

# Recap: small changes (N=2)

$f(x_0, y_0)$  = value of  $f$  at some reference point  $(x_0, y_0)$

$$f(x_0 + \Delta x, y_0 + \Delta y), \quad |\Delta x|, |\Delta y| \ll 1$$

Change in  $f$ :  $\Delta f = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$

We are actually interested in **relative changes**:

$$\varepsilon_f = \frac{\Delta f}{f_0}, \quad \varepsilon_x = \frac{\Delta x}{x_0}, \quad \varepsilon_y = \frac{\Delta y}{y_0}$$

Main formula (for the case N=2):

$$\varepsilon_f \simeq \left( \frac{x_0 f_x^0}{f_0} \right) \varepsilon_x + \left( \frac{y_0 f_y^0}{f_0} \right) \varepsilon_y$$

# Justification of the formula:

Taylor expansion theorem (as given previously)

$$f(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) + \dots \quad (2.1)$$

We expand about the reference value, so

$$a \longrightarrow x_0, \quad b \longrightarrow y_0, \quad x \longrightarrow x_0 + \Delta x, \quad y \longrightarrow y_0 + \Delta y$$

Hence, we can re-write (2.1) as

$$f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

or  $\Delta f = f_x^0 \Delta x + f_y^0 \Delta y$ . Divide through by  $f_0 \equiv f(x_0, y_0)$ :

$$\begin{aligned} \frac{\Delta f}{f_0} &= \frac{f_x^0}{f_0} \Delta x + \frac{f_y^0}{f_0} \Delta y = \frac{f_x^0}{f_0} \frac{x_0}{x_0} \frac{\Delta x}{x_0} + \frac{f_y^0}{f_0} \frac{y_0}{y_0} \frac{\Delta y}{y_0} \\ &= \left( \frac{x_0 f_x^0}{f_0} \right) \varepsilon_x + \left( \frac{y_0 f_y^0}{f_0} \right) \varepsilon_y \end{aligned}$$

### Example B.6:

An **ideal gas** is situated in a closed chamber. Find the approximate change in the **pressure** if the **temperature** of the gas and the **volume** of the chamber increase by 1% and 2%, respectively.

Solution: **The ideal gas law:**

$$pV = nRT \quad \longrightarrow \quad p = nR \left( \frac{T}{V} \right) \equiv f(T, V)$$

Apply the previous formula:  $\varepsilon_f \simeq \left( \frac{T_0 f_T^0}{f_0} \right) \varepsilon_T + \left( \frac{V_0 f_V^0}{f_0} \right) \varepsilon_V$

We are given:  $\varepsilon_T = 1\%$  and  $\varepsilon_V = 2\%$ .

All we are left to do is calculate the **pre-factors** of  $\varepsilon_T$  and  $\varepsilon_V$  above.

Recall that  $(T_0, V_0)$  is the reference value – if this is not given explicitly it is perhaps because it plays no role in the final result.

The partial derivatives of  $f(T, V) = nR \left( \frac{T}{V} \right)$ :

$$f_T^0 \equiv f_T(T_0, V_0) = \frac{nR}{V_0} \quad \text{and} \quad f_V^0 \equiv f_V(T_0, V_0) = -nR \left( \frac{T_0}{V_0^2} \right)$$

Evaluate the pre-factors:

$$\frac{T_0 f_T^0}{f_0} = T_0 \cdot \frac{nR}{V_0} \cdot \frac{V_0}{nRT_0} = 1$$

$$\frac{V_0 f_V^0}{f_0} = V_0 \cdot \left( -\frac{nRT_0}{V_0^2} \right) \cdot \frac{V_0}{nRT_0} = -1$$

Hence

$$\begin{aligned} \varepsilon_f &= \varepsilon_T - \varepsilon_V \\ &= 1\% - 2\% = -1\%. \end{aligned}$$

This means that the pressure **decreases** by 1%.

## 2.7 Small Errors (ctd)

Sometimes we are more interested in **maximum** changes (or errors) in functions, allowing the relative changes (or errors) in the independent variables to be a decrease or an increase.

In this case we speak of the **maximum percentage relative change** (or error). This is defined by taking absolute values of all the terms on the RHS in the formulae on the previous slide:

$$N = 1 : \quad \varepsilon_f^{max} \simeq \left| \frac{x_0 f'(x_0)}{f_0} \right| |\varepsilon_x|$$

$$N = 2 : \quad \varepsilon_f^{max} \simeq \left| \frac{x_0 f_x^0}{f_0} \right| |\varepsilon_x| + \left| \frac{y_0 f_y^0}{f_0} \right| |\varepsilon_y|$$

$$N = 3 : \quad \varepsilon_f^{max} \simeq \left| \frac{x_0 f_x^0}{f_0} \right| |\varepsilon_x| + \left| \frac{y_0 f_y^0}{f_0} \right| |\varepsilon_y| + \left| \frac{z_0 f_z^0}{f_0} \right| |\varepsilon_z|$$

### Example B.6 revisited:

An ideal gas is situated in a closed chamber. Find the **maximum** percentage change in the **pressure** if the **temperature** of the gas and the **volume** of the chamber change by  $\pm 1\%$  and  $\pm 2\%$ , respectively.

### Solution:

Apply the formula: 
$$\varepsilon_f \simeq \left| \frac{T_0 f_T^0}{f_0} \right| |\varepsilon_T| + \left| \frac{V_0 f_V^0}{f_0} \right| |\varepsilon_V|$$

We are given that:  $\varepsilon_T = \pm 1\%$  and  $\varepsilon_V = \pm 2\%$ . Hence,

$$\varepsilon_f^{max} = |(+1)| \cdot |1\%| + |(-1)| \cdot |2\%| = 3\%$$

The **maximum** change in pressure is 3%

### Example B.7:

The sides of a right triangles increase by 1% and 3%, respectively. Find the corresponding approximate change in the area of the triangle.

Solution:    **Area of right triangle:**

$$A = \frac{1}{2}bc \equiv f(b, c)$$

Apply the previous formula:  $\varepsilon_f \simeq \left( \frac{b_0 f_b^0}{f_0} \right) \varepsilon_b + \left( \frac{c_0 f_c^0}{f_0} \right) \varepsilon_c$

We are given:  $\varepsilon_b = 1\%$  and  $\varepsilon_c = 3\%$ .

All we are left to do is calculate the **pre-factors** of  $\varepsilon_b$  and  $\varepsilon_c$  above.

Recall that  $(b_0, c_0)$  is the reference value.....

The partial derivatives of  $f(b, c) = \frac{1}{2}bc$ :

$$f_b^0 \equiv f_b(b_0, c_0) = \frac{1}{2}c_0 \quad \text{and} \quad f_c^0 \equiv f_c(b_0, c_0) = \frac{1}{2}b_0$$

Evaluate the pre-factors:

$$\frac{b_0 f_b^0}{f_0} = b_0 \cdot \left( \frac{1}{2}c_0 \right) \cdot \frac{2}{b_0 c_0} = 1$$

$$\frac{c_0 f_c^0}{f_0} = c_0 \cdot \left( \frac{1}{2}b_0 \right) \cdot \frac{2}{b_0 c_0} = 1$$

Hence

$$\begin{aligned} \varepsilon_f &= \varepsilon_b + \varepsilon_c \\ &= 1\% + 3\% = 4\%. \end{aligned}$$

This means that the area of the triangle **increases** by 4%.

### Example B.8:

The area of an arbitrary triangle is given in terms of the lengths,  $b$  and  $c$ , of **two of its sides** and the **angle**,  $\alpha$ , between the sides as

$$A = \frac{1}{2}bc \sin \alpha .$$

If  $\alpha_0 = \frac{\pi}{4}$  and the errors in the measurements of  $b$ ,  $c$  and  $\alpha$  are

$$0.5\%, \quad -0.25\%, \quad \text{and} \quad 1.8\% ,$$

respectively, then find the approximate change (i.e. the **percentage error**) in the area of this triangle.

Express your answer to 2 significant figures.

# Solution:

Area of an arbitrary triangle:

$$A = \frac{1}{2}bc \sin \alpha \equiv f(b, c, \alpha)$$

Apply formula for  $N = 3$ : 
$$\varepsilon_f \simeq \left( \frac{b_0 f_b^0}{f_0} \right) \varepsilon_b + \left( \frac{c_0 f_c^0}{f_0} \right) \varepsilon_c + \left( \frac{\alpha_0 f_\alpha^0}{f_0} \right) \varepsilon_\alpha$$

We are given:  $\varepsilon_b = 0.5\%$ ,  $\varepsilon_c = -0.25\%$  and  $\varepsilon_\alpha = 1.8\%$ .

All we are left to do is calculate the **pre-factors**, etc.

Recall that  $(b_0, c_0, \alpha_0)$  is the reference value. The fact that  $b_0$  and  $c_0$  are not given suggests that perhaps we don't need them.

The partial derivatives of  $f(b, c, \alpha) = \frac{1}{2}bc \sin \alpha$  (next slide)

$$f_b^0 \equiv f_b(b_0, c_0, \alpha_0) = \frac{1}{2} c_0 \sin \alpha_0, \quad f_c^0 \equiv f_c(b_0, c_0, \alpha_0) = \frac{1}{2} b_0 \sin \alpha_0$$

$$f_\alpha^0 \equiv f_\alpha(b_0, c_0, \alpha_0) = \frac{1}{2} b_0 c_0 \cos \alpha_0$$

Pre-factors:

$$\frac{b_0 f_b^0}{f_0} = b_0 \cdot \left( \frac{1}{2} c_0 \sin \alpha_0 \right) \cdot \frac{2}{b_0 c_0 \sin \alpha_0} = 1$$

$$\frac{c_0 f_c^0}{f_0} = c_0 \cdot \left( \frac{1}{2} b_0 \sin \alpha_0 \right) \cdot \frac{2}{b_0 c_0 \sin \alpha_0} = 1$$

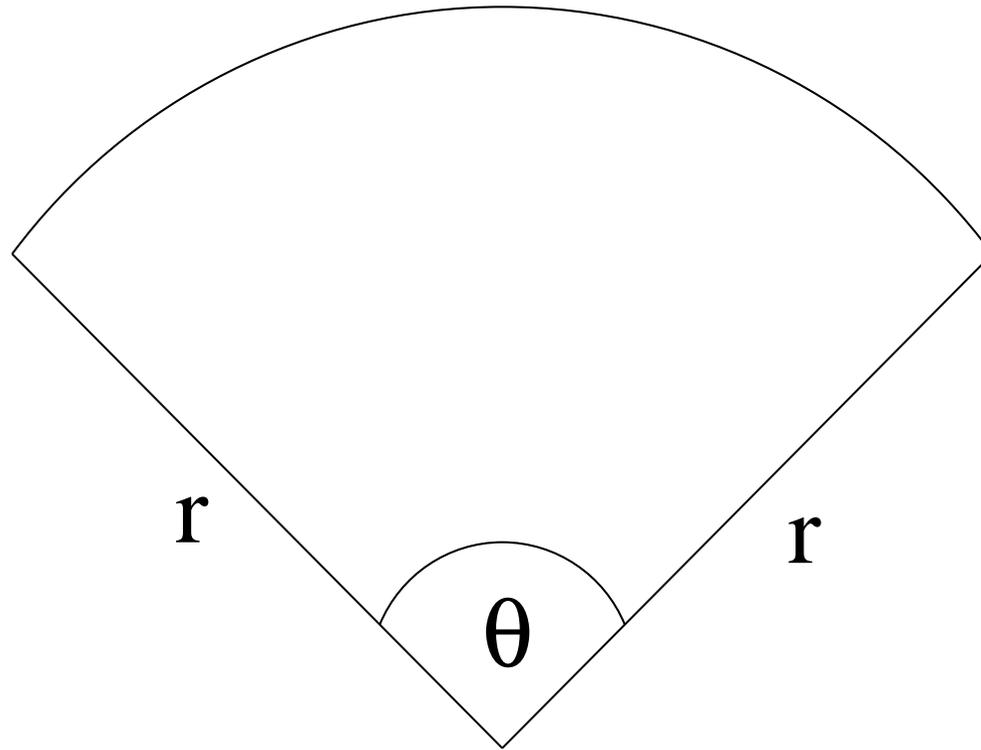
$$\frac{\alpha_0 f_\alpha^0}{f_0} = \alpha_0 \cdot \left( \frac{1}{2} b_0 \overset{c_0}{\cos \alpha_0} \right) \cdot \frac{2}{b_0 c_0 \sin \alpha_0} = \frac{\alpha_0}{\tan \alpha_0} = \frac{\pi}{4}$$

Hence

$$\begin{aligned} \varepsilon_f &= \varepsilon_b + \varepsilon_c + \frac{\pi}{4} \varepsilon_\alpha \\ &= 0.5\% - 0.25\% + \frac{\pi}{4} \times 1.8\% \simeq 1.7\%. \end{aligned}$$

This means that the area of the triangle **increases** by 1.7%.

## Example 2.7.1



What is the percentage error in the area  $A$  of the above sector if the actual value of  $r$  is 1% greater than the design value  $r_0$  and the actual value of  $\theta$  is 3% smaller than the design value  $\theta_0$ ?

**Solution.**  $A = \frac{1}{2} r^2 \theta = f(r, \theta)$

$$\varepsilon_A = \left( \frac{r_0 f_r}{f_0} \right) \varepsilon_r + \left( \frac{\theta_0 f_\theta}{f_0} \right) \varepsilon_\theta$$

We are given that:  $\varepsilon_r = 1\%$  and  $\varepsilon_\theta = -3\%$

Calculating the prefactors we get:

$$\frac{r_0 f_r}{f_0} = r_0 \left( r_0 \theta_0 \right) \frac{2}{r_0^2 \theta_0} = 2 \quad \Bigg| \quad \frac{\theta_0 f_\theta}{f_0} = \theta_0 \left( \frac{1}{2} r_0^2 \right) \frac{2}{r_0^2 \theta_0} = 1$$

Thus,

$$\begin{aligned} \varepsilon_A &= (2) \varepsilon_r + \varepsilon_\theta = 2(1\%) + (-3\%) \\ &= (2 - 3)\% = -1\% \end{aligned}$$

The area decreases by 1%

## Example 2.7.2

Suppose the design lengths of a box are  $x_0 = 2\text{m}$ ,  $y_0 = 1.5\text{m}$  and  $z_0 = 0.4\text{m}$ , but their measured lengths are  $x = 1.98\text{m}$ ,  $y = 1.51\text{m}$  and  $z = 0.41\text{m}$ . What is the percentage error in the volume  $V$ ? What about the maximum percentage error?

**Solution:**

$$V = xyz = f(x, y, z)$$

$$x_0 = 2\text{ m}$$

$$x_0 + \Delta x = 1.98\text{ m}$$

$$\frac{\Delta x}{x_0} = \frac{1.98 - 2}{2}$$

$$= -0.01$$

$$\varepsilon_x = \left( \frac{\Delta x}{x_0} \right) \times 100\%$$

$$y_0 = 1.5\text{ m}$$

$$y_0 + \Delta y = 1.51\text{ m}$$

$$\frac{\Delta y}{y_0} = \frac{1.51 - 1.5}{1.5}$$

$$= 0.0067$$

$$\varepsilon_y = \left( \frac{\Delta y}{y_0} \right) \times 100\%$$

$$z_0 = 0.4\text{ m}$$

$$z_0 + \Delta z = 0.41\text{ m}$$

$$\frac{\Delta z}{z_0} = \frac{0.41 - 0.4}{0.4}$$

$$= 0.025$$

$$\varepsilon_z = \left( \frac{\Delta z}{z_0} \right) \times 100$$

$$\epsilon_x = -1\%$$

$$\epsilon_y = 0.67\%$$

$$\epsilon_z = 2.5\%$$

Apply the formula:  $\epsilon_v = (\dots) \epsilon_x + (\dots) \epsilon_y + (\dots) \epsilon_z$   
( $N=3$ )

prefactors are all equal to 1 (CHECK IT!)

$$\epsilon_v = -1\% + 0.67\% + 2.5\% = 2.17\%$$

hence, volume increases by 2.17%

$$\begin{aligned}\epsilon_v^{\max} &= |-1\%| + |0.67\%| + |2.5\%| \\ &= (1 + 0.67 + 2.5)\% \\ &= 4.17\%\end{aligned}$$

# Chain-Rule Formulae

For multivariate functions there are several versions of the Chain Rule.

$F(u)$  = function of **one** real variable

$u = u(x, y)$  a function of **two** variables

Consider the **composite function**:

$$f(x, y) = F(u(x, y))$$

We have seen that

$$\frac{\partial f}{\partial x} = F'(u(x, y)) \frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y} = F'(u(x, y)) \frac{\partial u}{\partial y}$$

# Chain-Rule Formulae (ctd)

We are interested in two new cases.

## First case:

$f(x, y)$  = function of **two** variables

$x = x(t)$ ,  $y = y(t)$  two functions of **one** variable

Consider the composite function:

$$g(t) = f(x(t), y(t))$$

Question:

How can we calculate the (ordinary) derivative of  $g$  with respect to  $t$ ?

# Chain-Rule Formulae (ctd)

## Second case:

$f(x, y)$  = function of **two** variables

$x = x(u, v)$ ,  $y = y(u, v)$  two functions of **two** variables

Consider the composite function:

$$g(u, v) = f(x(u, v), y(u, v))$$

Question:

How can we calculate the partial derivatives

$$\frac{\partial g}{\partial u} \quad \text{and} \quad \frac{\partial g}{\partial v} ?$$

## 2.8 Total Derivatives

This corresponds to '**Case 1**' mentioned earlier

### Example 2.8.1

If  $f(x, y) = x^2 + y^3$ ,  $x(t) = \sqrt{t}$ ,  $y(t) = t^2$ , find  $g'(t)$ .

**One way** is to substitute:

$$g(t) = f(x(t), y(t)) = (\sqrt{t})^2 + (t^2)^3 = t + t^6.$$

$$\text{Hence } g'(t) = 1 + 6t^5.$$

## Alternative method - use the (new) Chain Rule

$$\frac{dg}{dt} = \frac{d}{dt}f(x(t), y(t)) = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.$$

$$[ \text{Reminder: } f(x, y) = x^2 + y^3, \quad x(t) = \sqrt{t}, \quad y(t) = t^2 ]$$

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 3y^2;$$

$$\frac{dx}{dt} = \frac{1}{2t^{\frac{1}{2}}}, \quad \frac{dy}{dt} = 2t.$$

$$\begin{aligned} \text{Hence } \frac{dg}{dt} &= \frac{d}{dt}f(x(t), y(t)) = 2x \frac{1}{2t^{\frac{1}{2}}} + 3y^2 2t \\ &= 2t^{\frac{1}{2}} \frac{1}{2t^{\frac{1}{2}}} + 3t^4 2t = 1 + 6t^5, \quad \text{as before.} \end{aligned}$$

## Example 2.8.2

Using the chain rule, find  $\frac{dg}{dt}$  where  $g(t) = f(x(t), y(t))$ ,

$$f(x, y) = x \cos y \text{ and } x(t) = t^2, \quad y(t) = 2t.$$

**Solution:**

$$\begin{aligned} g'(t) &= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = \cos(y)2t + (-x \sin y)2 \\ &= 2t \cos(2t) - 2t^2 \sin(2t). \end{aligned}$$

The same result can be obtained by calculating  $g(t)$  explicitly

$$g(t) = f(x(t), y(t)) = t^2 \cos(2t)$$

and differentiating this expression with respect to  $t$ .

### Example 2.8.3

Suppose the coordinates of a particle at time  $t$  satisfy

$$x(t) = a \cos(\omega t), \quad y(t) = a \sin(\omega t)$$

and that we consider  $f(x, y) = x^2 + y^2$ .

Show that  $f(x, y)$  is constant along the curve  $x(t), y(t)$ .

**Solution:**

$$\begin{aligned} \frac{d}{dt} f(x(t), y(t)) &= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \\ &= (2x)(-a\omega \sin(\omega t)) + (2y)(a\omega \cos(\omega t)) \\ &= -2a^2\omega \sin(\omega t) \cos(\omega t) + 2a^2\omega \sin(\omega t) \cos(\omega t) = 0. \end{aligned}$$

Hence  $f$  is constant.

At  $t = 0$ ,  $x = a$  and  $y = 0$ . Hence at  $t = 0$ ,  $f = a^2$ .

Since  $f$  is constant along the curve  $x(t), y(t)$ ,  $f = a^2$  for all  $t$ .