Mathematical Modelling III

Examples #4

Torsion of cylindrical bodies

1. (a) By considering the general expression of the torque,

$$\mathbf{M} = \int_{\Omega} \mathbf{r} \wedge (\mathbf{e}_3 \cdot \boldsymbol{\sigma})\, \mathrm{dA}\,,$$

and using the boundary condition for the warping function Ψ, show that this integral can be cast as

$$\mu \mathbf{e}_3 \left[\left(\frac{\alpha}{L}\right) \int_{\Omega} r^2\, \mathrm{dA} - \int_{\partial\Omega} \mathbf{e}_3 \wedge \mathrm{d}\mathbf{r} \cdot (r \Psi \mathbf{e}_\theta) \right]. \tag{1}$$

[Hint: Use Stokes' Theorem; if you cannot use the direct notation, assume Cartesian coordinates and do everything componentwise -- of course, you'll have to replace the stress tensor by what we found in the lectures.]

(b) Use the two-dimensional version of Green's theorem to show that the torsional rigidity can be expressed as

$$D = \mu \left(\frac{L}{\alpha}\right) \int_{\Omega} X_1 \frac{\partial \Psi}{\partial X_2} - X_2 \frac{\partial \Psi}{\partial X_1}\, \mathrm{dA} + \mu \int_{\Omega} X_1^2 + X_2^2\, \mathrm{d}A\,. \tag{2}$$

(c) From (2) deduce that the torsional rigidity D of a solid cylinder of cross-section $\Omega \subset \mathbb{R}^2$ can also be given by the formula

$$D = \mu I_p - \mu \left(\frac{L}{\alpha}\right)^2 \int_{\Omega} (\boldsymbol{\nabla}\Psi) \cdot (\boldsymbol{\nabla}\Psi)\, \mathrm{dA}\,,$$

where

$$I_p \equiv \int_{\Omega} r^2\, \mathrm{dA}$$

is the polar second moment of the section with respect to the centre of torsion.

[This result shows that the torsional rigidity is always strictly positive and its maximum occurs when there is no warping.]

(d) Show that the same result can be found using direct notation and starting from (1).

2. Consider the torsion problem for a cylinder of length $2L$ whose cross-section is the rectangle defined by

$$\Omega \equiv \left\{ (X_1, X_2) \in \mathbb{R}^2 \;\middle|\; -a \le X_1 \le a\,,\ \ -b \le X_2 \le b \right\}.$$

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(a) Show that the appropriate boundary conditions for the warping function Ψ are

$$\frac{\partial \Psi}{\partial X_1} = \beta X_2 \,, \qquad X_1 = \pm a \,,$$

$$\frac{\partial \Psi}{\partial X_2} = -\beta X_1 \,, \qquad X_2 = \pm b \,,$$

where $\beta \equiv \alpha / L$.

(b) Defining $\overline{\Psi} := \beta X_1 X_2 - \Psi$, transform the original boundary-value problem for Ψ in terms of this new function. Use the method of separable variables to find an explicit representation for $\overline{\Psi}$.

(c) Using the standard result,

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$$

together with (2) show that the torsional rigidity of the rectangular cylinder is given by

$$\mu a b^3 H(\eta) \,,$$

where

$$\eta := \frac{a}{b} \quad \text{and} \quad H(\zeta) \equiv \frac{16\zeta^2}{3} \left[1 - \frac{192}{\pi^5} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^5} \tanh \frac{(2n+1)\pi}{2\zeta} \right] .$$

3. The cross-section of a cylinder is a triangularly-shaped region Ω. Assume that $\partial\Omega$ is the triangle formed by the intersections of the following lines

$$(d_1): \qquad \left\{ (X_1, X_2) \in \mathbb{R}^2 \;\middle|\; X_2 = (2c/3) + m X_1 \right\} ,$$

$$(d_2): \qquad \left\{ (X_1, X_2) \in \mathbb{R}^2 \;\middle|\; X_2 = (2c/3) - m X_1 \right\} ,$$

$$(d_3): \qquad \left\{ (X_1, X_2) \in \mathbb{R}^2 \;\middle|\; X_2 = -c/3 \right\} ,$$

for some $c \in \mathbb{R}$.

(a) Show that

$$\widetilde{\Psi}(X_1, X_2) = A \left(X_2 + \frac{c}{3} \right) \left[\left(\frac{2c}{3} - X_2 \right)^2 - m^2 X_1^2 \right] \tag{3}$$

is an acceptable choice for the stress function for suitably defined $m, A \in \mathbb{R}$. Sketch the cross-section for which (3) plays the role of stress function.

(b) Show that the torsional rigidity of the cylinder is equal to

$$\frac{\mu}{15\sqrt{3}} c^4 \,,$$

and find the maximal shear stress in the cross-section.

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4. Show that the torsional rigidity for a hollow cylinder of cross-section

$$\left\{(r,\theta)\;\middle|\;R_1 \le r \le R_2\,,\;\; 0 \le \theta < 2\pi\right\} \qquad (0 < R_1 < R_2)$$

is

$$\frac{1}{2}\mu\pi(R_2^4 - R_1^4)\,.$$

If the hollow cylinder has $R_2 = 6$ cm and $R_1 = 5$ cm, find the radius of the solid cylinder which has the same torsional rigidity. Assuming that both configurations involve the same elastic material, show that the hollow cylinder is (approximately) 58% lighter than the solid cylinder.

5. An axisymmetric composite cylinder is composed of a solid inner shaft, of radius a and shear modulus μ_1, and an outer sleeve of outer radius b and shear modulus μ_2. The shaft and sleeve are ideally bonded at their interface and the composite cylinder is subjected to an applied torque M.

   (a) Determine the distribution of stresses within the composite cylinder in terms of the twist per unit length (α/L).

   (b) Use the following general expression for the torque,

   $$M = \int_\Omega (X_1\sigma_{32} - X_2\sigma_{31})\,\mathrm{dA}\,,$$

   to find an expression for the twist per unit length α in terms of the applied torque. Determine the torsional rigidity of this configuration.

6. Consider the hollow cylinder whose cross-section is bounded by two concentric, similar ellipses $\partial\Omega^1$ and $\partial\Omega^2$,

$$\partial\Omega^1 \equiv \left\{(X_1, X_2) \in \mathbb{R}^2 \;\middle|\; \frac{X_1^2}{a^2} + \frac{X_2^2}{b^2} = k^2\right\},$$

$$\partial\Omega^2 \equiv \left\{(X_1, X_2) \in \mathbb{R}^2 \;\middle|\; \frac{X_1^2}{a^2} + \frac{X_2^2}{b^2} = 1\right\},$$

where $0 < a < b$ and $0 < k < 1$ are real constants.

   (a) Show that

   $$\widetilde{\Psi}(X_1, X_2) = m\left(\frac{X_1^2}{a^2} + \frac{X_2^2}{b^2} - 1\right)$$

   may be used as a stress function to describe the torsion of this cross-section, and determine the value of the constant $m \in \mathbb{R}$ in terms of the twist per unit length α.

   (b) Find the torsional rigidity of this configuration.

   Hint: Make use of the following formula found in your notes,

   $$D = \frac{2\mu}{(\alpha/L)}\left[\int_\Omega \widetilde{\Psi}\,\mathtt{dA} + \sum_j A_j\widetilde{\Psi}_j\right].$$

7. Consider the following two disks,

$$\mathcal{D}_1 \equiv \left\{ (X_1, X_2) \in \mathbb{R}^2 \;\middle|\; (X_1 - A)^2 + X_2^2 \leq A^2 \right\},$$

$$\mathcal{D}_2 \equiv \left\{ (X_1, X_2) \in \mathbb{R}^2 \;\middle|\; X_1^2 + X_2^2 \leq a^2 \right\},$$

where $0 < a < A$. A cylindrical body whose cross-section is given by $\mathcal{D}_1 \setminus \mathcal{D}_2$ is loaded at its ends by two torsional couples $\pm\mathbf{M} = \pm M\mathbf{e_3}$.

(a) Considering the polar coordinates $(r, \theta) \in (0, \infty) \times [0, 2\pi)$ defined by

$$X_1 = r\cos\theta\,, \qquad X_2 = r\sin\theta\,,$$

where

$$r = \sqrt{X_1^2 + X_2^2}\,, \qquad \theta = \tan^{-1}\frac{X_2}{X_1}\,,$$

show that

$$\widetilde{\Psi}(r, \theta) = C\left(r^2 - a^2 - 2Ar\cos\theta + 2\frac{Aa^2}{r}\cos\theta \right), \qquad C \in \mathbb{R}$$

may be used as a potential stress function. Determine the value of the constant C in terms of the twist per unit length.

(b) Calculate the stress distribution in the cross-section and find the maximum shear stress.

(c) Find the torsional rigidity.

8. (**Leibenzon's Theorem**) The transverse cross-section of a long beam subjected to terminal torsional moments is a simply connected domain $\Omega \subset \mathbb{R}^2$ bounded by a piecewise smooth curve $\partial\Omega$. The shear stress vector is defined in the usual way,

$$\boldsymbol{\tau} = \sigma_{31}\,\mathbf{e_1} + \sigma_{32}\,\mathbf{e_2}\,,$$

where $\mathbf{e_1}$ and $\mathbf{e_2}$ are unit vectors along the X_1 and, respectively, X_2. Show that

$$\oint_{\partial\Omega} \boldsymbol{\tau}\cdot \mathrm{d}\mathbf{r} \propto (\alpha/L) \times (\text{area of } \Omega)\,,$$

and find the proportionality constant.

[Note that the left-hand side of the above relation is the *circulation of* $\boldsymbol{\tau}$.]

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