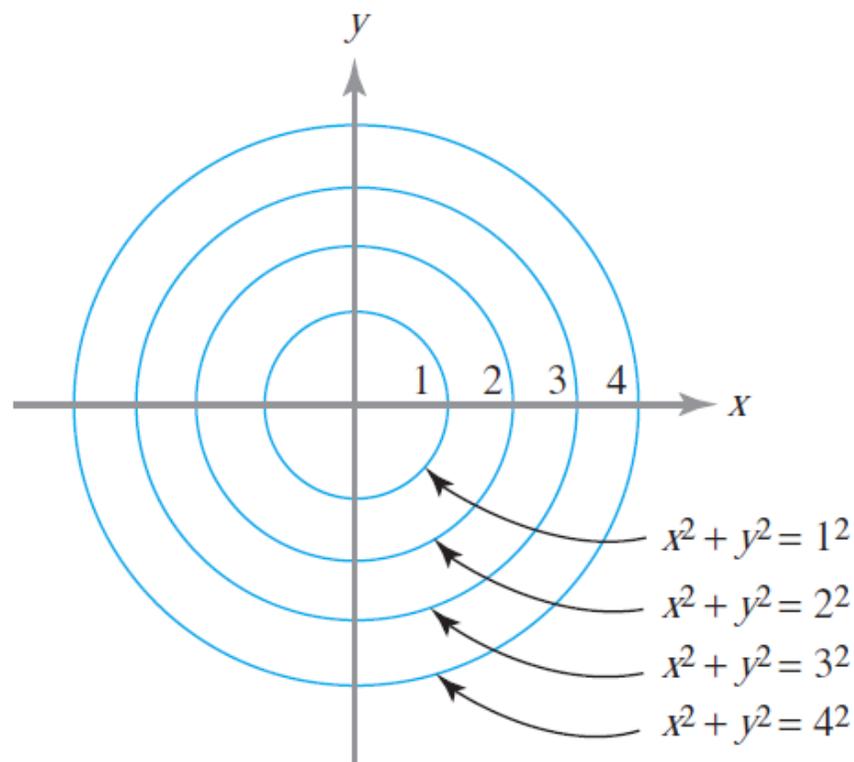




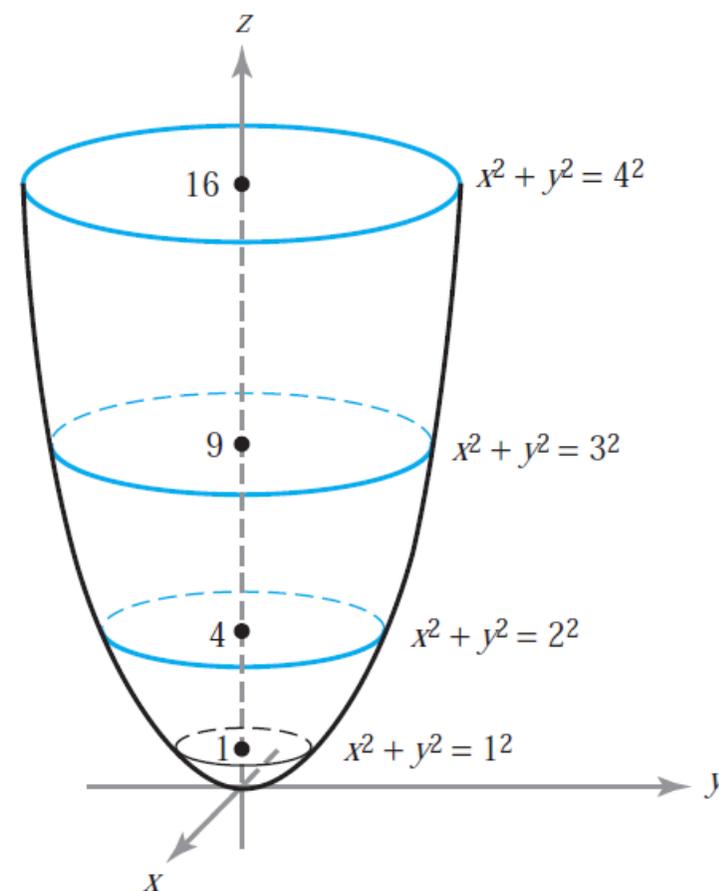
# Level sets

Some level curves for the function

$$f(x, y) = x^2 + y^2$$



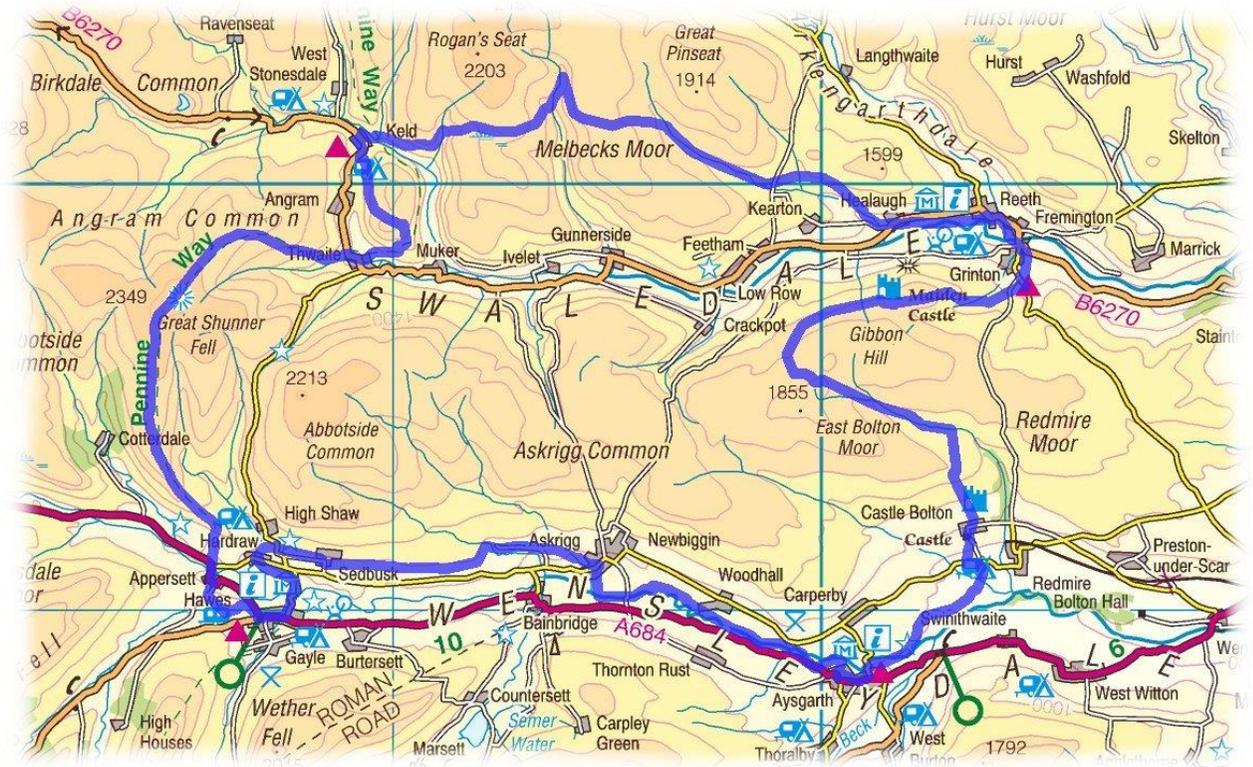
The level curves on the left are “raised” to the graph of  $f(x, y)$ . These are the so-called **contour curves**.



# Level sets



Level contours are used extensively in topographical maps.



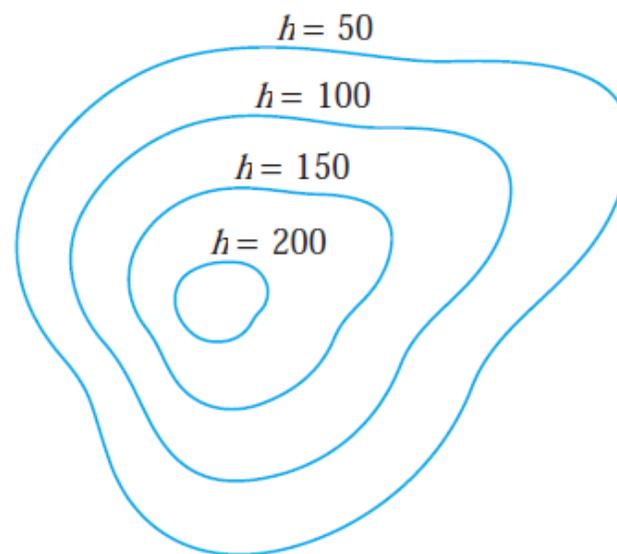
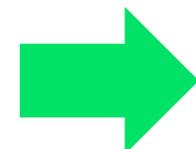
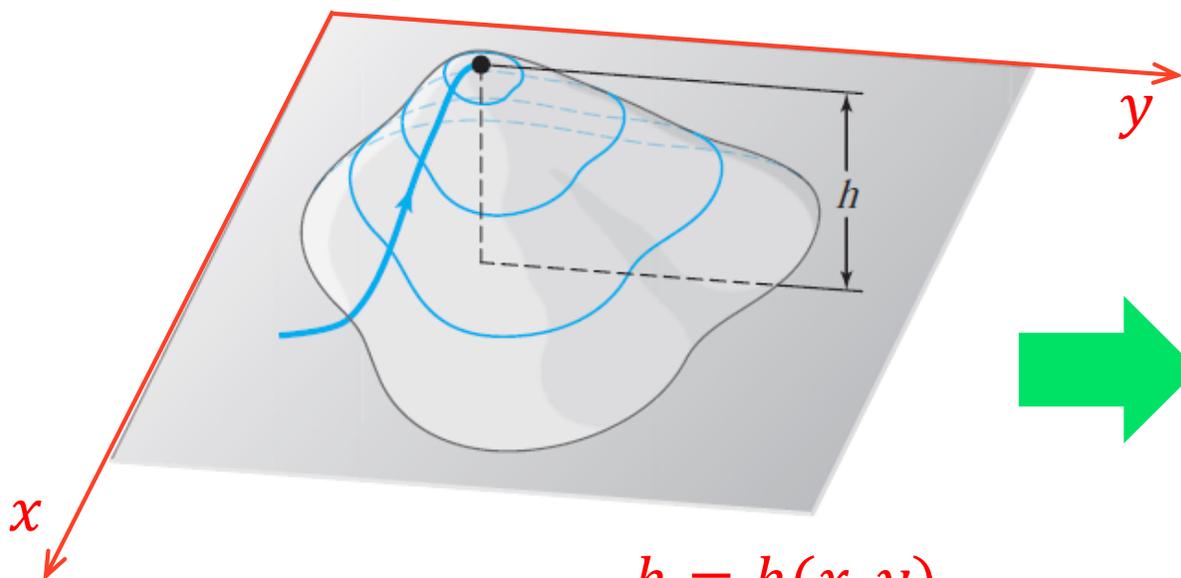


# Level sets

**Level contours** provide a means to visualise a 3D object in two dimensions:

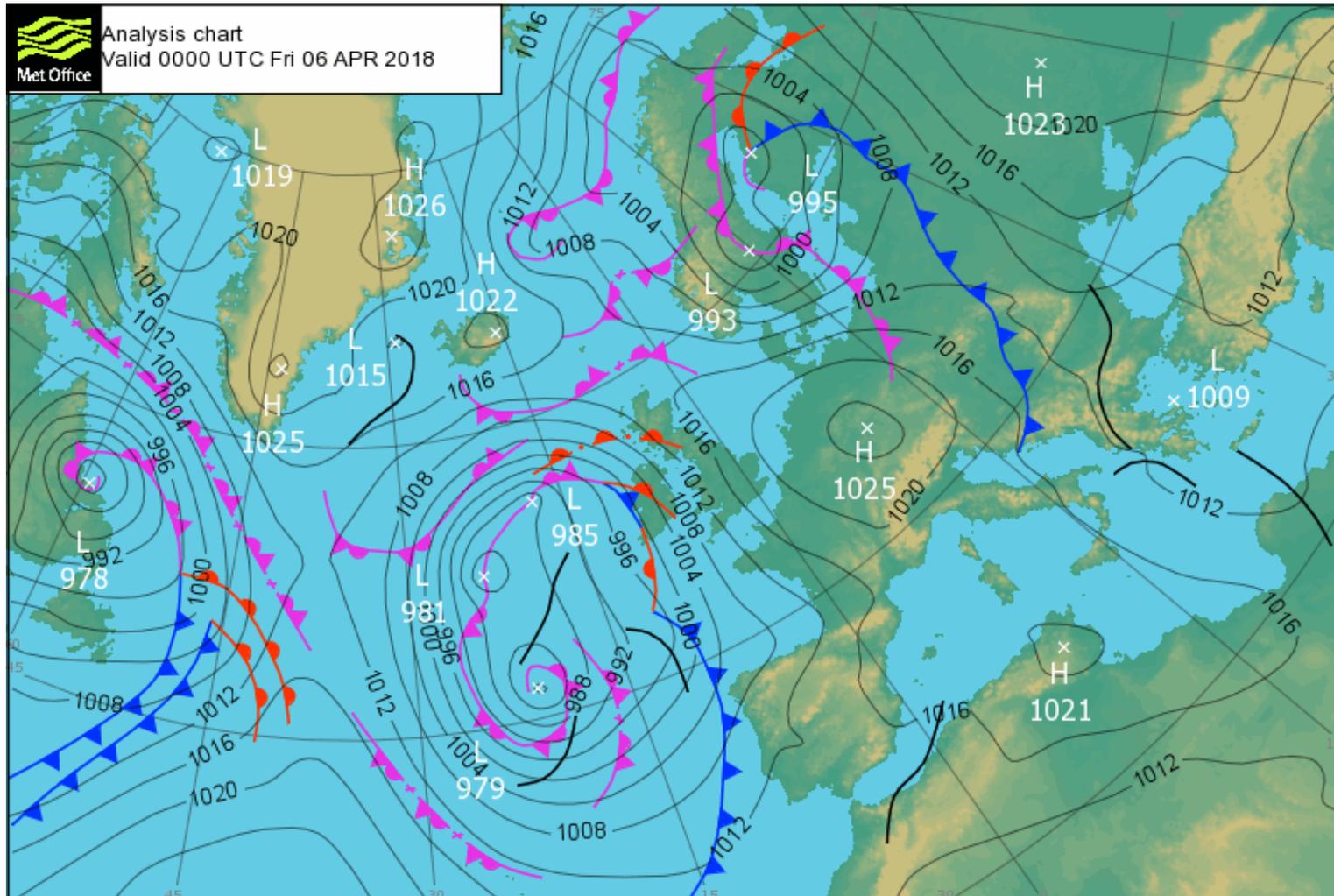
3D

2D



$$h = h(x, y)$$

# Surface pressure chart





# Surface pressure chart

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The chart on the previous slide shows the surface pressure pattern using **isobars** (lines of equal pressure) and indicate areas of **high** (H) and **low** pressure (L) along with their central pressure value.

Isobars are represented by solid lines. Note that they are **level contours** for the pressure function  $p = p(x, y)$ , where  $(x, y)$  is a generic point in the plane of the map/chart.

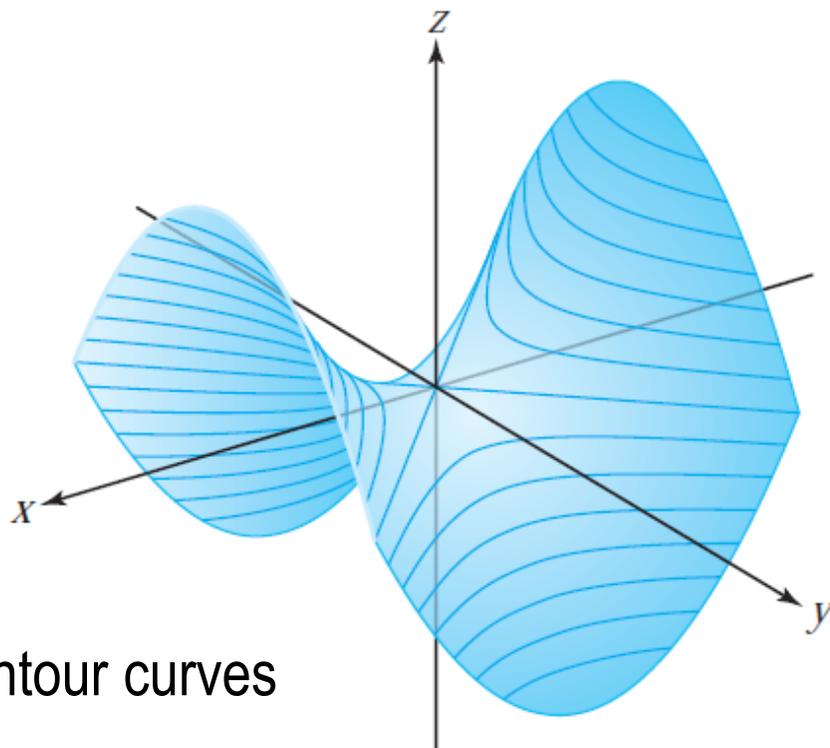
High pressure is usually associated with settled weather while low pressure is normally associated with unsettled weather. Fronts are also displayed.

For more info follow this [link](#)

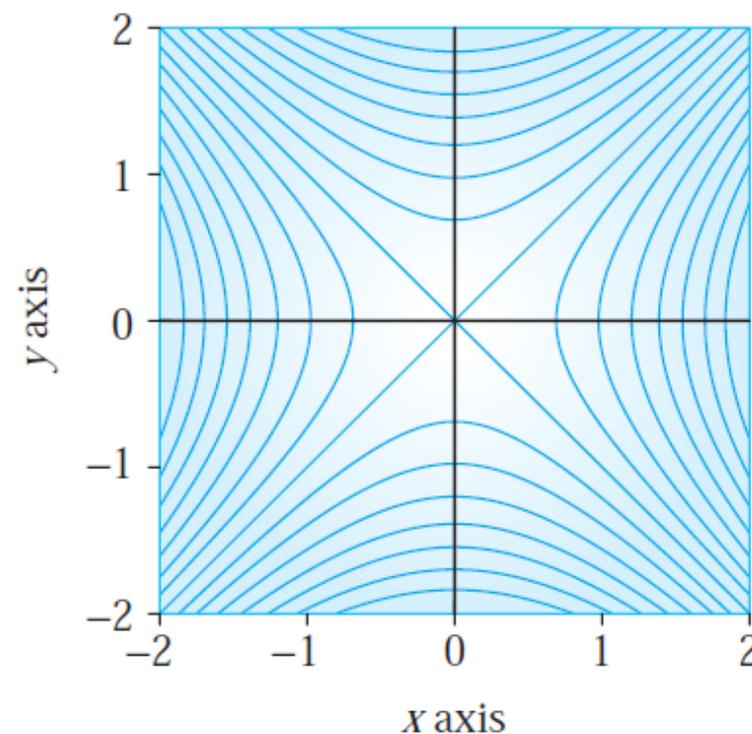


# Level sets

Some **level curves** on the graph of  $f(x, y) = x^2 - y^2$



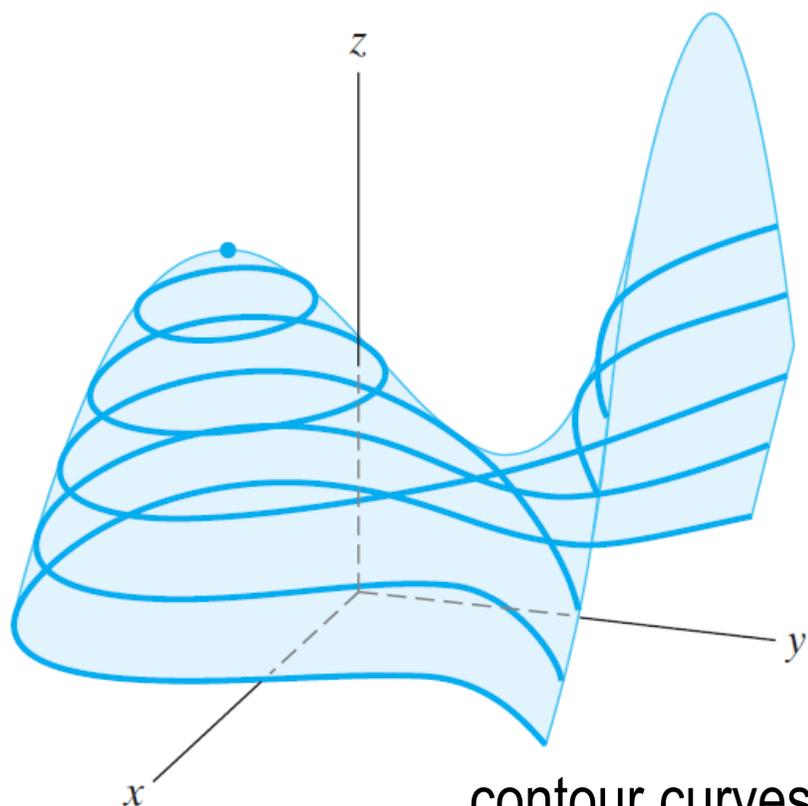
contour curves



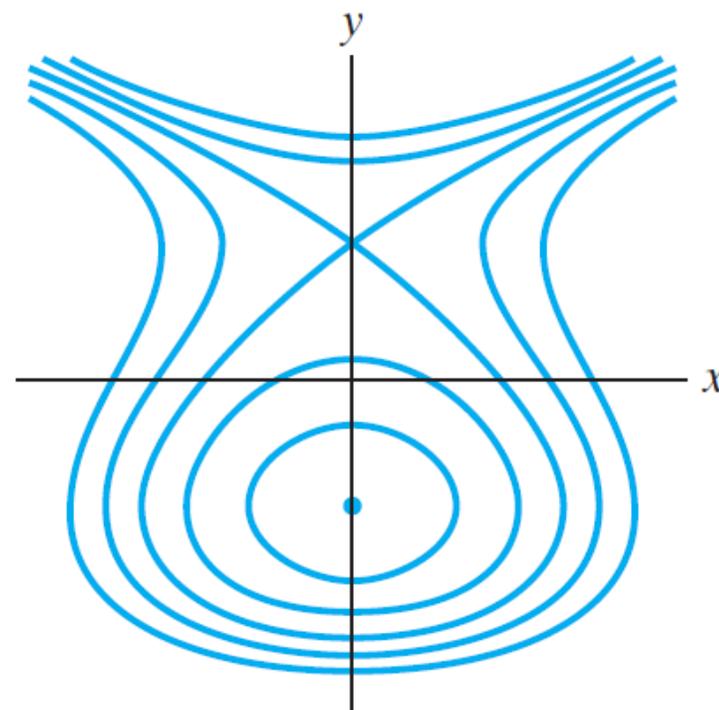


# Level sets

Some **level curves** on the graph of  $f(x, y) = \frac{1}{12}y^3 - y - \frac{1}{4}x^2 + \frac{7}{2}$



contour curves





# Level sets

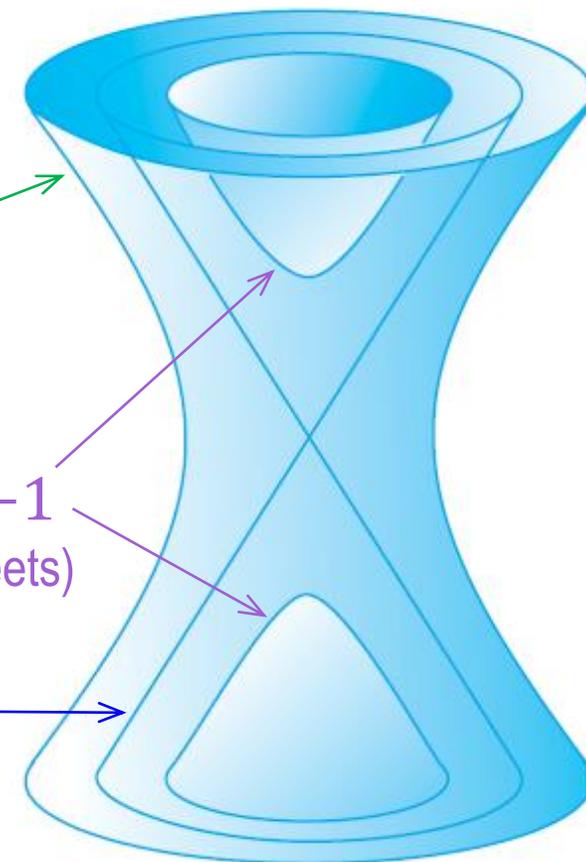
For functions  $f(x, y, z)$  the level sets will be surfaces in 3D.  
It is a lot harder to visualise them.

E.G.:  $f(x, y, z) = \frac{z^2}{c^2} - \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} \right)$

$f(x, y, z) = -1$   
(hyperboloid of one sheet)

$f(x, y, z) = +1$   
(hyperboloid of two sheets)

$f(x, y, z) = 0$   
(cone)





# Vector differential operators

The vector differential operator  $\nabla$  (called 'del' or 'nabla') is given by

$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}$$

(in a cartesian coordinate system), and can act (similar to a vector) on scalar or vector functions.

Let's look at how it acts on a scalar field first.



# Gradient of a scalar field

Suppose  $f(x, y, z) = f(\mathbf{x})$  is a scalar field.

Then  $\nabla$  transforms it into a vector field, called the **gradient of  $f$**

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$$

**E.G.:**

If  $f(x, y, z) = x^2 + y^2 + z$ , then

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y, \quad \frac{\partial f}{\partial z} = 1$$

and

$$\begin{aligned} \nabla f &= (2x, 2y, 1) \\ &= 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k} \end{aligned}$$



# 'DEL' identities

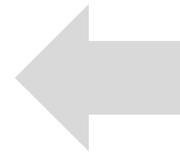
$$\left. \begin{aligned} f &= f(x, y, z) \\ g &= g(x, y, z) \end{aligned} \right\} \text{SCALAR FIELDS}$$

$$\nabla(f + g) = \nabla f + \nabla g$$

$$\nabla(\alpha f) = \alpha(\nabla f), \alpha \in \mathbb{R}$$

$$\nabla(fg) = g(\nabla f) + f(\nabla g)$$

$$\nabla\left(\frac{f}{g}\right) = \frac{1}{g^2}(g\nabla f - f\nabla g)$$



note the analogy with the  
standard properties for ordinary  
and partial derivatives



## An important observation (2D)

If we work with a function  $z = f(x, y)$ , then

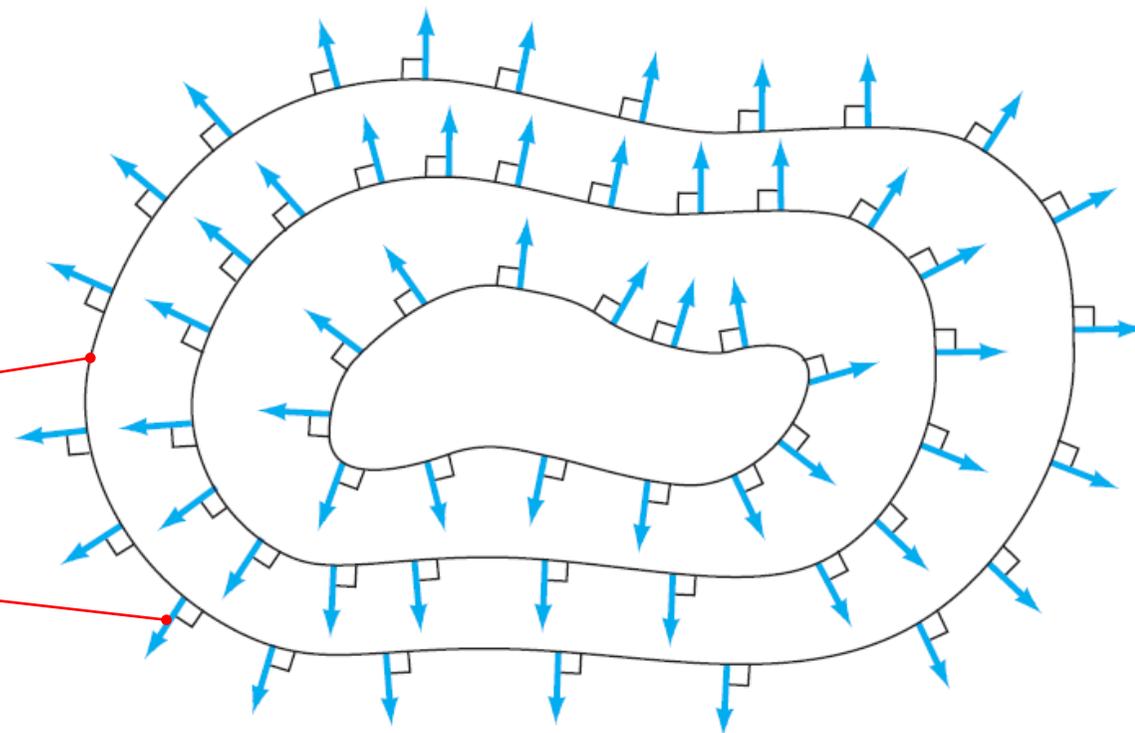
$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}$$

a 2D vector field in the  $xy$ -plane

We are going to see later on that this vector field is perpendicular to the **level contours** of such surfaces

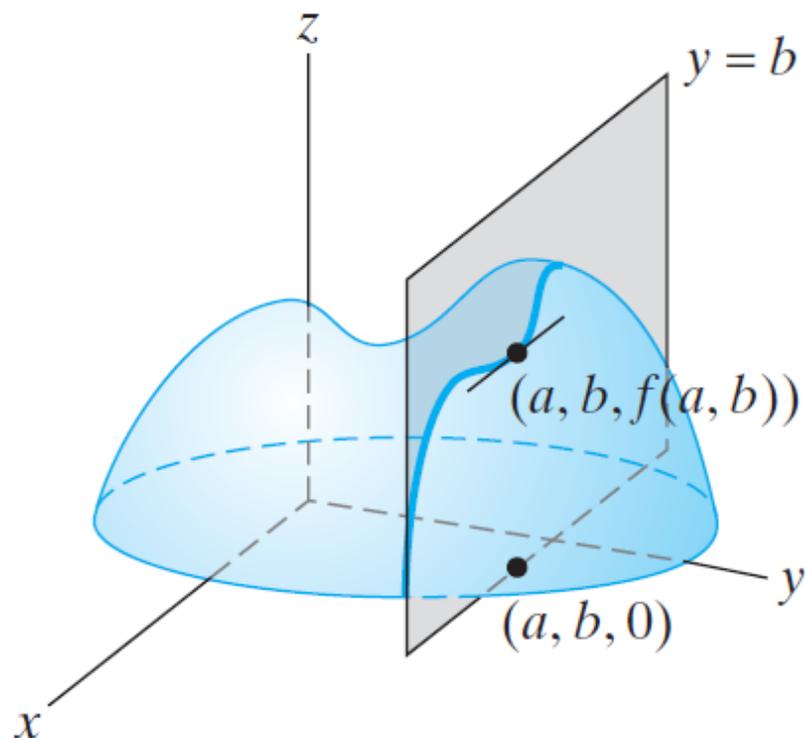
$$f(x, y) = \text{const.}$$

$$\nabla f(x, y)$$

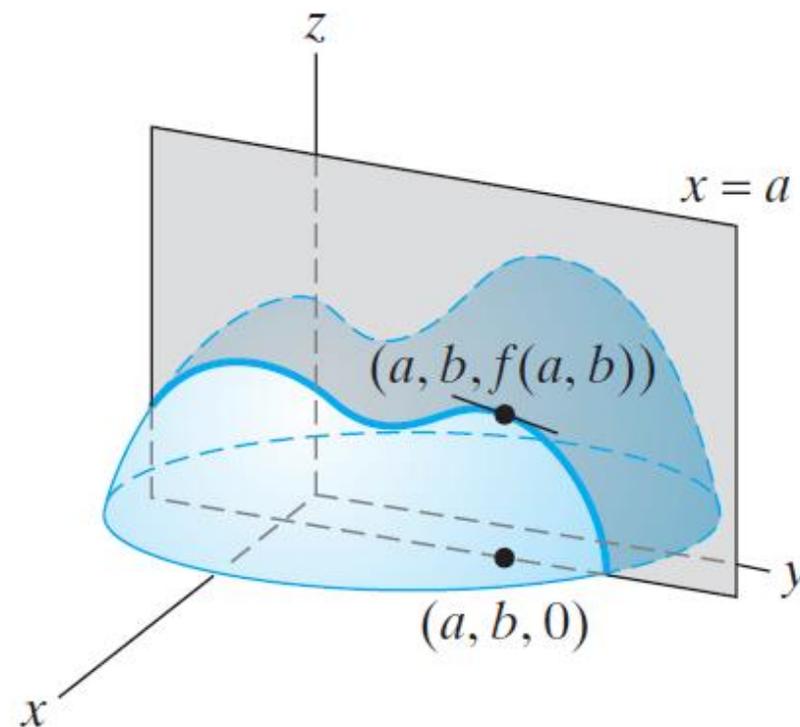




# Partial derivatives: reminders



$\frac{\partial f}{\partial x}(a, b)$  represents the slope at the point  $(a, b, f(a, b))$  of the curve obtained by intersecting the surface  $z = f(x, y)$  with the plane  $y = b$



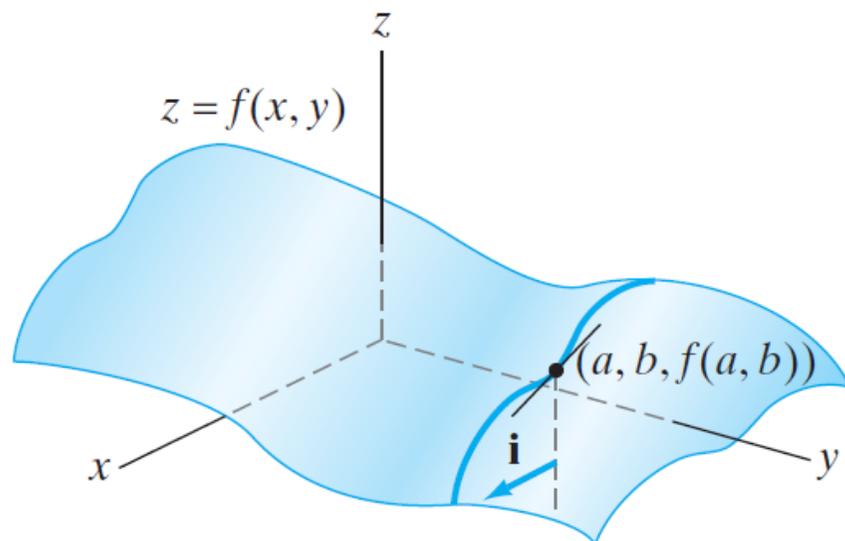
$\frac{\partial f}{\partial y}(a, b)$  represents the slope at the point  $(a, b, f(a, b))$  of the curve obtained by intersecting the surface  $z = f(x, y)$  with the plane  $x = a$



# Another way to look at things....

We can also view  $\frac{\partial f}{\partial x}(a, b)$

as the rate of change of  $f$  as we move “infinitesimally” from  $\mathbf{a} = (a, b)$  in the  $\mathbf{i}$ -direction.



$$\frac{\partial f}{\partial x}(a, b) = \lim_{h \rightarrow 0} \frac{f(a + h, b) - f(a, b)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f((a, b) + (h, 0)) - f(a, b)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f((a, b) + h(1, 0)) - f(a, b)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(\mathbf{a} + h\mathbf{i}) - f(\mathbf{a})}{h}.$$

In a similar way,

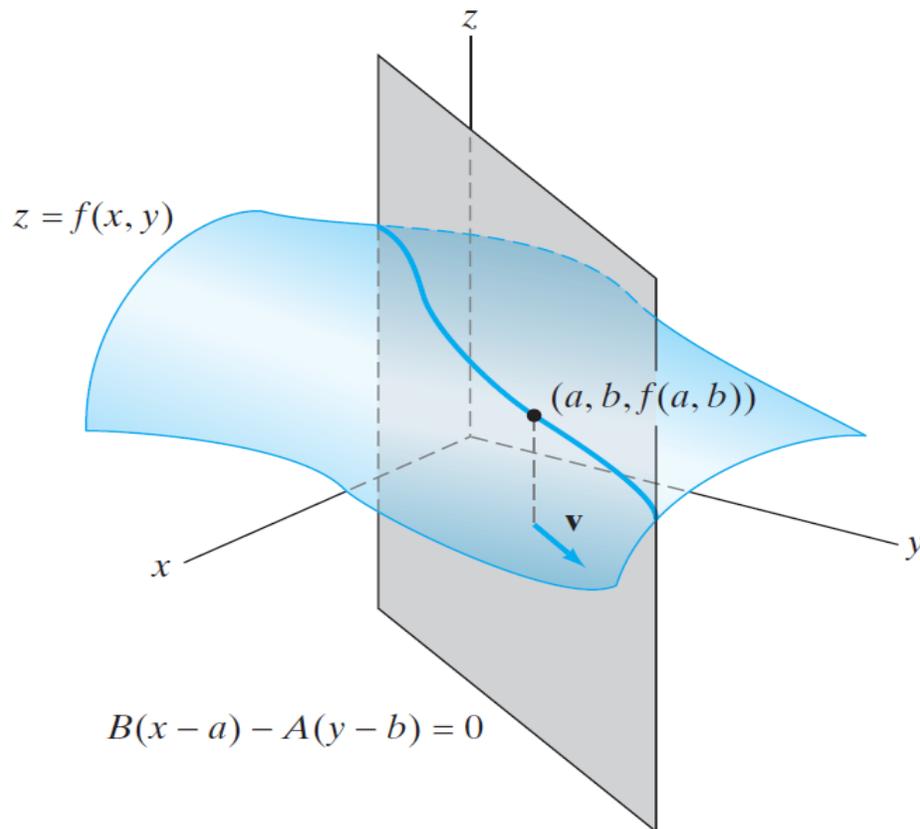
$$\frac{\partial f}{\partial y}(a, b) = \lim_{h \rightarrow 0} \frac{f(\mathbf{a} + h\mathbf{j}) - f(\mathbf{a})}{h}$$



# Directional derivative

Consider  $f$  a scalar field and  $\mathbf{a}$  a point where this function is defined.

If  $\mathbf{v}$  is any unit vector (i.e.,  $|\mathbf{v}| = 1$ ), then the **directional derivative of  $f$  at  $\mathbf{a}$  in the direction of  $\mathbf{v}$** , denoted  $D_{\mathbf{v}}f(\mathbf{a})$ , is



$$D_{\mathbf{v}}f(\mathbf{a}) = \lim_{h \rightarrow 0} \frac{f(\mathbf{a} + h\mathbf{v}) - f(\mathbf{a})}{h}$$

the rate of change of  $f$  as we move  
(infinitesimally) from  $\mathbf{a} = (a, b)$   
in the direction of

$$\mathbf{v} = (A, B) = A\mathbf{i} + B\mathbf{j}$$



# Directional derivative

$$\varphi(h) = f(\mathbf{a} + h\mathbf{v}) \quad \Rightarrow \quad \varphi'(0) = \lim_{h \rightarrow 0} \frac{\varphi(h) - \varphi(0)}{h}$$

$\varphi: \mathbb{R} \rightarrow \mathbb{R}$

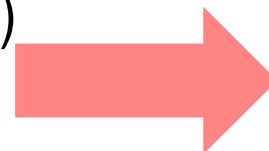
can be evaluated  
using the **Chain Rule**  
(for the total derivative)

this is just the RHS  
in the definition of  $D_{\mathbf{v}}f(\mathbf{a})$

$$\varphi(h) = f(a_1 + hv_1, a_2 + hv_2, a_3 + hv_3)$$

$$\mathbf{a} = (a_1, a_2, a_3)$$
$$\mathbf{v} = (v_1, v_2, v_3)$$

$$\begin{aligned} \varphi'(h) = & \frac{\partial f}{\partial x}(a_1 + hv_1, a_2 + hv_2, a_3 + hv_3) \frac{d}{dh}(a_1 + hv_1) + \\ & + \frac{\partial f}{\partial y}(a_1 + hv_1, a_2 + hv_2, a_3 + hv_3) \frac{d}{dh}(a_2 + hv_2) + \\ & + \frac{\partial f}{\partial z}(a_1 + hv_1, a_2 + hv_2, a_3 + hv_3) \frac{d}{dh}(a_3 + hv_3) \end{aligned}$$





# Directional derivative

Set  $h = 0$  in the long equation that results from the application of the CR:

$$\begin{aligned}\varphi'(0) &= \frac{\partial f}{\partial x}(\mathbf{a})v_1 + \frac{\partial f}{\partial y}(\mathbf{a})v_2 + \frac{\partial f}{\partial z}(\mathbf{a})v_3 \\ &= \left( \frac{\partial f}{\partial x}(\mathbf{a}), \frac{\partial f}{\partial y}(\mathbf{a}), \frac{\partial f}{\partial z}(\mathbf{a}) \right) \cdot (v_1, v_2, v_3) \\ &= \nabla f(\mathbf{a}) \cdot \mathbf{v}\end{aligned}$$

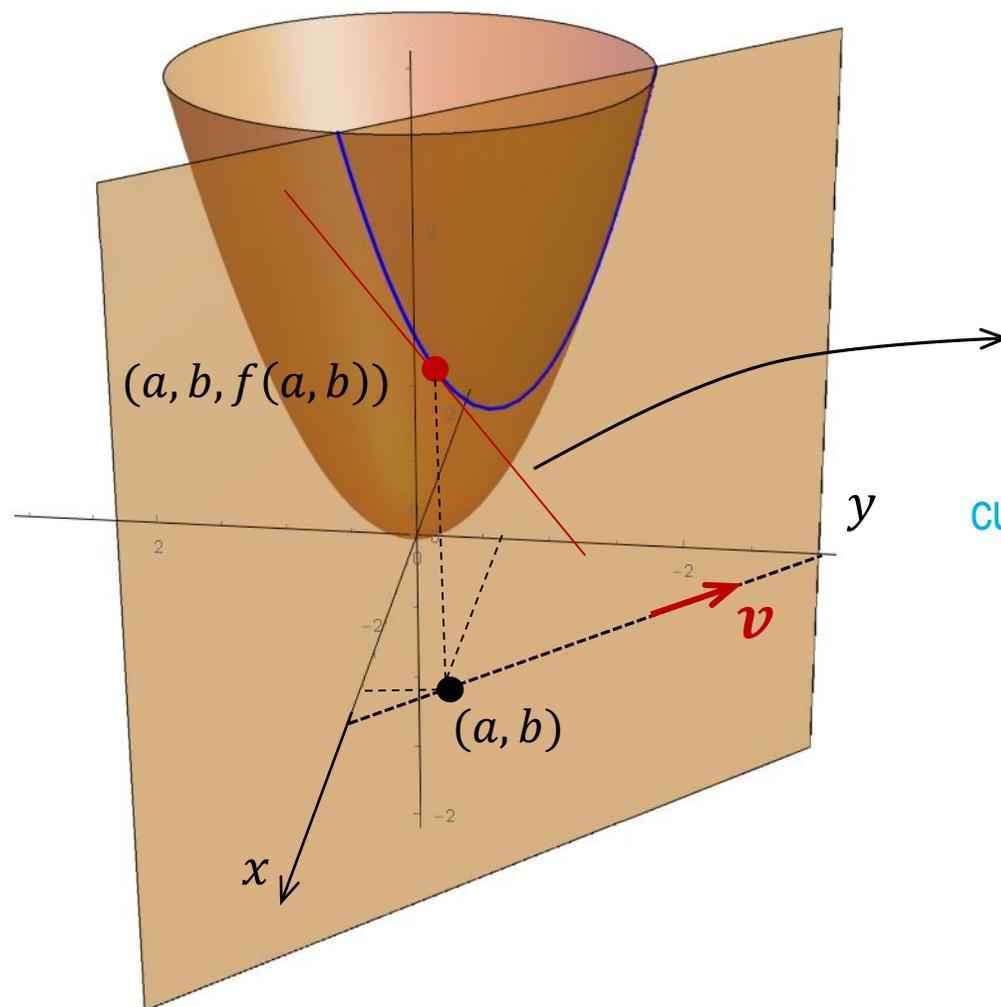
In conclusion

$$D_{\mathbf{v}}f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{v}$$

in this formula  $\mathbf{v}$   
must be a  
UNIT vector



# Directional derivatives (more examples)



$$z = f(x, y) = x^2 + y^2$$

$$v = (-1, 1)/\sqrt{2}$$

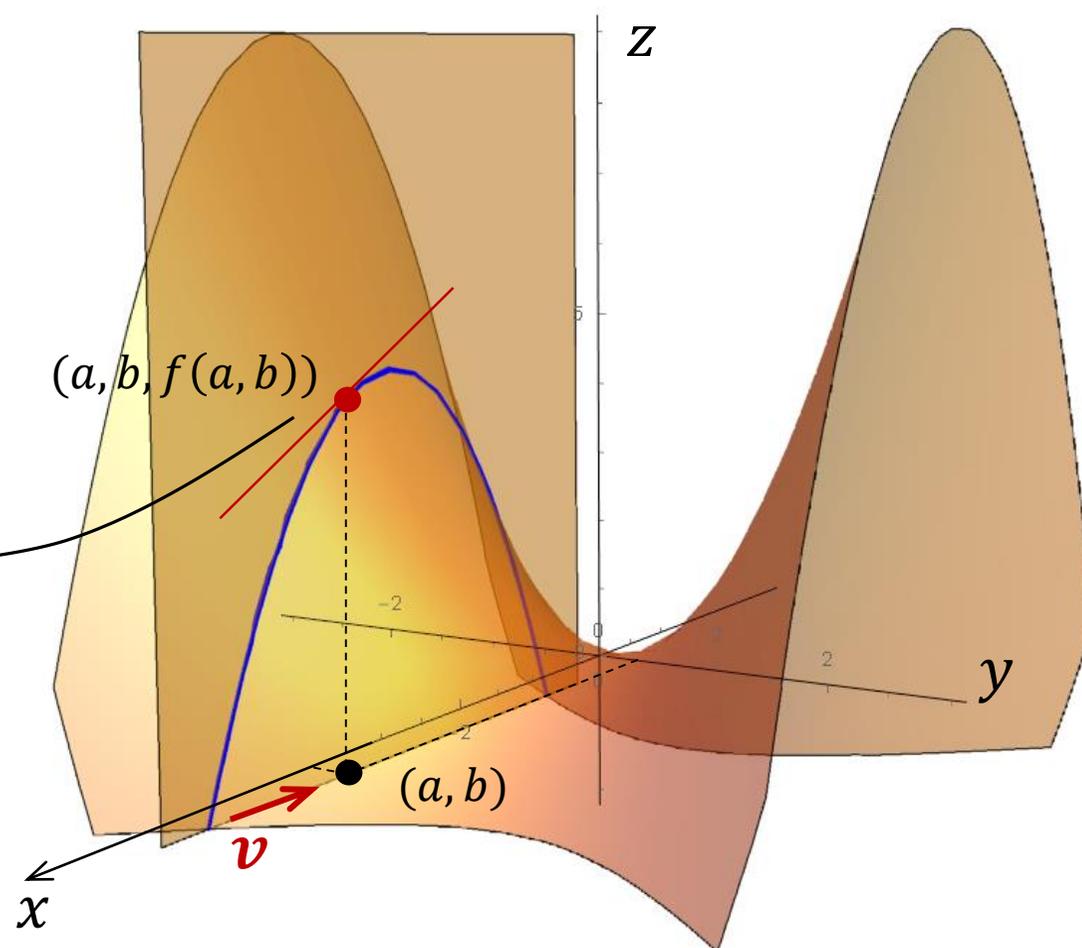
$D_v f(a, b)$   
is the **slope** of the  
tangent to the blue  
curve, at the point identified  
by the red marker

# Directional derivatives (more examples)

$$z = f(x, y) = x^2 - y^2$$

$$\mathbf{v} = (1, 0)$$

$D_{\mathbf{v}}f(a, b)$   
is the **slope** of the  
tangent to the blue  
curve, at the point identified  
by the red marker



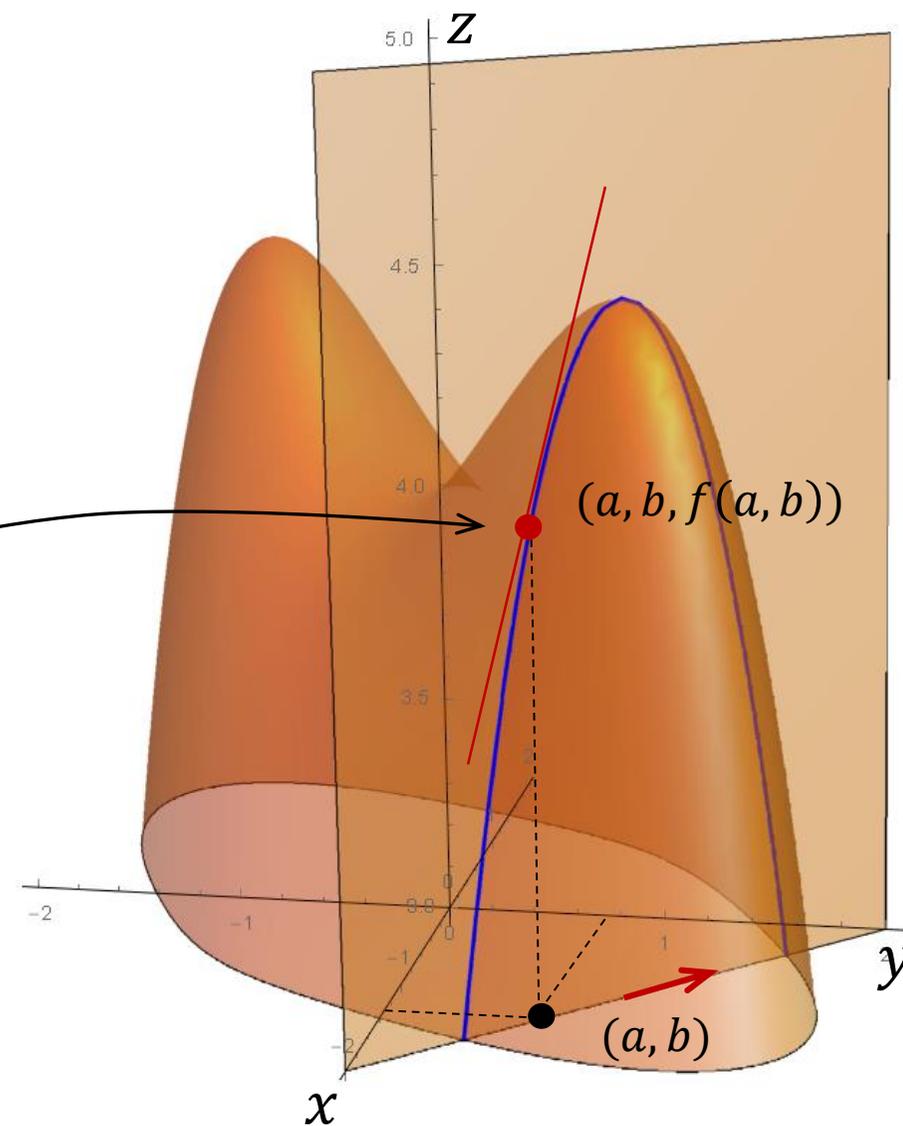


# Directional derivatives (more examples)

$$z = f(x, y) = 4 - xy - \frac{1}{4}(x^4 + y^4)$$

$$\mathbf{v} = (-1, 1)/\sqrt{2}$$

$D_{\mathbf{v}}f(a, b)$   
is the **slope** of the  
tangent to the blue  
curve, at the point identified  
by the red marker





# Engineering notation

The standard engineering notation for directional derivative is different from the above (**but the definition is the same**):

$$\frac{\partial f}{\partial s} = (\nabla f) \cdot \mathbf{s}$$



$D_s f$

(in our notation)

$$\frac{\partial f}{\partial n} = (\nabla f) \cdot \mathbf{n}$$



$D_n f$

(in our notation)

As usual, in the above formulae  $\mathbf{s}$  and  $\mathbf{n}$  are always unit vectors.