Mathematical Modelling III

Examples #3

Plane elasticity/Airy stress function

1. Verify the following relations for the case of plane strain with constant body force,

$$\frac{\partial}{\partial X_2}\left(\nabla^2 u_1\right) = \frac{\partial}{\partial X_1}\left(\nabla^2 u_2\right) ,$$

$$\frac{\partial}{\partial X_1}\left(\nabla^2 u_1\right) = -\frac{\partial}{\partial X_2}\left(\nabla^2 u_2\right) ,$$

$$\nabla^4 u_1 = \nabla^4 u_2 = 0 \,,$$

where $\mathbf{u} \equiv u_1(X_1, X_2)\mathbf{e}_1 + u_2(X_1, X_2)\mathbf{e}_2$ is the two dimensional displacement field. The first two relations show that $\nabla^2 u_1$ and $\nabla^2 u_2$ are the real and, respectively, the imaginary parts of an analytic function.

[Hint: Use the Navier-Lamé system for the first two relations; the last one follows as a consequence of the compatibility relation in 2D.]

2. Show that the general two-dimensional stress transformation relations can be used to generate relations between the normal and shear stresses in a polar coordinate system in terms of Cartesian components

$$S_{rr} = \frac{1}{2}(S_{11} + S_{22}) + \frac{1}{2}(S_{11} - S_{22})\cos 2\theta + S_{12}\sin 2\theta \,,$$

$$S_{\theta\theta} = \frac{1}{2}(S_{11} + S_{22}) - \frac{1}{2}(S_{11} - S_{22})\cos 2\theta - S_{12}\sin 2\theta \,,$$

$$S_{r\theta} = \frac{1}{2}(S_{22} - S_{11}) + S_{12}\cos 2\theta \,.$$

3. In polar coordinates the plane stress system of equations of linear elasticity in the absence of inertial and body forces has the well-known form

$$\frac{\partial S_{rr}}{\partial r} + \frac{1}{r}\frac{\partial S_{r\theta}}{\partial \theta} + \frac{1}{r}(S_{rr} - S_{\theta\theta}) = 0 \,, \tag{1a}$$

$$\frac{\partial S_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial S_{\theta\theta}}{\partial \theta} + \frac{2}{r}S_{r\theta} = 0 \,. \tag{1b}$$

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The *Airy stress function* $\Phi = \Phi(r,\theta)$ is introduced by demanding that

$$S_{rr} = \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2}\,, \tag{2a}$$

$$S_{\theta\theta} = \frac{\partial^2 \Phi}{\partial r^2}\,, \tag{2b}$$

$$S_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Phi}{\partial \theta}\right)\,. \tag{2c}$$

(a) Show that (1) are identically satisfied by this choice, but the compatibility relation requires that Φ is **biharmonic**, i.e.

$$\nabla^4 \Phi = 0\,,$$

where $\nabla^4(\dots) = \nabla^2(\nabla^2(\dots))$ is the **bi-Laplacian** and

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\,.$$

(b) Establish (2) by using the definition of the Airy stress function in Cartesian coordinates.

[Hint: Use the chain rule to relate partial differentiation between (X_1, X_2) and (r,θ), together with the formulae recorded in Q.2.]

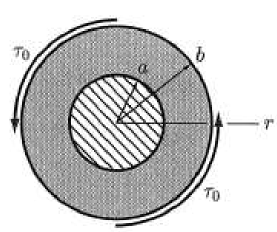
4. (a) By considering `Q.13` from *Examples #1*, show that in plane strain the components of the infinitesimal 2D strain tensor $\mathbf{E}$ with respect to a polar system of coordinates are

$$E_{rr} = \frac{\partial u_r}{\partial r}\,,$$

$$E_{\theta\theta} = \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right)\,,$$

$$E_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)\,.$$

(b) Consider the plane strain hollow circular shaft shown in cross-section below.



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A cylindrical coordinate system (r, θ, z) is defined with its origin at the centre of the cross-section. The inner surface of the shaft at $r = a$ is ideally bonded to a fixed *rigid* core, and the outer boundary at $r = b$ is subjected to uniform *shearing traction* τ_0. Assuming that the components of the stress tensor depend only on r, i.e. $S_{ij} = S_{ij}(r)$ $(i, j \in \{r, \theta\})$, and all the other fields of interest have the same property, use the semi-inverse method to find the displacements and stresses in the shaft.

[Hint: Use the equilibrium equations in the form given in (1). Also, in 2D problems with radial symmetry instead of using the compatibility relation in terms of stresses, it is more helpful to use the following equation

$$E_{rr} = \frac{d}{dr}(rE_{\theta\theta}),$$

which follows trivially from the first part of this question.]

5. A large elastic plate in equilibrium and made of an isotropic elastic material occupies the region

$$\{(X_1, X_2) \in \mathbb{R}^2 \mid -A < X_1 < A, \ -B < X_2 < B\}.$$

The plate is subjected to opposite shearing tractions R on the edges $X_1 = \pm A$ and, respectively, $X_2 = \pm B$, and no body forces are assumed to be present. The plate has a small hole of radius $r = a$ and centre $(0, 0)$, which is assumed to be traction free.

If the plate is very large (i.e., $A \to \infty$ and $B \to \infty$) show that the local stress field near the hole admits the polar coordinate representation

$$S_{rr}(r, \theta) = R\left[1 - 4\left(\frac{a}{r}\right)^2 + 3\left(\frac{a}{r}\right)^4\right]\sin 2\theta,$$

$$S_{r\theta}(r, \theta) = R\left[1 + 2\left(\frac{a}{r}\right)^2 - 3\left(\frac{a}{r}\right)^4\right]\cos 2\theta,$$

$$S_{\theta\theta}(r, \theta) = -R\left[1 + 3\left(\frac{a}{r}\right)^4\right]\sin 2\theta.$$

6. Consider an elastic plate in equilibrium which occupies the region

$$\{(X_1, X_2) \in \mathbb{R}^2 \mid -\infty < X_1 < \infty, \ -B < X_2 < B\}.$$

The plate is submitted to a uniform tension of magnitude P in the X_1-direction, i.e. $S_{11}(X_1, X_2) \to P$ as $X_1 \to \pm\infty$. A small hole of radius $r = a$ is made in the middle of the plate, where $a \ll B$ and the rim of the hole is assumed to be traction free.

Using an Airy stress function of the form

$$\Phi(r, \theta) = \left(Ar^2 + Br^4 + \frac{C}{r^2} + D\right)\cos 2\theta, \qquad (A, B, C, D \in \mathbb{R}),$$

show that the local stress distribution near the central hole is given by

$$S_{rr}(r,\theta) = \frac{P}{2}\left[1 - \left(\frac{a}{r}\right)^2\right] + \frac{P}{2}\left[1 - 4\left(\frac{a}{r}\right)^2 + 3\left(\frac{a}{r}\right)^4\right]\cos 2\theta\,,$$

$$S_{\theta\theta}(r,\theta) = \frac{P}{2}\left[1 + \left(\frac{a}{r}\right)^2\right] - \frac{P}{2}\left[1 + 3\left(\frac{a}{r}\right)^4\right]\cos 2\theta\,,$$

$$S_{r\theta}(r,\theta) = -\frac{P}{2}\left[1 + 2\left(\frac{a}{r}\right)^2 - 3\left(\frac{a}{r}\right)^4\right]\sin 2\theta\,.$$

[Hint: You need to identify the appropriate boundary conditions and then use (2) to fix the arbitrary constants A, B, C and D.]

7. A thin elastic plate in the shape of a circular annulus,

$$\big\{(r,\theta)\;\big|\; R_1 \le r \le R_2\,,\; 0 \le \theta < 2\pi\big\}\,,$$

is *stretched* by applying in-plane *radial displacements* of magnitude U_1 along the boundary $r = R_1$, and of magnitude U_2 along the boundary $r = R_2$.

(a) By assuming a radially symmmetric state of stress and deformation, show that the stress distribution in the stretched configuration admits the representation

$$S_{rr} = \frac{E}{1-\nu^2}\left[(1+\nu)\frac{U_1R_1 + U_2R_2}{R_2^2 - R_1^2} + (1-\nu)\frac{U_1R_2 + U_2R_1}{R_2^2 - R_1^2}\left(\frac{R_1R_2}{r^2}\right)\right], \qquad (3a)$$

$$S_{\theta\theta} = \frac{E}{1-\nu^2}\left[(1+\nu)\frac{U_1R_1 + U_2R_2}{R_2^2 - R_1^2} - (1-\nu)\frac{U_1R_2 + U_2R_1}{R_2^2 - R_1^2}\left(\frac{R_1R_2}{r^2}\right)\right], \qquad (3b)$$

where ν denotes the Poisson's ratio and E stands for the Young's modulus of the plate.

(b) Let

$$\lambda := \frac{U_1}{U_2}\,, \qquad \eta := \frac{R_1}{R_2}\,, \qquad \overline{\eta} := \sqrt{\frac{1-\nu}{1+\nu}}\,.$$

Prove that if $\eta < \overline{\eta}$ then the azimuthal stresses (3b) become compressive in the annular region

$$\big\{(r,\theta)\;\big|\; R_1 \le r \le R_2\,\overline{\rho}\,,\; 0 \le \theta < 2\pi\big\}\,,$$

where

$$\overline{\rho} := \sqrt{\left(\frac{1-\nu}{1+\nu}\right)\frac{\eta^2 + \lambda\eta}{1 + \lambda\eta}}\,. \qquad (4)$$

Find the minimum value of λ that corresponds to the onset of compressive azimuthal stresses in the annulus. Are there any compressive stresses when $\eta > \overline{\eta}$?

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8. An annular elastic plate of inner radius R_1, outer radius R_2 and thickness h ($h/R_2 \ll 1$) is initially stretched by imposing the uniform displacement field $U_0 > 0$ along the outer edge, $r = R_2$, while the inner boundary, $r = R_1$, is uniformly rotated through a small angle by applying a torque per unit volume of magnitude equal to M. This in-plane rotation is achieved by means of a *rigid* shaft.

   Using a cylindrical system of coordinates (r, θ, z) defined in an obvious manner, and assuming radial symmetry, show that the stress distribution in the plate is given by

$$S_{rr}(r) = \frac{E}{1+\nu}\left(\frac{U_0 R_2}{R_2^2 - R_1^2}\right)\left[\frac{1+\nu}{1-\nu} + \frac{R_1^2}{r^2}\right],$$

$$S_{\theta\theta}(r) = \frac{E}{1+\nu}\left(\frac{U_0 R_2}{R_2^2 - R_1^2}\right)\left[\frac{1+\nu}{1-\nu} - \frac{R_1^2}{r^2}\right],$$

$$S_{r\theta}(r) = \left(\frac{M}{2\pi h}\right)\frac{1}{r^2}\,.$$

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