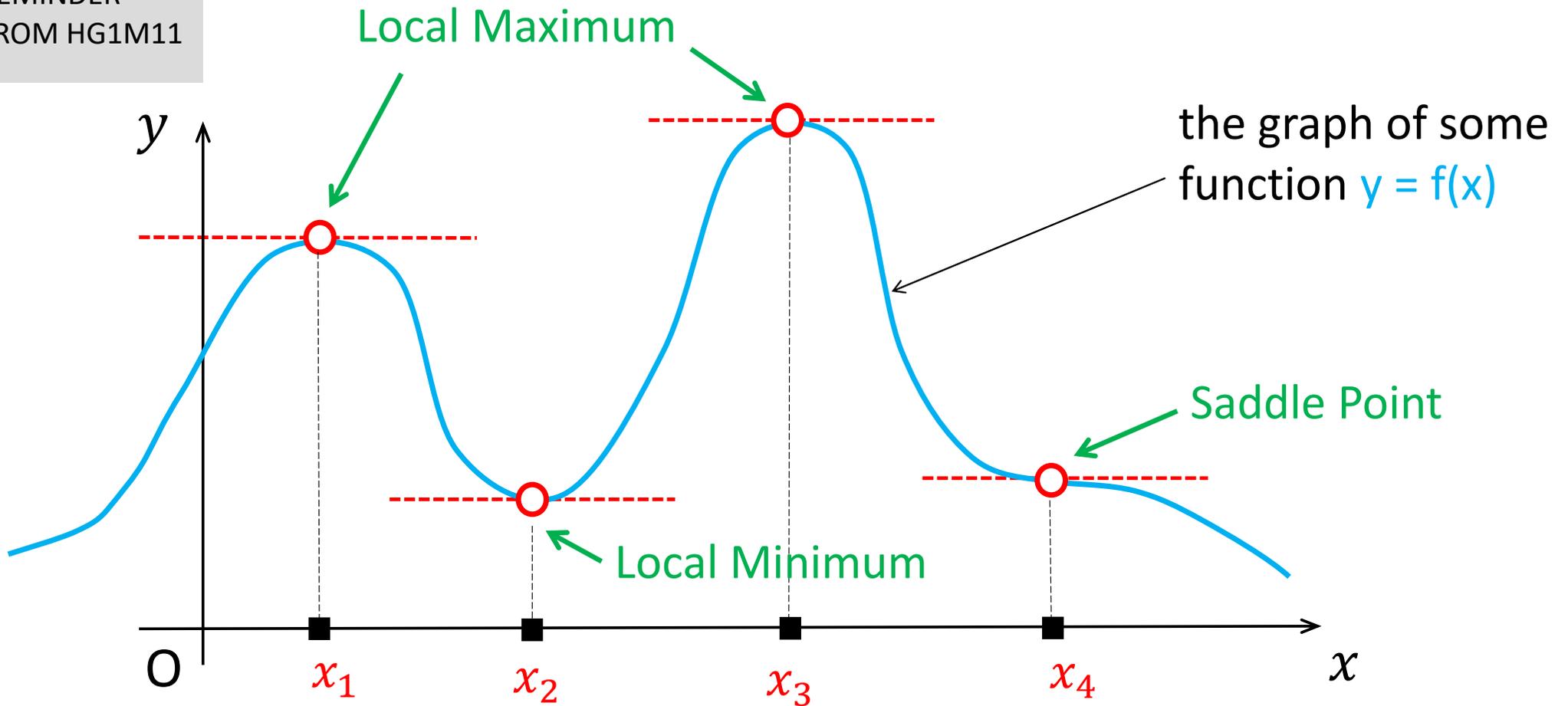


REMINDER
FROM HG1M11



CRITICAL POINTS: x_1, x_2, x_3, x_4

$$f'(x_j) = 0$$

$(j = 1, 2, 3, 4)$

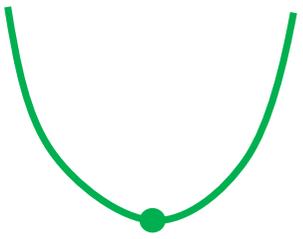
● If $f(x)$ is a function of x , **critical points** occur where

$$f'(x) = 0$$

● Let x_0 be a critical point, i.e. $f'(x_0) = 0$

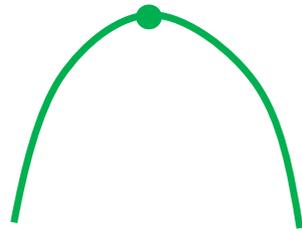
● To classify the critical points we must calculate $f''(x)$ and then evaluate it at each critical point

$$f''(x_0) > 0$$



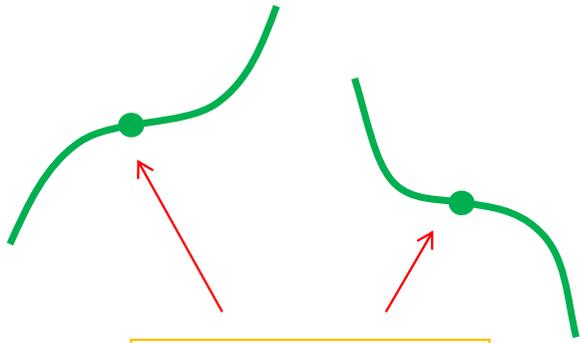
minimum

$$f''(x_0) < 0$$



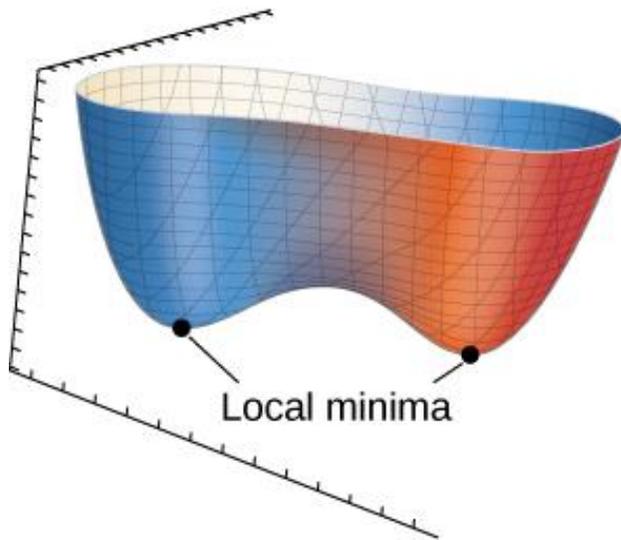
maximum

$$f''(x_0) = 0$$

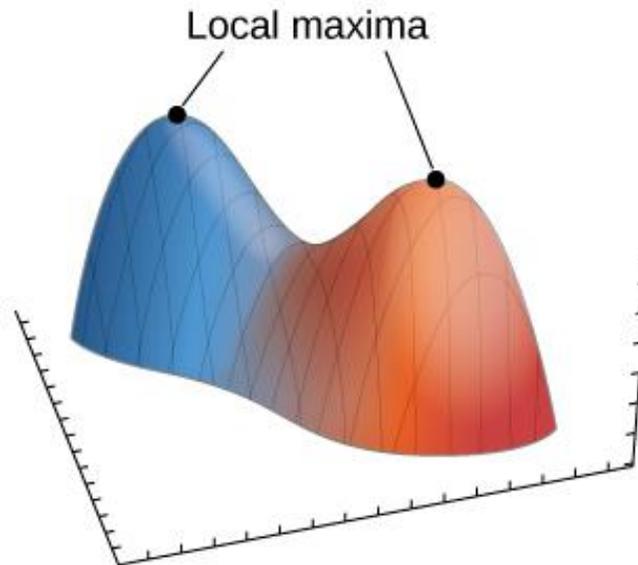


saddle points

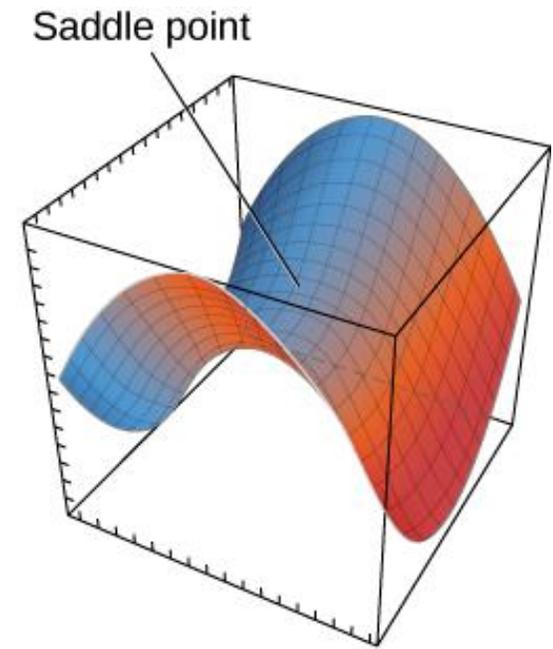
Extension to 2D



(a)



(b)

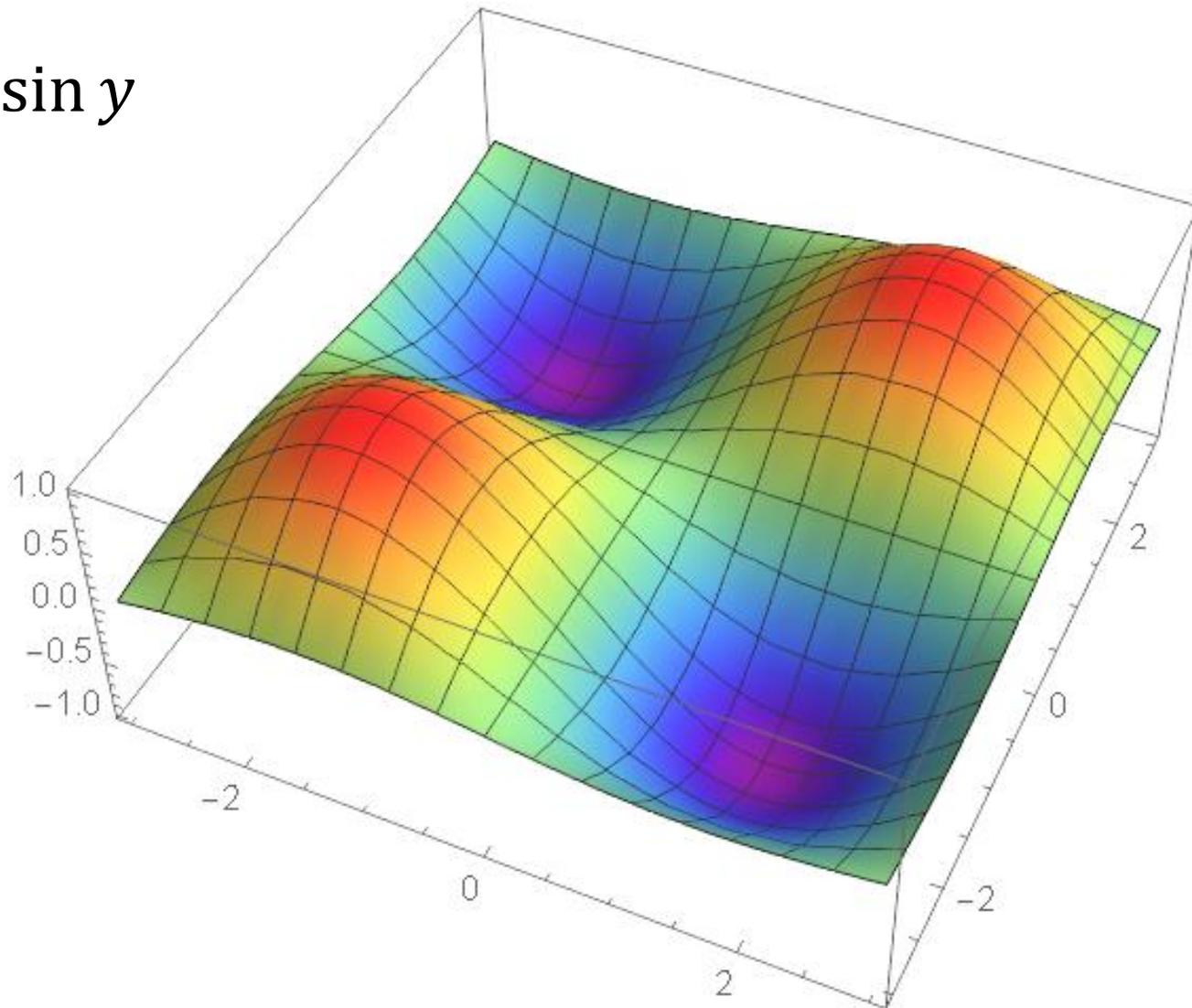


(c)



Example

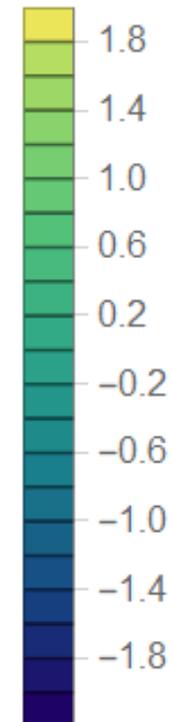
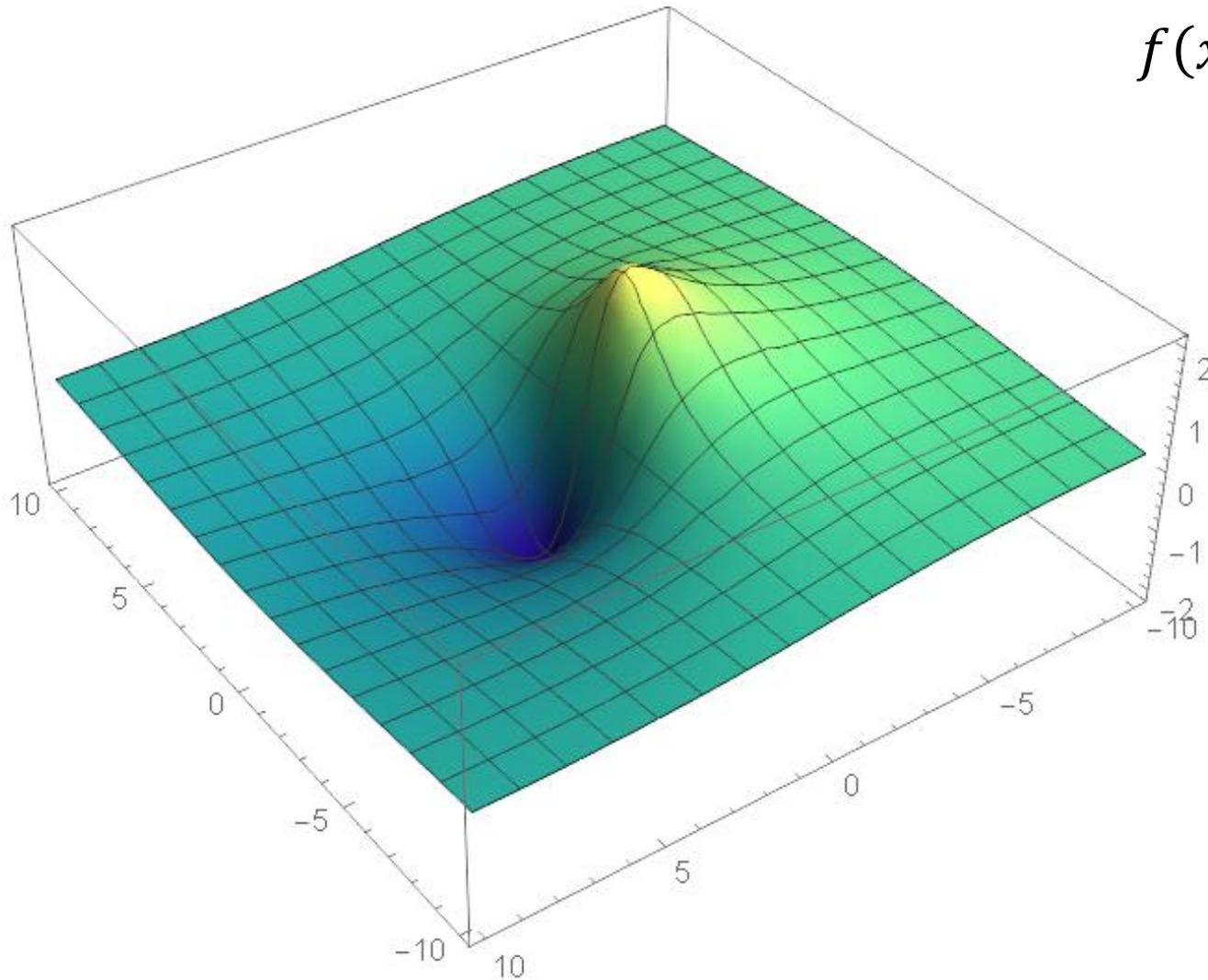
$$f(x, y) = \sin x \sin y$$



Example



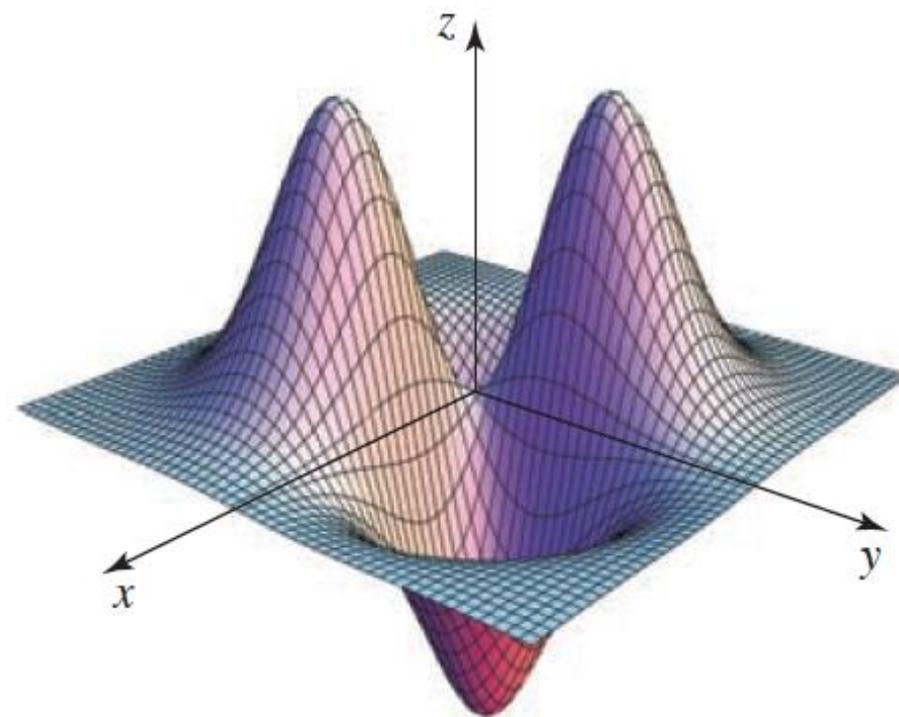
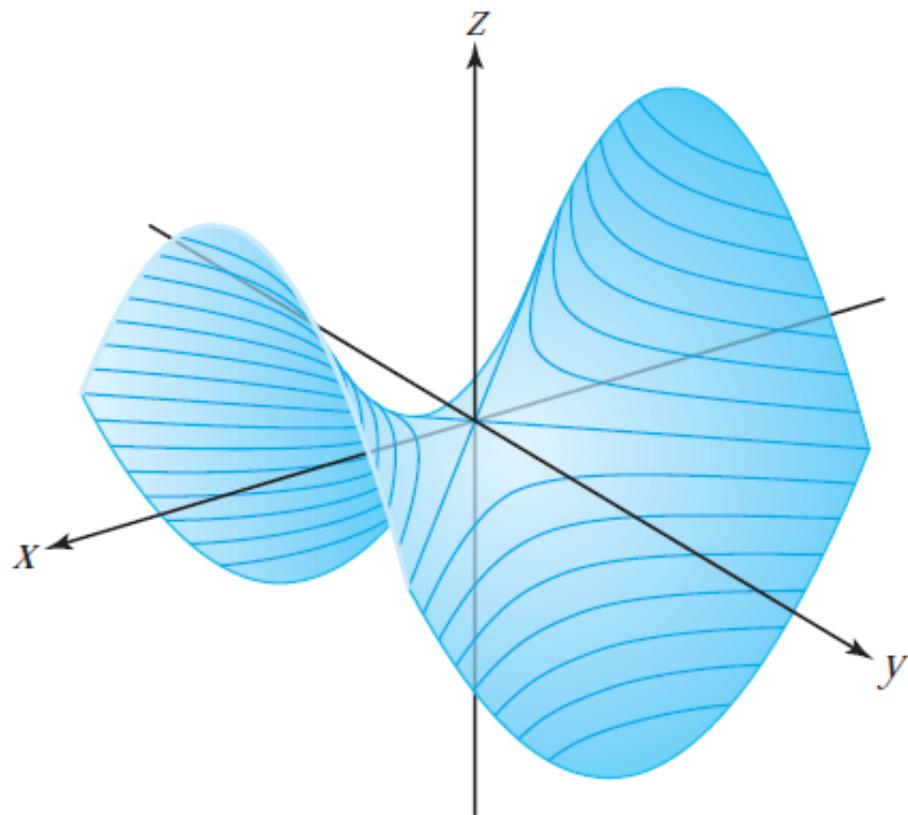
$$f(x, y) = \frac{-6y}{2 + x^2 + y^2}$$





Example

$$f(x, y) = x^2 - y^2$$



(c) Graph of $f(x, y) = -xye^{-x^2-y^2}$

2.7 Stationary Points for Functions of Two Variables

There are three types of stationary (or critical) points on surfaces: **maxima**, **minima** and **saddle points**.

We now show how to locate and classify them.

2.7.1 Conditions for Stationary Points

Step One

For any stationary point of the function $f(x, y)$, we must have

$$\frac{\partial f}{\partial x} = 0 \quad \mathbf{AND} \quad \frac{\partial f}{\partial y} = 0.$$

So we calculate these derivatives and find which values of x and y make them zero.

Step Two

We calculate the three second-order partial derivatives at each of these points, and then calculate the value of the '*Hessian*' Δ , where

$$\Delta = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \left(\frac{\partial^2 f}{\partial x^2} \right) \left(\frac{\partial^2 f}{\partial y^2} \right) - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2.$$

2.7.1 Conditions for stationary points (ctd..)

Step Three

- If $\Delta > 0$ **AND** $\frac{\partial^2 f}{\partial x^2} > 0$ (or $\frac{\partial^2 f}{\partial y^2} > 0$)

then the stationary point is a local **minimum**.

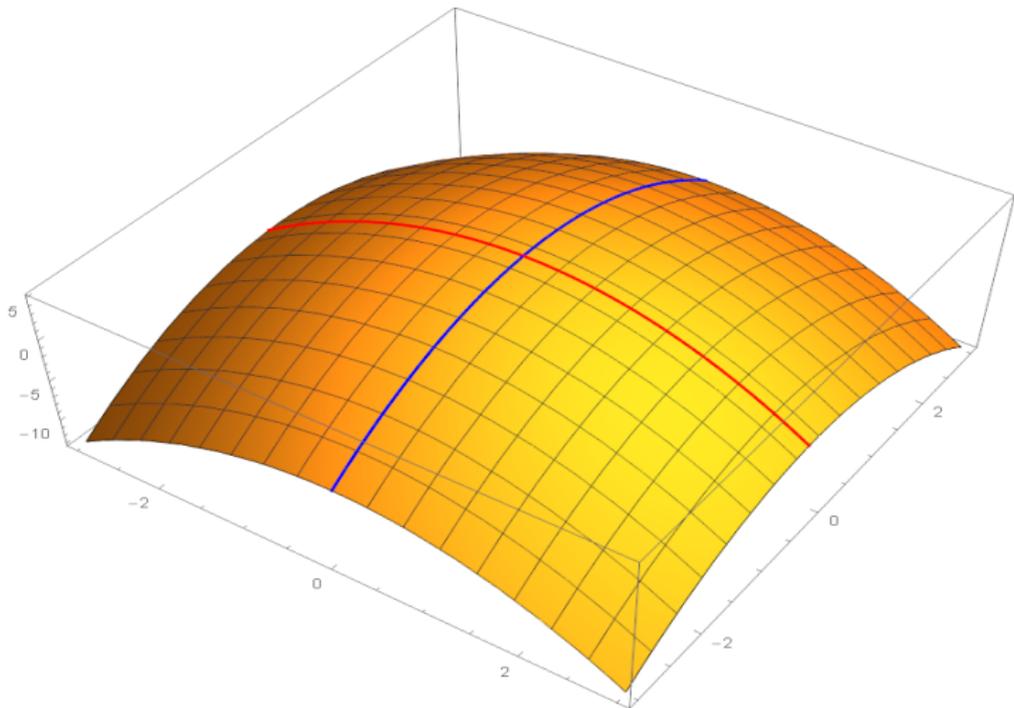
- If $\Delta > 0$ **AND** $\frac{\partial^2 f}{\partial x^2} < 0$ (or $\frac{\partial^2 f}{\partial y^2} < 0$)

then the stationary point is a local **maximum**.

- If $\Delta < 0$ then the stationary point is a **saddle point**.
- If $\Delta = 0$ the procedure is inconclusive.

Example 2.7.1

Verify that $(0, 0)$ is a stationary point of $f(x, y) = 5 - x^2 - y^2$, and determine its classification (as a maximum, minimum or saddle point).



Solution

$$\frac{\partial f}{\partial x} = -2x, \quad \frac{\partial f}{\partial y} = -2y.$$

Both derivatives vanish when $x = 0$ and $y = 0$.

The only stationary point for this function is the origin $(0, 0)$.

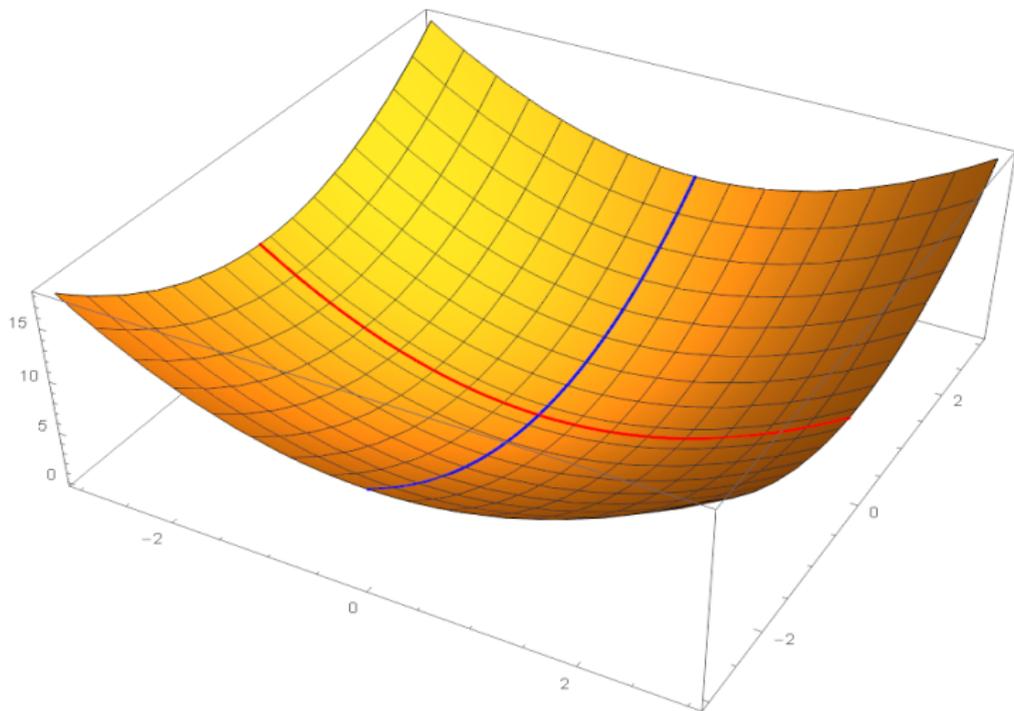
$$\text{Also, } \frac{\partial^2 f}{\partial x^2} = -2 = \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 f}{\partial x \partial y} = 0,$$

$$\text{and so } \Delta = \left(\frac{\partial^2 f}{\partial x^2} \right) \left(\frac{\partial^2 f}{\partial y^2} \right) - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2 = (-2)(-2) - 0 = 4 > 0.$$

Further, $\frac{\partial^2 f}{\partial x^2} = -2 < 0$ and so $(0, 0)$ is a local maximum.

Example 2.7.2

Verify that $(0, 0)$ is a critical point of $f(x, y) = x^2 + y^2$, and determine its classification.



Solution

$\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 2y$. Both derivatives vanish when $x = 0$ and $y = 0$. The only stationary point for this function is the origin $(0, 0)$.

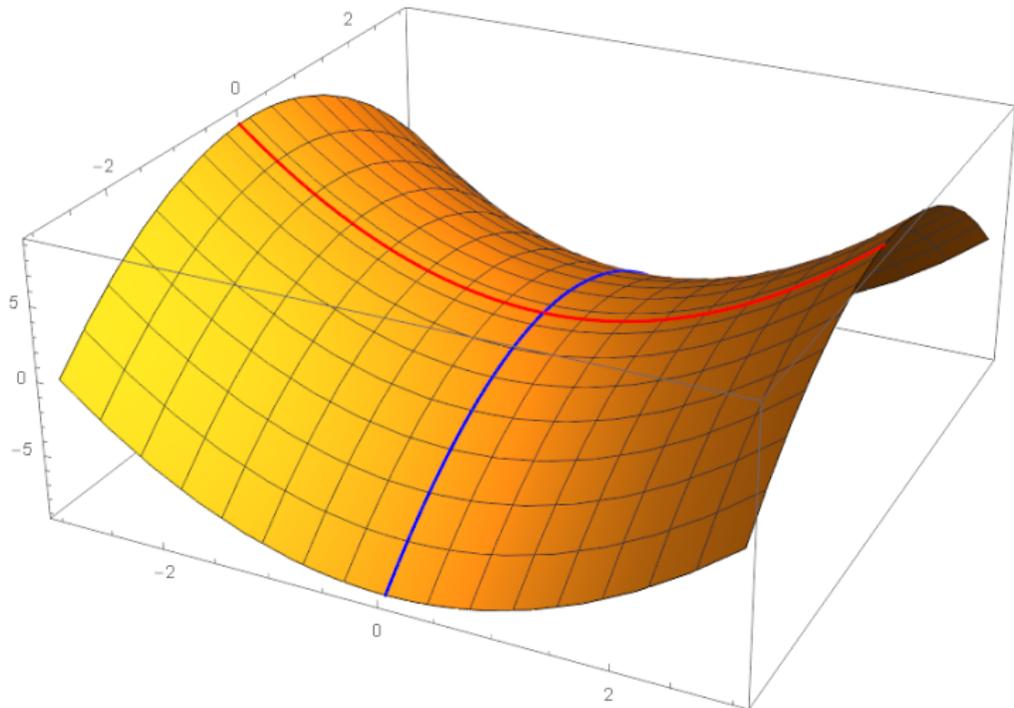
$$\text{Also, } \frac{\partial^2 f}{\partial x^2} = 2 = \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 f}{\partial x \partial y} = 0,$$

$$\text{and so } \Delta = \left(\frac{\partial^2 f}{\partial x^2} \right) \left(\frac{\partial^2 f}{\partial y^2} \right) - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2 = (2)(2) - 0 = 4 > 0.$$

Further, $\frac{\partial^2 f}{\partial x^2} = 2 > 0$ and so $(0, 0)$ is a local minimum.

Example 2.7.3

Verify that $(0, 0)$ is a critical point of $f(x, y) = x^2 - y^2$, and determine its classification.



Solution

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = -2y.$$

Both derivatives vanish when $x = 0$ and $y = 0$.

The only stationary point for this function is the origin $(0, 0)$.

$$\text{Also, } \frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial y^2} = -2, \quad \frac{\partial^2 f}{\partial x \partial y} = 0,$$

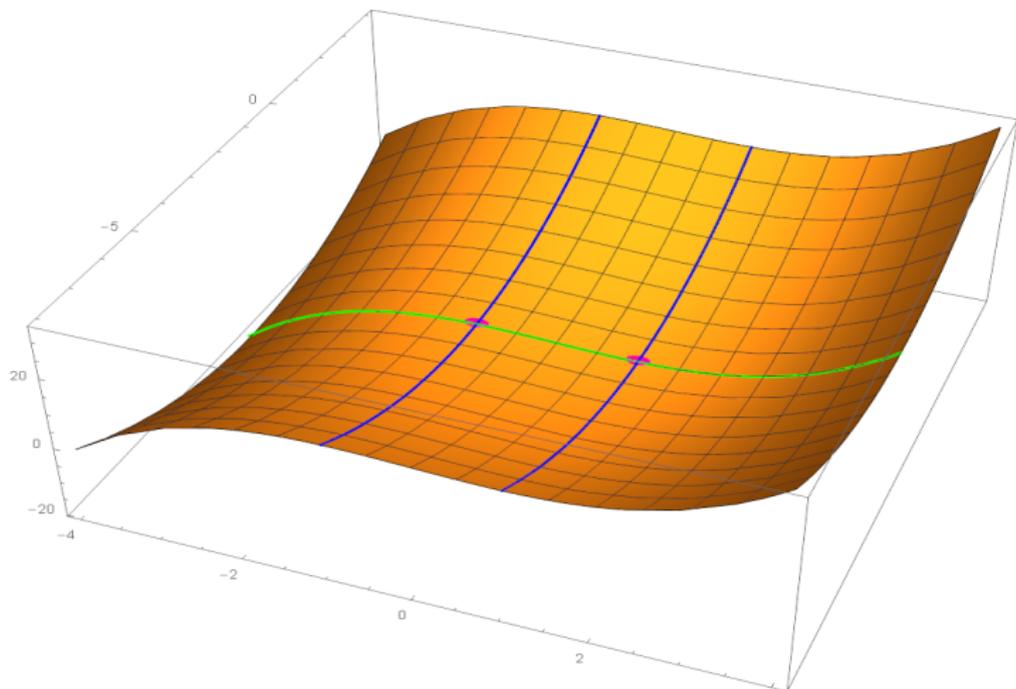
$$\text{and so } \Delta = \left(\frac{\partial^2 f}{\partial x^2} \right) \left(\frac{\partial^2 f}{\partial y^2} \right) - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2 = (2)(-2) - 0 = -4 < 0.$$

Hence, $(0, 0)$ is a saddle point.

Example 2.7.4

Locate and classify the stationary points of the function

$$f(x, y) = \frac{1}{3}x^3 - x + \frac{1}{2}y^2 + 2y.$$



Solution

$$\frac{\partial f}{\partial x} = x^2 - 1, \quad \frac{\partial f}{\partial y} = y + 2,$$

and so at a stationary point,

$$x^2 - 1 = 0 \quad \text{and} \quad y + 2 = 0.$$

that is, $x = 1$ or -1 , and $y = -2$.

The stationary points are at $(-1, -2)$ and $(1, -2)$.

$$\frac{\partial^2 f}{\partial x^2} = 2x, \quad \frac{\partial^2 f}{\partial y^2} = 1, \quad \frac{\partial^2 f}{\partial x \partial y} = 0, \quad \Delta = 2x.$$

We now consider each point in turn.

2.7.1 Example (ctd)

Point 1: at $(-1, -2)$, $\Delta = (2x).1 - 0 = -2 < 0$.

Hence, $(-1, -2)$ is a saddle point.

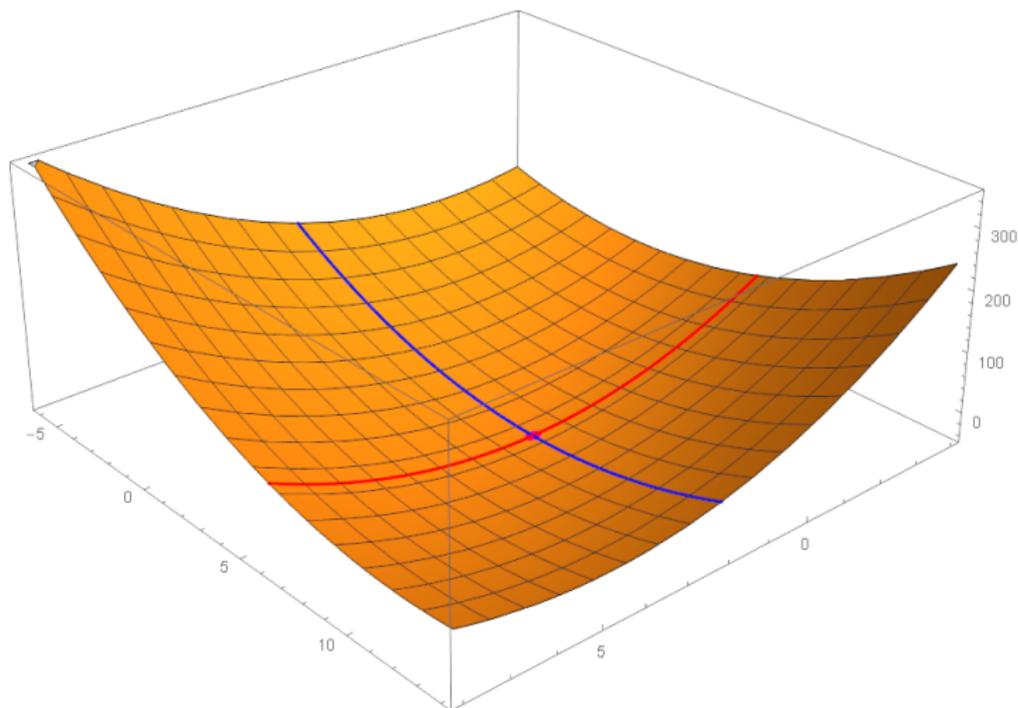
Point 2: at $(1, -2)$, $\Delta = (2x).1 - 0 = 2 > 0$.

Also, $\frac{\partial^2 f}{\partial x^2} = 2x = 2 > 0$, hence, $(1, -2)$ is a local minimum.

Example 2.7.5

Locate and classify the stationary points of

$$f(x, y) = 3x^2 - 2xy + y^2 - 8y.$$



Solution

$$\frac{\partial f}{\partial x} = 6x - 2y, \quad \frac{\partial f}{\partial y} = -2x + 2y - 8;$$

$$\frac{\partial^2 f}{\partial x^2} = 6, \quad \frac{\partial^2 f}{\partial x \partial y} = -2, \quad \frac{\partial^2 f}{\partial y^2} = 2.$$

Hence, $\Delta = 6 \times 2 - 4 = 8 > 0$, and $\frac{\partial^2 f}{\partial x^2} = 6 > 0$,
so any stationary point (if there is one) must be a minimum.
At the stationary point, we require

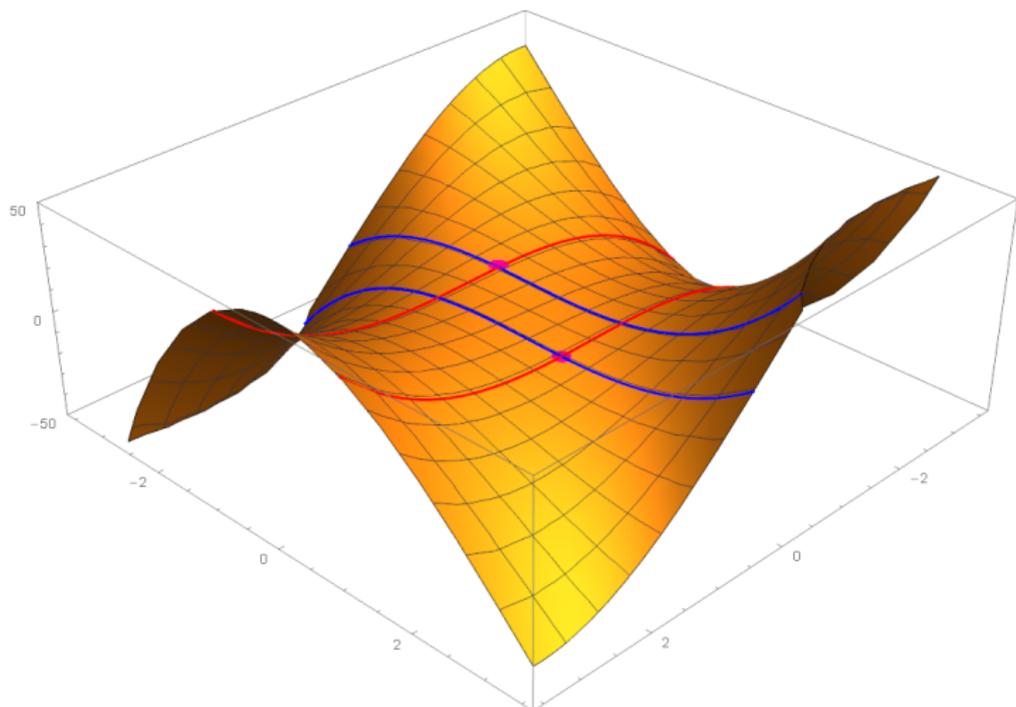
$$6x - 2y = 0 \quad \text{and} \quad -2x + 2y - 8 = 0.$$

Solving these simultaneous equations yields that the minimum occurs at the point (2, 6).

Example 2.7.6

Locate and classify the stationary points of

$$f(x, y) = x^3 + y^3 - 3xy^2 + 2x.$$



Solution

$$\frac{\partial f}{\partial x} = 3x^2 - 3y^2 + 2; \quad \frac{\partial f}{\partial y} = 3y^2 - 6xy.$$

Hence, at a stationary point,

$$3x^2 - 3y^2 + 2 = 0 \quad \text{and} \quad 3y^2 - 6xy = 0.$$

The second equation gives $y(y - 2x) = 0$.

The only possibilities are $y = 0$ and $y = 2x$.

Substituting $y = 0$ into the first equation gives $3x^2 + 2 = 0$, which has no real solutions for x .

Hence, the only possibility is $y = 2x$.

2.7.1 Example (ctd)

If $y = 2x$, then substituting into the first equation gives

$$3x^2 - 3(2x)^2 + 2 = 0,$$

which simplifies to $9x^2 = 2$,

with solutions

$$x = +\frac{\sqrt{2}}{3} \quad \text{and} \quad x = -\frac{\sqrt{2}}{3}.$$

Since $y = 2x$, the stationary points are $(\frac{\sqrt{2}}{3}, \frac{2\sqrt{2}}{3})$ and $(-\frac{\sqrt{2}}{3}, -\frac{2\sqrt{2}}{3})$.

2.7.1 Example (ctd..)

To classify them, we calculate that

$$\frac{\partial^2 f}{\partial x^2} = 6x; \quad \frac{\partial^2 f}{\partial x \partial y} = -6y; \quad \frac{\partial^2 f}{\partial y^2} = 6y - 6x.$$

$$\begin{aligned} \text{Hence, } \Delta &= \left(\frac{\partial^2 f}{\partial x^2} \right) \left(\frac{\partial^2 f}{\partial y^2} \right) - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2 \\ &= 6x(6y - 6x) - 36y^2 = 36(xy - x^2 - y^2). \end{aligned}$$

$$\text{At } \left(\frac{\sqrt{2}}{3}, \frac{2\sqrt{2}}{3} \right), \quad \Delta = 36 \left(\frac{4}{9} - \frac{2}{9} - \frac{8}{9} \right) = -24 < 0.$$

Hence, this point is a saddle point.

$$\text{At } \left(-\frac{\sqrt{2}}{3}, -\frac{2\sqrt{2}}{3} \right), \quad \Delta = 36 \left(\frac{4}{9} - \frac{2}{9} - \frac{8}{9} \right) = -24 < 0.$$

Hence, this point is also a saddle point.

Example 2.7.7

Calculate the dimensions of an open rectangular box of volume 4m^3 and with minimum surface area.

Solution. Suppose the box has sides of length x , y and z . Then its volume is $V = xyz$ and its surface area is $A = 2zy + 2zx + xy$. Note that the box is open, so one face is missing.

$$V = 4 \Rightarrow z = \frac{4}{xy}. \quad \text{Hence } A = \frac{8}{x} + \frac{8}{y} + xy,$$

$$\text{To minimise } A: \quad \frac{\partial A}{\partial x} = -\frac{8}{x^2} + y; \quad \frac{\partial A}{\partial y} = -\frac{8}{y^2} + x.$$

$$\text{So } -\frac{8}{x^2} + y = 0 \quad \text{and} \quad -\frac{8}{y^2} + x = 0.$$

$$\text{Hence } \frac{x^2y}{y^2x} = \frac{8}{8} = 1 \Rightarrow \frac{x}{y} = 1, \quad \text{or } x = y.$$

$$x^2y = 8 \Rightarrow x^3 = 8, \quad \text{so } x = y = 2. \quad \text{Finally, } xyz = 4 \Rightarrow z = 1.$$

N.B. We should really check that the surface area is *minimised*.

$$\frac{\partial^2 A}{\partial x^2} = \frac{16}{x^3} = 2 > 0.$$

Also,

$$\frac{\partial^2 A}{\partial y^2} = 2 \quad \text{and} \quad \frac{\partial^2 A}{\partial x \partial y} = 1$$

so that

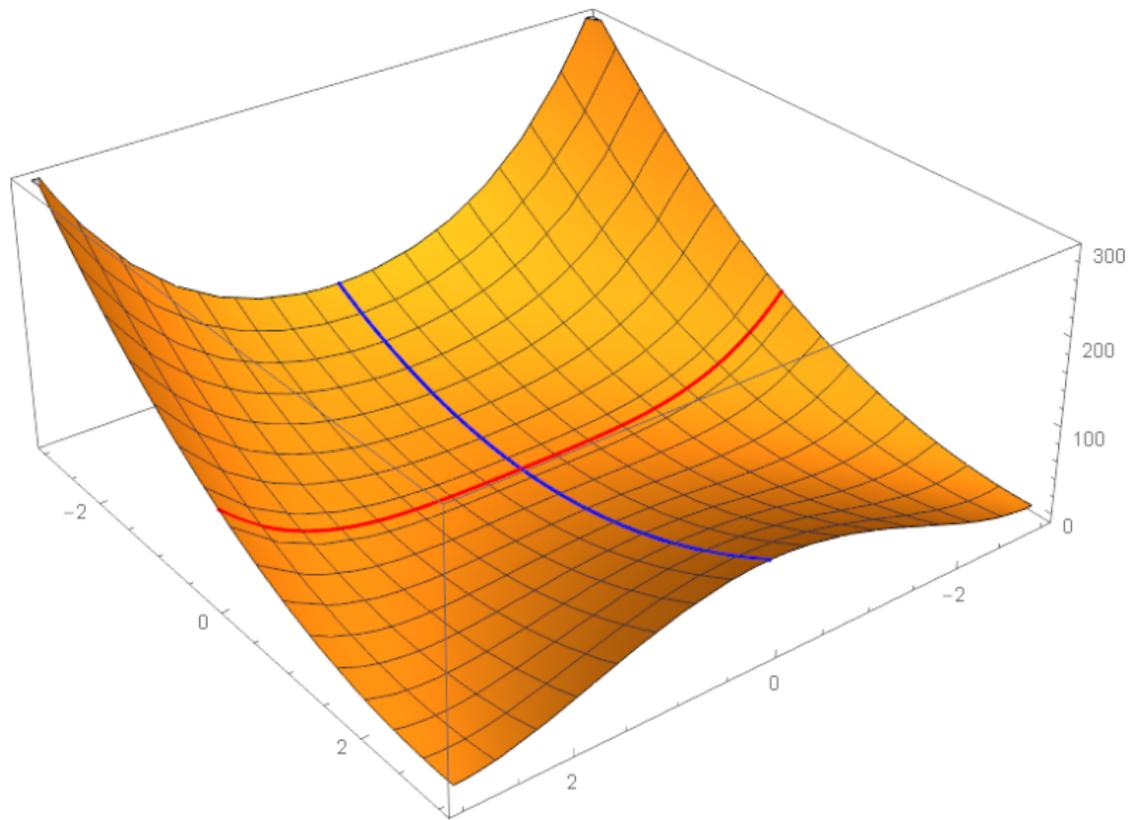
$$\Delta = \left(\frac{\partial^2 A}{\partial x^2} \right) \left(\frac{\partial^2 A}{\partial y^2} \right) - \left(\frac{\partial^2 A}{\partial x \partial y} \right)^2 = 4 - 1 = 3.$$

$$\text{i.e. } \Delta > 0.$$

Hence, all of the conditions for this stationary point to be a minimum are satisfied.

Example B.3: The function $f(x, y) = (x^2 - 3y)^2 + 3y$ has

- (I). One maximum at $(0, -1/6)$
- (II). One minimum at $(3, 3)$
- (III). One saddle point at $(0, 0)$
- (IV). No stationary points
- (V). None of these



Example B.5: Find and classify the stationary points of the function

$$f(x, y) = -x^2 - 12y + x^2y + y^3.$$

