



HG1MC1

Mathematics for Chemistry 1

Chapter 6 (I) **(Probability & statistics)**



Chapter 6: outline

- Basic rules for calculating probabilities (today)
- Discrete random variables & PDFs
- Continuous random variables & PDFs
- Linear regression & related concepts



A typical situation

There are **5** keys in a drawer: **2** back-door keys and **3** front-door keys.

Consider the following scenario:

I put my hand in the drawer and take out a key without looking at it.

Questions:

- What is the *chance* of it being a **front-door key**?
- What is the *chance* of it being a **back-door key**?
- What is the *chance* of it being a **garage key**?



Terminology

experiment (or trial) = any situation, *like taking out a key*, where we know all the possibilities but we do not know which one will occur on any given occasion

outcome = each of the possibilities of the experiment

event = a group of outcomes considered together and given a neat verbal description (events will be denoted by capital letters: A , B , C , etc.)

$$A \cup B = \text{"A or B"}$$

$$A \cap B = \text{"A and B"}$$

$$A \cup B \cup C = \text{"A or B or C"}$$

$$A \cap B \cap C = \text{"A and B and C"}$$

etc.

etc.

In the above example:

experiment = “ take out a key from the drawer ”

event = “ the key obtained is a front-door key ”, etc
(there are 3 **outcomes** for this event)

Probability theory assigns to each event A , a positive number $P(A)$, which is designed to provide a **theoretical** measure of how likely A is to occur.

$P(A)$ is known as the **probability** of A

The set of *all* possible outcomes is called the **sample space** of the experiment.

E.G.: Driving to work, a commuter passes through a sequence of 3 intersections with traffic lights. At each light he either **stops** (S) or **continues** (C).

The sample space is the set of all possible outcomes:

$$\Omega = \{CCC, CCS, CSS, CSC, SSS, SSC, SCC, SCS\}$$

Some events:

$$A = \{\underline{SSS}, \underline{SSC}, \underline{SCC}, \underline{SCS}\}$$

$$A \cup B = \{\underline{SSS}, \underline{SSC}, \underline{SCC}, \underline{SCS}, \underline{CCS}, \underline{CSS}\}$$

$$B = \{\underline{SSS}, \underline{SCS}, \underline{CCS}, \underline{CSS}\}$$

$$A \cap B = \{\underline{SSS}, \underline{SCS}\}$$



The Fundamental Law of Probability

Let \mathcal{E} be an experiment that has n possible outcomes, all of which are **equally likely** to occur. Let A be an event that corresponds to r of the outcomes. Then

$$P(A) = \frac{r}{n} = \frac{\text{total \# of outcomes (event)}}{\text{total \# of outcomes (experiment)}}$$

OBSERVATION:

To find the two numbers in the above formula we will always have to count outcomes. This requires using various **principles of counting** and **combinatorics**.



ASIDE (combinatorics)

- △ The number of permutations of N objects is $N!$
(i.e., the # of different ways of ordering them)
- △ Consider a process for which there are N possible results each time is performed. If it is performed k times, then the total number of possible combined outcomes, when the order *does* matter, is N^k
- △ Given N people, the number of different ways to choose an k -person committee where the order *does* matter equals

$$\frac{N!}{(N - k)!}$$

- △ If the order *doesn't* matter the answer is

$$\binom{N}{k} = \frac{N!}{k! (N - k)!}$$



Basic properties for probabilities

✘ $0 \leq P(A) \leq 1$ for any event A

✘ $P(A) = 0$ if and only if A is an impossible event, i.e. an event that cannot actually occur

✘ $P(A) = 1$ if and only if A is an event that is absolutely certain to occur

✘ $P(\bar{A}) = 1 - P(A)$ or $P(A) = 1 - P(\bar{A})$

“not A ”

the **opposite** or
complementary
event to A



Examples

Example 6.1: I throw a die twice and note down the two numbers obtained. What is the probability that the sum of these numbers is greater than 3?

Example 6.2: How many people must you ask in order to have a 50:50 chance of finding someone who shares your birthday?

Example 6.3: Suppose that a room contains N people. What is the probability that at least two of them have the same birthday?



Examples

6.1 Let $A =$ "sum of the 2 numbers > 3 "

$\bar{A} =$ "sum is ≤ 3 "

Note that \bar{A} corresponds to 3 of the 36 outcomes



1
1
2



1
2
1

$$P(\bar{A}) = \frac{3}{36} = \frac{1}{12} \Rightarrow P(A) = 1 - P(\bar{A}) = 1 - \frac{1}{12} = \frac{11}{12}$$

6 possibilities 1st #
6 possibilities 2nd # \rightarrow 36 possibilities altogether
(equally likely to occur)



Examples

6.2 \mathcal{E} = "ask n people"

A = "someone's birthday is the same as yours"

\bar{A} = "no one's birthday"

total # outcomes $\mathcal{E} = 365^n$

total # outcomes $\bar{A} = 364^n$

$$P(\bar{A}) = \frac{364^n}{365^n} = \left(\frac{364}{365}\right)^n \Rightarrow P(A) = 1 - P(\bar{A}) = 1 - \left(\frac{364}{365}\right)^n$$

$$P(A) = \frac{1}{2} \Rightarrow \frac{1}{2} = \left(\frac{364}{365}\right)^n \Rightarrow -\ln 2 = n \cdot \ln\left(\frac{364}{365}\right)$$



$$n \approx 253$$



Example 6.3 (solution)

Disregard leap years and assume that every day of the year is equally likely to be a birthday.

A = “at least 2 people have the same birthday”

\bar{A} = “no two people have a common birthday”

possible
outcomes
(experiment)

$$\Rightarrow 365^N$$

possible
outcomes
for \bar{A}

$$\Rightarrow 365 \times 364 \times \dots \times (365 - N + 1)$$

$$P(A) = 1 - \frac{365 \times 364 \times \dots \times (365 - N + 1)}{365^N}$$

N	P(A)
4	0.016
16	0.284
23	0.507
32	0.753
40	0.891
56	0.988



Further concepts

Two events A and B are said to be **mutually exclusive** if they cannot happen at the same time.

Rules:

$$P(A \cup B) = P(A) + P(B)$$

$A, B =$ mutually exclusive

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

$A, B, C =$ mutually exclusive

\vdots

(i.e., no two of them can occur simultaneously)

$$P(A_1 \cup A_2 \cup \dots \cup A_n) = P(A_1) + P(A_2) + \dots + P(A_n)$$

$A_1, A_2, \dots, A_n =$ mutually exclusive



Conditional probability

$A, B =$ events

$P(B|A)$ = the probability of B occurring, given that we know that A has occurred

Example 6.4: 6 men and 5 women have dark (D) and fair (F) hair according to the scheme

Men: 5 D & 1 F
Women: 2 D & 3 F

A person is chosen at random. Let A and B be the events defined by

$A =$ “ person chosen is fair “

$B =$ “ person chosen is a woman “

Find $P(A)$, $P(B)$, $P(A \cap B)$, $P(A|B)$, $P(B|A)$



Conditional probability

$$P(A) = \frac{4}{11} \quad P(A \cap B) = \frac{3}{11} \quad P(A|B) = \frac{3}{5}$$

$$P(B) = \frac{5}{11} \quad P(B|A) = \frac{3}{4}$$

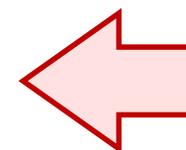
Note that

$$P(A) \cdot P(B|A) = \frac{4}{11} \times \frac{3}{4} = \frac{3}{11} = P(A \cap B)$$
$$P(A) \cdot P(A|B) = \frac{5}{11} \times \frac{3}{5} = \frac{3}{11} = P(A \cap B)$$

} not a coincidence!

$$P(A \cap B) = P(A) \cdot P(B|A)$$

$$P(A \cap B) = P(B) \cdot P(A|B)$$



this is true for any two events A & B



Conditional probability

Example 6.5: A bag contains **6 silver** coins and **4 gold**. Two coins are drawn at random, one after the other, without replacement.

- a). What is the probability of drawing first a **silver** then a **gold** coin?
b). Assume that we draw 3 coins without replacement. What is the probability of **silver** then **gold** then **silver**?

a). Let S_1 and G_2 be the events:

$S_1 =$ "1st coin is silver"

$G_2 =$ "2nd coin is gold"

$$\begin{aligned} P(S_1 \text{ and } G_2) &= P(S_1 \cap G_2) = P(S_1) \cdot P(G_2 | S_1) \\ &= \frac{6}{10} \times \frac{4}{9} = \frac{4}{15} \end{aligned}$$

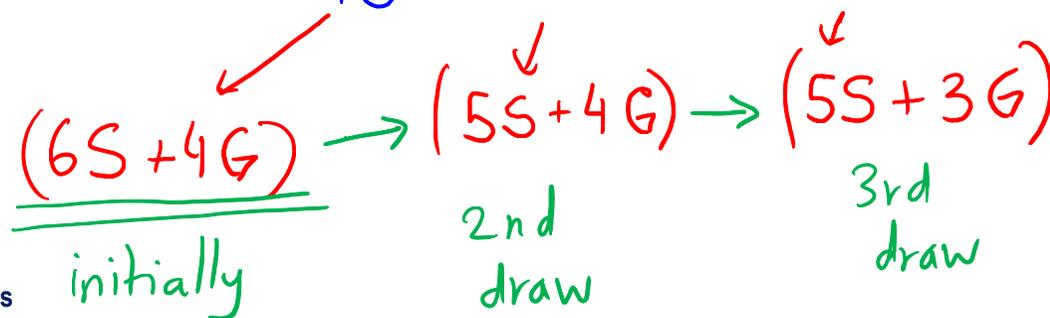


Conditional probability

b). In addition to S_1 and G_2 already defined, let us consider the event:

$S_3 =$ "3rd coin is silver"

$$\begin{aligned} P(\underline{S_1} \text{ and } G_2 \text{ and } \underline{S_3}) &= P(S_1 \cap G_2 \cap S_3) \\ &= P(S_1) P(G_2 | S_1) P(S_3 | G_1 \text{ and } G_2) \\ &= \frac{6}{10} \times \frac{4}{9} \times \frac{5}{8} \end{aligned}$$





Independent events

If the likelihood of an event B occurring is unaffected by whether an event A has or has not occurred, then we say that A and B are **independent events**.

In this case:

$$\left. \begin{aligned} P(B|A) &= P(B) \\ P(A \cap B) &= P(A) \cdot P(B|A) \end{aligned} \right\} \Rightarrow P(A \cap B) = P(A) \cdot P(B)$$

Similarly: if A, B, C are *independent events* then

$$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$$



Independent events

Example 6.6: A die is thrown repeatedly. What is the probability that a “4” occurs for the first time on the second throw.

Example 6.7: For the experiment of throwing a dice twice, consider the following events:

A = “6 comes up”

B = “the numbers obtained are different”

What is the conditional probability $P(A|B)$?



Independent events

6.6 $A =$ "not 4 on 1st throw"
 $B =$ "4 on 2nd throw" } note that these events are independent

$$\text{So } P(A \text{ and } B) = P(A \cap B) = P(A) \cdot P(B) \\ = \frac{5}{6} \times \frac{1}{6} = \frac{5}{36}$$

6.7 A and B are defined already
 $P(A|B) = \frac{P(A \cap B)}{P(B)}$ } must calculate these numbers ☒

$A \cap B =$ 6 on 1st throw
AND
 not 6 on 2nd throw

E_1

OR

not 6 on 1st throw
AND
 6 on 2nd throw

E_2

$$= E_1 \cup E_2$$



Independent events

Note that E_1 and E_2 are mutually exclusive, so

$$P(A \cap B) = P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

$$P(E_1) = \frac{1}{6} \times \frac{5}{6} = \frac{5}{36} \text{ (why?)}$$

$$P(E_2) = \frac{5}{6} \times \frac{1}{6} = \frac{5}{36}$$

} \Rightarrow

$$\Rightarrow P(A \cap B) = \frac{5}{36} + \frac{5}{36} = \frac{5}{18}$$

\Rightarrow \boxtimes

$$P(A|B) = \frac{\cancel{5}}{\cancel{18}_3} \times \frac{\cancel{6}}{\cancel{5}} = \frac{1}{3}$$

$$\text{Also } P(B) = \frac{30}{36} = \frac{5}{6}$$

In conclusion

$$P(A|B) = \frac{1}{3}$$



The binomial pattern

This is the name for the pattern that occurs when we repeat an experiment n times and look at how often a particular event occurred.

Let A be an event associated with an experiment \mathcal{E} .

Suppose that this experiment is performed n times, and that the n performances are independent of each other (so that the result of one performance does not affect the result of any other).

Suppose that each time \mathcal{E} is performed, $P(A) = p$. Then

$$P\left(\begin{array}{l} A \text{ occurs exactly } k \text{ times} \\ \text{in } n \text{ performances of } \mathcal{E} \end{array}\right) = \binom{n}{k} p^k (1-p)^{n-k}$$



The binomial pattern

Note that the result stated on the previous slide can be applied very widely.

It can be used whenever a routine is repeated **with unchanged probabilities**:

- repeatedly tossing a coin
- testing items from a production line for defects
- drawing articles from a bag with replacement (because at each draw the contents of the bag are the same as they were at the start).



The binomial pattern

Example 6.8: A dice is thrown **5 times**. What is the probability of obtaining precisely **two sixes**?

Example 6.9: **20%** of the cars travelling on a certain motorway have defective tyres. One day the police stop at random **8 cars** using the motorway and inspect their tyres. What is the probability that they find at least **2 cars** with defective tyres?



The binomial pattern

6.8 In the binomial formula take

$$h=5$$

$$k=2$$

$$p=\frac{1}{6}$$

Σ = "one throw of a die"

A = "a 6 comes up"

$$P\left(\begin{array}{l} \text{A occurs exactly 2 times} \\ \text{in 5 throws} \end{array}\right) = \binom{5}{2} \left(\frac{1}{6}\right)^2 \cdot \left(1 - \frac{1}{6}\right)^3 = \frac{5!}{2! \cdot 3!} \times \frac{1}{6^2} \times \frac{5^3}{6^3}$$

$$= \frac{\cancel{1 \cdot 2 \cdot 3} \cdot 4 \cdot 5}{1 \cdot 2 \cdot \cancel{1 \cdot 2 \cdot 3}} \times \frac{1}{6^2} \times \frac{5^3}{6^3} = \frac{2}{6} \left(\frac{5}{6}\right)^4 = \frac{1}{3} \left(\frac{5}{6}\right)^4$$

$$\approx 0.1608 \quad (16\%)$$



The binomial pattern

6.9 ξ = "the police stop at random a car. . . ."

A = "car has defective tyres"

B = "at least 2 cars with defective tyres"

It is easier to calculate $P(\bar{B})$

\bar{B} = "# of cars with defective tyres < 2"
i.e. 0 or 1

$$\bar{B} = E_1 \cup E_0$$

E_1 = "1 car with defective tyres"

E_0 = "0 cars with defective tyres"

← these are mutually exclusive!

$$P(\bar{B}) = P(E_1 \cup E_0) = \underbrace{P(E_1)} + \underbrace{P(E_0)}$$

↳ to calculate these we use the binomial formula



The binomial pattern

$$P(E_1) = \binom{8}{1} \cdot \left(\frac{1}{5}\right)^1 \cdot \left(1 - \frac{1}{5}\right)^{8-1} = 8 \cdot \frac{1}{5} \cdot \frac{4^7}{5^7} = 2 \left(\frac{4}{5}\right)^8$$

$$\boxed{h=8} \quad \boxed{p=\frac{1}{5} = \frac{20}{100}}$$

$$P(E_0) = \binom{8}{0} \cdot \left(\frac{1}{5}\right)^0 \cdot \left(1 - \frac{1}{5}\right)^8 = \left(\frac{4}{5}\right)^8$$

$$\boxed{h=8} \quad \boxed{p=\frac{1}{5} = \frac{20}{100}}$$

$$P(\bar{B}) = P(E_1) + P(E_0) = 3 \cdot \left(\frac{4}{5}\right)^8$$

$$\underline{P(B)} = 1 - P(\bar{B}) = 1 - 3 \left(\frac{4}{5}\right)^8 \approx \underline{\underline{0.497}}$$

required answer