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Mathematical Modelling III

Examples #5

Potential flows in 2D; applications of complex variables

1. The velocity field $\boldsymbol{v} = v_r \boldsymbol{e}_r + v_\theta \boldsymbol{e}_\theta$,

$$v_r = \frac{Q}{2\pi r}\,, \qquad v_\theta = 0\,,$$

where Q is a constant, is called a *line source* if $Q > 0$ and a *line sink* if $Q < 0$. Show that it is irrotational and that it satisfies $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$, except at $r = 0$, where it is not defined. Find the velocity potential and the streamfunction, and show that the complex potential is

$$w(z) = \frac{Q}{2\pi}\log z\,.$$

2. Fluid occupies the region $x_1 \geq 0$ and there is a plane rigid boundary at $x = 0$. A line source is placed at $z_0 = d + \mathrm{i}0$. Show that the pressure at $x_1 = 0$ decreases to a minimum at $|x_2| = d$ and thereafter is an increasing function of $|x_2|$.

3. Consider a line source of strength Q located at the origin and a line sink of strength Q located at $z = \delta e^{\mathrm{i}\alpha}$ ($\alpha, \delta \in \mathbb{R}$). Write down the complex potential for this *doublet*. In the limit $\delta \to 0$ with $\mu = Q\delta/2\pi$ fixed, show that the complex potential becomes

$$w(z) = \frac{\mu e^{\mathrm{i}\alpha}}{z}\,.$$

[Note: This combination of a source and sink is called a dipole.]

4. Show that the cylinder $|z| = (\mu/V)^{1/2}$ is a streamline for the flow whose complex potential is given by

$$w(z) = Vz + \frac{\mu}{z}\,.$$

Sketch the streamlines and deduce that the flow due to a dipole in a uniform stream is equivalent to flow past a circular cylinder.

5. As an alternative to the method used in lectures, show that the problem of a cylinder (radius $r = a$, centre the origin) in an irrotational flow at speed V in the direction of the x_1-axis may be found in terms of the velocity potential $\phi(r, \theta)$ satisfying

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0\,,$$

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$$\phi(r,\theta) \sim Vr\cos\theta\,, \qquad r\to\infty\,,$$

$$\frac{\partial\phi}{\partial r} = 0\,, \qquad \text{on } r = a\,.$$

Consider also the case when there is a circulation Γ around the cylinder.

[HINT: Seek a solution of the form $\phi(r,\theta) = R(r)\Theta(\theta)$.]

6. Find the complex potential $w(z)$ for the flow external to the cylinder $|z| = a > 0$ due to a source of strength Q at $z = 2a$. Prove that

   (a) the force on the cylinder is $\rho Q^2/12\pi a$ and find its direction;

   (b)
   $$\oint_{\mathcal{C}} \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 \mathrm{d}z = 0\,,$$
   where $\mathcal{C} = \{z \in \mathbb{C} \,:\, |z| = 3a\}$; and hence that

   (c) in this flow, the equal and opposite force which acts on the source is given by Blasius' formula evaluated for a contour which encloses the source alone.

7. Prove that the pressure distribution around a circular cylinder fixed with its axis orthogonal to a uniform stream V is given by
   $$p = p_\infty + \frac{1}{2}\rho V^2(1 - 4\sin^2\theta)\,,$$
   where p_∞ is the reference pressure (at infinity).

8. Show that the mapping
   $$Z = \mathrm{i}e^{\pi z/2a}\,,$$
   transforms a strip of breadth $2a$ into a half-plane.
   Use this transformation to show that the complex potential of a $2D$ irrotational flow between two parallel infinite plates at $\mathrm{Im}\, z = \pm 2a$, due to a line source of strength $Q > 0$ at the origin is given by
   $$w(z) = \frac{Q}{2\pi}\log\left[\sinh\left(\frac{\pi z}{2a}\right)\right]\,.$$
   [HINT: The fluid emerging from $z = 0$ must flow off in equal amounts to $\mathrm{Re}\, z = \pm\infty$, so there must be sinks of strength $(-Q/2)$ at $\mathrm{Re}\, z = \pm\infty$.]

9. By applying the transformation
   $$Z = z + \frac{a^2}{z}e^{2\mathrm{i}\alpha}\,,$$
   to the flow defined by
   $$w(z) = V\left(z + \frac{a^2}{z}\right)\,,$$
   show that
   $$W(Z) = \frac{1}{2}V(1 + e^{-2\mathrm{i}\alpha})Z + \frac{1}{2}V(1 - e^{-2\mathrm{i}\alpha})(Z^2 - 4a^2e^{2\mathrm{i}\alpha})^{1/2}$$

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is the complex potential for a plate of length $4a$ in a uniform stream with the plate at an angle α to the undisturbed flow.

10. Show that the Joukowski transformation maps the circle $|z| = a$ $(0 < c < a$ – see your notes) onto an ellipse.

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