2.10 Integration of PDEs

In a later part of the module we shall need to 'integrate' partial derivatives.

For example, consider the **ordinary** differential equation $\dfrac{df}{dx} = 1$.

Its solution is $f(x) = x + c$, $\quad$ (c is the 'constant of integration').

But suppose now that f is a function of two variables x and y, so that $f = f(x, y)$. Suppose also that we were given the **partial** differential equation

$$\frac{\partial f}{\partial x} = 1.$$

What is the solution this time?

2.10 Integration of PDEs (ctd)

Given $\frac{\partial f}{\partial x} = 1$, clearly, $f(x, y) = x + c$ is still a solution.

But is this the most general form of the solution?

Recall that differentiating partially with respect to x treats y as a constant (and vice versa). So in the solution we have just obtained, **the constant** c **might depend upon** y **and nothing will change**.

If we write

$$f(x, y) = x + g(y),$$

then

$$\frac{\partial f}{\partial x} = \frac{\partial x}{\partial x} + \frac{\partial g(y)}{\partial x} = 1 + 0 = 1.$$

So when integrating partial derivatives, the 'constant' of integration will generally depend upon the **other** variable.

Example 2.10.1

Find the most general form of the function $f = f(x, y)$ which satisfies

$$\text{(a)}\ \frac{\partial^2 f}{\partial x \partial y} = 0\,; \qquad \text{(b)}\ \frac{\partial^2 f}{\partial x \partial y} = x^2 y.$$

Solution: (a) $\dfrac{\partial^2 f}{\partial x \partial y} = 0 \Rightarrow \dfrac{\partial}{\partial x}\underbrace{\left(\dfrac{\partial f}{\partial y}\right)}_{g(x,y)} = 0$ $\qquad \dfrac{\partial g}{\partial x} = 0$

Hence $g(x,y) = G(y)$ for some arbitrary function $G(y)$

Recalling the notation introduced earlier

$$\frac{\partial f}{\partial y} = G(y) \Rightarrow f(x,y) = \underbrace{\int G(y)\,dy}_{H(y)} + h(x)$$

arbitrary function

In conclusion, the answer to (a) is

$f(x,y) = h(x) + H(y)$ for some arbitrary functions h and H

(b) $\dfrac{\partial^2 f}{\partial x \partial y} = x^2 y \Rightarrow \dfrac{\partial}{\partial x} \underbrace{\left(\dfrac{\partial f}{\partial y}\right)}_{g(x,y)} = x^2 y$ $\qquad \dfrac{\partial g}{\partial x} = x^2 y$

Integrate w.r.t. x the last equation: $g(x,y) = \int x^2 y \, dx + \overset{\text{arbitrary}}{G(y)}$

$$\Rightarrow \underbrace{g(x,y)}_{\frac{\partial f}{\partial y} = \frac{1}{3} x^3 y + G(y)} = \frac{1}{3} x^3 y + G(y)$$

$$\frac{\partial f}{\partial y} = \frac{1}{3} x^3 y + G(y) \Rightarrow f(x,y) = \frac{1}{6} x^3 y^2 + \underbrace{\int G(y)\, dy}_{H(y)} + \underset{\text{arbitrary}}{h(x)}$$

In conclusion, the answer to (b) is

$$f(x,y) = \frac{1}{6} x^3 y^2 + h(x) + H(y)$$

arbitrary functions

Example 2.10.2

Find the most general form of the function $f = f(x, y)$ which simultaneously satisfies

$$\frac{\partial f}{\partial x} = 2x^3y^4 + x \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 2x^4y^3 + y$$

Solution: Integrate one of the equations & then substitute into the other

$$\frac{\partial f}{\partial x} = 2x^3y^4 + x \Rightarrow f(x,y) = \frac{1}{2}x^4y^4 + \frac{x^2}{2} + g(y)$$

$g(y)$ arbitrary/unknown at this stage

substitute into the second equation

$$\frac{\partial f}{\partial y} = 2x^4y^3 + 0 + g'(y) = 2x^4y^3 + y \Rightarrow g'(y) = y \Rightarrow g(y) = \frac{y^2}{2} + C$$

Final answer:

$$f(x,y) = \frac{1}{2}x^4y^4 + \frac{1}{2}(x^2 + y^2) + C$$

(C = arbitrary constant)

Example 2.10.3

Find the most general form of the function $f = f(x, y)$ which simultaneously satisfies

$$\frac{\partial f}{\partial x} = 2xy\cos(x^2y) + x^2 \qquad \text{and} \qquad \frac{\partial f}{\partial y} = x^2\cos(x^2y) + y^2\,.$$

Solution: We'll follow the same strategy as before.

$$\frac{\partial f}{\partial x} = 2xy\cos(x^2y) + x^2 \Rightarrow f(x,y) = \sin(x^2y) + \frac{1}{3}x^3 + g(y)$$

↑ arbitrary function....

substitute in the second equation

$$\frac{\partial f}{\partial y} = x^2\cos(x^2y) + 0 + g'(y) = x^2\cos(x^2y) + y^2 \Rightarrow g'(y) = y^2$$

$$\Rightarrow g(y) = \frac{1}{3}y^3 + C$$

Final answer:

$$f(x,y) = \sin(x^2y) + \frac{1}{3}(x^3 + y^3) + C \qquad (C = \text{arbitrary constant})$$

Example 2.10.4

Find the most general form of the function $f = f(x, y)$ which simultaneously satisfies

$$\frac{\partial f}{\partial x} = 2x\mathrm{e}^{xy} + x^2y\mathrm{e}^{xy} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = x^3\mathrm{e}^{xy} + 2y\,.$$

Solution: It is easier to start integrating the 2nd equation.

$$\frac{\partial f}{\partial y} = x^3 e^{xy} + 2y \Rightarrow f(x,y) = x^2 e^{xy} + y^2 + g(x)$$

arbitrary function.....

substitute into the first equation

$$\frac{\partial f}{\partial x} = \cancel{2xe^{xy}} + \cancel{x^2ye^{xy}} + 0 + g'(x) = \cancel{2xe^{xy}} + \cancel{x^2ye^{xy}} \Rightarrow g'(x) = 0$$

OR $g(x) = C$

In conclusion:

$$f(x,y) = x^2e^{xy} + y^2 + C$$

(C = arbitrary constant)

An important observation

If $P(x, y)$ and $Q(x, y)$ are given expressions, the existence of a function $f = f(x, y)$ which satisfies

$$\frac{\partial f}{\partial x} = P(x, y) \qquad \text{and} \qquad \frac{\partial f}{\partial y} = Q(x, y)$$

"Consistency condition"

requires that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. **If this condition is not satisfied then we cannot find any function with the above properties.**

E.G.: Consider an $f(x, y)$ s.t. $\frac{\partial f}{\partial x} = x^2 + xy^3$ and $\frac{\partial f}{\partial y} = x^4$

In this case $P(x, y) = x^2 + xy^3$ and $Q(x, y) = x^4$, so

$$\frac{\partial P}{\partial y} = 3xy^2 \neq 4x^3 = \frac{\partial Q}{\partial x}$$

Check that the consistency condition is satisfied in the previous examples