UNIVERSITY
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Mathematical Modelling III

Examples #4

1. Inviscid fluids

1. Let $\partial\mathcal{P}$ be the boundary of a fixed (in space) volume $\mathcal{P} \subset \mathcal{B}_t$ (i.e., inside a moving fluid, but not in motion); let $\mathbf{n}$ denote the outward unit normal to $\partial\mathcal{P}$; and let $\mathrm{d}a$ denote the area element on $\partial\mathcal{P}$. The *volume flow rate* across this boundary *per unit area* is $\mathbf{v}\cdot\mathbf{n}$ and the *mass flow rate per unit area* is $\rho(\mathbf{v}\cdot\mathbf{n})$.
   Show that the *Principle of Mass Conservation* can be cast in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{P}} \rho \, \mathrm{d}v = -\int_{\partial\mathcal{P}} \rho(\mathbf{v}\cdot\mathbf{n}) \, \mathrm{d}a \, .$$

   [Note: This says that the rate of increase of mass in $\mathcal{P}$ equals the rate at which the mass is crossing $\partial\mathcal{P}$ in the inward direction.]

2. Consider the same setting as above. By using Cauchy's equation of motion establish the *Momentum Integral Equation*

$$\int_{\mathcal{P}} \frac{\partial}{\partial t}(\rho\mathbf{v}) \, \mathrm{d}v = -\int_{\partial\mathcal{P}} \rho\mathbf{v}(\mathbf{v}\cdot\mathbf{n}) \, \mathrm{d}a + \int_{\mathcal{P}} \rho\mathbf{b} \, \mathrm{d}v + \int_{\partial\mathcal{P}} \boldsymbol{\sigma}\cdot\mathbf{n} \, \mathrm{d}a \, . \tag{1}$$

   [Note: The term on the left-hand side is the rate of change of momentum inside the fixed volume $\mathcal{P}$; the first term on the right-hand side expresses the flux of momentum through the surface $\partial\mathcal{P}$; the second term on the right-hand side expresses the total body force acting on the fixed volume and, finally, the last term expresses the resultant of the surface forces acting on $\partial\mathcal{P}$ due to the surrounding fluid.]

3. A fluid flows steadily from infinity with velocity $-U\mathbf{e}_1$ (U =const.) past the fixed sphere $|\mathbf{x}| = \alpha$. Given that the resultant velocity field $\mathbf{v}$ of the fluid at any point is given by

$$-U\left\{\left[1+\left(\frac{\alpha}{r}\right)^3\right]\mathbf{e}_1 - \frac{3\alpha^3 x_1}{r^4}\mathbf{x}\right\}, \qquad (r = |\mathbf{x}|)\,,$$

   find the acceleration $\mathbf{a}$ at any point $\mathbf{x} = \beta\mathbf{e}_1$ ($\beta > \alpha$) and evaluate the maximum value of $\mathbf{a}$ for variations in β.

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4. Take the Euler equations for an incompressible fluid of (spatially) constant density, cast it into an appropriate form, and perform suitable operations to obtain the *Energy Equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{P}} \frac{1}{2}\rho|\mathbf{v}|^2\,\mathrm{d}v = -\int_{\partial\mathcal{P}} \left(p' + \frac{1}{2}\rho|\mathbf{v}|^2\right)\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}a\,,$$

where $\mathcal{P}$ is the region enclosed by a fixed surface $\partial\mathcal{P}$ drawn in the fluid, $p' := p + \rho\mathcal{V}$, and $\mathbf{b} = -\nabla\mathcal{V}$ (i.e., the body force is conservative with $\mathcal{V}$ the associated potential). p' is sometimes referred to as the non-hydrostatic part of the pressure field.

5. By using the expression of the gradient of a vector field in cylindrical polar coordinates, calculate the acceleration $\mathbf{a} \equiv \dfrac{D\mathbf{v}}{Dt}$ for the radially symmetric flow

$$\mathbf{v} = G(r,t)\mathbf{e}_\theta\,.$$

6. Consider the flow

$$\mathbf{v} = \alpha(x_1\mathbf{e}_1 - x_2\mathbf{e}_2) \qquad (\alpha > 0)\,,$$

and let the concentration of some pollutant advected by the flow be

$$c(x_1, x_2, t) = \beta x_1^2 x_2 \mathrm{e}^{-\alpha t}\,,$$

for $x_2 > 0$ and $\beta > 0$ a constant. Does the pollutant concentration for any particular fluid element change with time?

7. Sketch the streamlines for the flow

$$\mathbf{v} = \frac{1}{r}\mathbf{e}_\theta\,, \tag{2}$$

and indicate the direction of the flow.
Let the concentration of some pollutant in the flow (2) be given by

$$c(r,\theta,t) = \sin\left(\theta - \frac{t}{r^2}\right)\,.$$

Does the pollutant concentration for any particular fluid element change with time?

8. Consider the flow

$$\mathbf{v} = \mathrm{e}^{x_2}\mathbf{e}_1 + \mathrm{e}^{x_1}\mathbf{e}_2\,.$$

Show that the acceleration is given by

$$\mathbf{a} = \mathrm{e}^{x_1+x_2}(\mathbf{e}_1 + \mathbf{e}_2)\,.$$

9. (a) Relative to fixed Cartesian axes, a flow is defined as

$$\mathbf{v} = (-\Omega x_2, \Omega x_1, 0)\,.$$

Show that in polar coordinates this flow can be written as $\mathbf{v} = (0, \Omega r, 0)$.

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(b) The *Rankine vortex* is defined by

$$v_r = v_z = 0, \qquad v_\theta = \begin{cases} \Omega r\,, & r < a\,, \\ \dfrac{\Omega a^2}{r}\,, & r > a\,. \end{cases}$$

Find the pressure both inside and outside its core. Show that the pressure at $r = 0$ is lower than that at $r = \infty$ by an amount $\rho\Omega^2 a^2$ (*hence the very low pressure at the centre of a tornado*). Deduce that if there is a free surface to the fluid and gravity is acting, then the surface $r = 0$ is a depth $\Omega^2 a^2/g$ below the surface at $r = \infty$ (*hence the dimples in a cup of tea accompanying the vortices that are shed by the edges of the spoon*).

10. For the irrotational motion of a liquid in two dimensions under conservative forces, prove that

$$\nabla^2(\log(\nabla^2 p)) = 0\,,$$

where p represents the pressure field.

11. Let $\mathcal{D}$ be the region between two co-axial (concentric) cylinders of radii R_1 and R_2, where $R_1 < R_2$. Consider the velocity field $\mathbf{v}$ defined by its components

$$v_r = v_z = 0 \qquad \text{and} \qquad v_\theta = \frac{A}{r} + Br\,,$$

where

$$A = \frac{R_1^2 R_2^2(\Omega_2 - \Omega_1)}{R_2^2 - R_1^2}\,, \qquad B = \frac{R_1^2\Omega_1 - R_2^2\Omega_2}{R_2^2 - R_1^2}\,.$$

Describe the streamlines of this flow.
Assuming that the pressure field is radially symmetric, show that

(a) $\mathbf{v}$ is a stationary solution of Euler's equation with $\rho \equiv 1$ and zero body force;

(b) the vorticity vector $\boldsymbol{\omega} = (0, 0, 2B)$;

(c) the angular velocity of the flow on the two cylinders is Ω_1 and Ω_2.

Finally, calculate the stretching tensor $\mathbf{D}$.

[Note: This type of fluid motion between co-axial cylinders is known as *Couette flow*.]

12. A fluid flow of constant density rotates about a vertical axis, the angular velocity of a particle of fluid about the axis being constant and equal to Ω; motion takes place under the action of gravity alone. Show that the free surface is a paraboloid of revolution, with axis vertical.

13. A circular cylinder the radius R of which varies with time (i.e., $R = R(t)$), is mounted vertically in an inviscid fluid of constant density and of infinite extent in directions normal to the cylinder. If (r, θ, z) are cylindrical polar coordinates with the axis of the cylinder

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as the $z-$axis and $\mathbf{v} = v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta + v_z\mathbf{e}_z$ is the velocity field, verify that the equation of mass continuity and the boundary conditions on the cylinder are satisfied by the field

$$v_r = R\dot{R}/r\,, \qquad v_\theta = g(t)f(r)\,, \qquad v_z \equiv 0\,,$$

where the dot in the first term denotes differentiation with respect to time. Show that the vorticity is everywhere in the vertical direction. If gravity is the only external force on the fluid, and all the field quantities are independent of the azimuthal angle θ, determine the forms of the functions $g(t)$ and $f(r)$.

14. A layer of fluid of constant density ρ sits at rest in a large sealed container with air at the surface maintained at twice atmospheric pressure. A small hole is drilled in the side of the container a vertical distance h below the fluid surface.
Calculate the speed with which the fluid emerges from the hole.

15. A *siphon* consists of a narrow pipe with one end immersed in a reservoir of fluid and the other held at a vertical distance h below the surface of the fluid in the reservoir. The maximum height of the pipe above the fluid surface is H. If fluid fills the pipe, show that it will flow out of the free end with speed v given by

$$v^2 = 2gh\,.$$

Determine the minimum pressure in the pipe. What will happen if

$$\rho g(H+h) > p_0\,,$$

where ρ is the density of the fluid and p_0 is atmospheric pressure?

16. The flow along a pipe is $Q\ m^3s^{-1}$. The pipe has scross-sectional area A, narrowing to cross-section B, then expanding to A again. The fluid discharges from the end of the second wide section. Determine the pressure p_B at the narrow section. If a thin, straight pipe is attached to the narrow section, connecting it with a reservoir a distance h vertically down, show that fluid will be sucked up through the pipe if

$$h < \frac{Q^2}{2g}\left(\frac{1}{B^2} - \frac{1}{A^2}\right)\,.$$

2. Viscous fluids

1. Viscous incompressible fluid of viscosity μ flows along a stationary circular pipe of radius r_0 under a constant pressure gradient $\mathrm{d}P/\mathrm{d}z = -\Lambda$. Starting from the Navier-Stokes equations, and assuming a steady flow $\mathbf{v} = v(r,z)\mathbf{e}_z$ (in cylindrical polar coordinates), show that

   (a) the velocity gradient $\mathbf{L}$ is given by

$$\mathbf{L} = \frac{\partial v}{\partial r}\mathbf{e}_z \otimes \mathbf{e}_r + \frac{\partial v}{\partial z}\mathbf{e}_z \otimes \mathbf{e}_z\,;$$

(b) v is independent of z, i.e. $v = v(r)$;

(c)

$$\nabla^2 P = \mu(\nabla^2 v)\mathbf{e}_z \, .$$

Assuming the no slip boundary condition, conclude that the velocity of the fluid is given by

$$\mathbf{v} = \frac{\Lambda}{4\mu}(r_0^2 - r^2)\mathbf{e}_z \, .$$

Finally, determine the volume flux of fluid through the pipe associated with this flow.

[Note: This type of fluid motion between co-axial cylinders is known as *Poiseuille flow*; that is, the steady flow of an incompressible viscous fluid parallel to the axis of a circular pipe of infinite length, produced by a pressure gradient along the pipe.]

2. Fluid of kinematic viscosity ν is confined in the region $0 < x_2 < h$. Initially the fluid is at rest. At $t = 0$, the boundary $x_2 = 0$ is suddenly brought into motion with constant speed U in the x_1-direction. Meanwhile the upper boundary $x_2 = h$ is held fixed. The flow in $0 < x_2 < h$ for $t > 0$ is driven by the motion of the lower boundary. There is no imposed pressure gradient. Show that the flow $\mathbf{v} = (v(x_2, t), 0, 0)$ satisfies

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial x_2^2} \, , \tag{3}$$

subject to initial and boundary conditions

$$v(x_2, 0) = 0 \qquad \text{for} \quad 0 < x_2 < h \, , \qquad v(0,t) = U \qquad \text{and} \qquad v(h,t) = 0 \qquad \text{for} \quad t > 0 \, . \tag{4}$$

Show that the steady-state solution for this system is

$$v = U\left(1 - \frac{x_2}{h}\right) \, . \tag{5}$$

Then, writing

$$v = U\left(1 - \frac{x_2}{h}\right) + v^* \, , \tag{6}$$

and substituting this into (3) and (4), show that v^* satisfies (3) and determine the initial and boundary conditions satisfied by v^*. By using the method of separation of variables, solve for v^* and hence show that

$$v = U\left(1 - \frac{x_2}{h}\right) - \frac{2U}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp\left(-\frac{n^2\pi^2\nu t}{h^2}\right)\sin\left(\frac{n\pi x_2}{h}\right) \, . \tag{7}$$

After what time does the solution (7) reduce approximately to the steady-state solution (5). Discuss how the evolution of vorticity evolves with time.

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3. Fluid of depth h flows down a plane inclined at an angle α to the horizontal. In a coordinate system where x_1 is in the direction of steepest descent and x_2 is perpendicular to the inclined plane, show that for a steady, incompressible flow of the form

$$\mathbf{v} = (v(x_2), 0, 0)\,,$$

the Navier-Stokes equations in the usual notation simplify to

$$0 = -\frac{\partial p}{\partial x_1} + \nu\frac{\partial^2 v}{\partial x_2^2} + g\sin\alpha\,, \qquad 0 = -\frac{\partial p}{\partial x_2} - g\cos\alpha\,, \qquad 0 = -\frac{\partial p}{\partial x_3}\,,$$

and state the appropriate boundary conditions. Hence, derive the equation

$$\frac{d^2 v}{dx_2^2} = -\frac{g}{\nu}\sin\alpha\,,$$

and thus find v.

4. Two incompressible viscous fluids of the same density ρ flow, on top of each other, down an inclined plane making an angle α with the horizontal. Their viscosities are μ_1 and μ_2, the lower fluid has depth h_1 and the upper fluid is of depth h_2. Seeking a steady-state solution, show that the velocity v_1 of the lower fluid is

$$v_1(x_2) = \left[(h_1 + h_2)x_2 - \frac{1}{2}x_2^2\right]\frac{\rho g\sin\alpha}{\mu_1}\,,$$

Explain why this is dependent on the depth h_2, but not the viscosity of the upper fluid.

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