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Mathematical Modelling III

Examples #2

Solutions of simple problems

1. Let $\mathbf{u}$ be a vector field. Show that $(\boldsymbol{\nabla} \otimes \mathbf{u}) \cdot \boldsymbol{\nabla} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u})$ and $\nabla^2(\boldsymbol{\nabla} \otimes \mathbf{u}) = \boldsymbol{\nabla} \otimes (\nabla^2 \mathbf{u})$.

2. If $\mathbf{A}$ is a tensor field, $\mathbf{u}$, $\mathbf{v}$ are two vector fields, and ϕ represents a scalar field, prove the following identities

$$\nabla^2(\phi \mathbf{u}) = (\nabla^2 \phi)\mathbf{u} + 2\boldsymbol{\nabla}\phi \cdot (\boldsymbol{\nabla} \otimes \mathbf{u}) + \phi(\nabla^2 \mathbf{u})\,, \tag{1}$$

$$\nabla^2(\mathbf{u} \cdot \mathbf{v}) = (\nabla^2 \mathbf{u}) \cdot \mathbf{v} + 2(\boldsymbol{\nabla} \otimes \mathbf{u}) : (\boldsymbol{\nabla} \otimes \mathbf{v}) + (\nabla^2 \mathbf{v}) \cdot \mathbf{u}\,, \tag{2}$$

$$\nabla^2(\mathbf{A} \cdot \mathbf{u}) = (\nabla^2 \mathbf{T}) \cdot \mathbf{u} + 2(\boldsymbol{\nabla} \otimes \mathbf{T})^T : (\boldsymbol{\nabla} \otimes \mathbf{u})^T + \mathbf{T} \cdot (\nabla^2 \mathbf{u})\,. \tag{3}$$

3. In the absence of body forces the equilibrium Navier-Lamé system has the form

$$(\lambda + \mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u} = \mathbf{0}\,.$$

   (a) Assuming that $\phi \equiv \phi(\mathbf{r})$ is a scalar field, verify that $\mathbf{u} = \boldsymbol{\nabla}\phi$ is a solution, if $\nabla^2 \phi = 0$.

   (b) Similarly (but involves more work), show that $\mathbf{u} \equiv \mathbf{u}(\mathbf{r})$ defined by

$$\mathbf{u} = 4(1 - \nu)\mathbf{f} - \boldsymbol{\nabla}(\mathbf{f} \cdot \mathbf{r})$$

   is a solution for some vector function $\mathbf{f} \equiv \mathbf{f}(\mathbf{r})$, if $\nabla^2 \mathbf{f} = \mathbf{0}$.

   (c) Explain why

$$\mathbf{u} = 4(1 - \nu)\mathbf{f} - \boldsymbol{\nabla}(\phi + \mathbf{f} \cdot \mathbf{r})$$

   is then also a solution.

   [Hint: Direct substitution and use of Q.2 above; you can avoid using the vector & tensor identities by working directly in indicial notation.]

4. (a) Show that the general Navier-Lamé equations can be cast in the equivalent form

$$(\lambda + 2\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) - \mu(\boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} \wedge \mathbf{u})) + \rho_0 \mathbf{b} = \rho_0 \ddot{\mathbf{u}}\,.$$

   (b) If $\mathbf{f}$ and $\mathbf{g}$ are two vector fields, check the following identity

$$\mathbf{f} \wedge (\boldsymbol{\nabla} \wedge \mathbf{g}) = \mathbf{f} \cdot (\mathbf{g} \otimes \boldsymbol{\nabla}) - (\mathbf{g} \otimes \boldsymbol{\nabla}) \cdot \mathbf{f}\,.$$

   Hence, or otherwise, prove that the stress vector $\mathbf{t} = \mathbf{n} \cdot \mathbf{S}$ admits the representation

$$\mathbf{t} = \lambda \mathbf{n}(\boldsymbol{\nabla} \cdot \mathbf{u}) + 2\mu(\mathbf{n} \cdot \boldsymbol{\nabla})\mathbf{u} + \mu(\mathbf{n} \wedge (\boldsymbol{\nabla} \wedge \mathbf{u}))\,.$$

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5. The displacement components in an isotropic linear elastic body are given by

$$u_1 \equiv 0\,, \qquad u_2 \equiv 0\,, \qquad u_3 = K\tan^{-1}(X_2/X_1)\,,$$

with $K \in \mathbb{R}$ a constant.
Calculate the strain and stress components in Cartesian coordinates. Hence, if there are no body forces, verify that the equilibrium equations are satisfied.

6. In an isotropic linear elastic solid the displacement field is given by

$$\mathbf{u} = -\tau X_2 X_3 \mathbf{e}_1 + \tau X_1 X_3 \mathbf{e}_2 + \tau\phi(X_1, X_2)\mathbf{e}_3\,,$$

where $\tau \in \mathbb{R}$ is an arbitrary constant. Show that the equilibrium equations in the absence of body forces are satisfied if ϕ is a harmonic function, i.e. $\nabla^2\phi = 0$.

[Hint: Direct substitution in the Navier-Lamé system. The same strategy works for the previous question as well.]

7. All the remaining questions on this sheet can be solved following the model discussed in your notes (pressurised thick spherical shell).

Consider the case of purely radial displacement in an isotropic linear elastic solid,

$$\mathbf{u} = f(r)\mathbf{r} \equiv r f(r)\mathbf{e}_r\,,$$

where $r = |\mathbf{r}|$ and f is some function.

(a) Show that the linearised strain tensor is given by

$$\mathbf{E} = f(r)\mathbf{I} + \frac{f'(r)}{r}\mathbf{r}\otimes\mathbf{r}\,.$$

(b) Furthermore, prove that

$$\mu\nabla^2\mathbf{u} + (\lambda+\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) = (\lambda+2\mu)\left[r f''(r) + 4f'(r)\right]\mathbf{e}_r\,.$$

8. Pressure $P > 0$ is applied to a spherical cavity of radius $r = R_1$ in an infinite isotropic elastic medium, which is unloaded at infinity. Assuming radial symmetry and that the centre of the sphere is at the origin, show that the radial displacement is

$$\mathbf{u} = \frac{P}{4\mu}\left(\frac{R_1}{r}\right)^3\mathbf{r}\,.$$

9. (this is the question discussed in your notes) A spherical shell of isotropic elastic material with internal radius $r = R_1$ and external radius $r = R_2$ is subjected to an internal pressure P_1 and external pressure P_2.

(a) Assuming *radially symmetric deformation* show that the only equilibrium equation which is not trivially satisfied is

$$\frac{\partial S_{rr}}{\partial r} + \frac{1}{r}\left(2S_{rr} - S_{\theta\theta} - S_{\varphi\varphi}\right) = 0\,.$$

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(b) By taking $\mathbf{u} = U(r)\mathbf{e}_r$ deduce that

$$r^2 \frac{d^2U}{dr^2} + 2r\frac{dU}{dr} - 2U = 0\,.$$

Find the general solution of this equation and establish that

$$S_{rr} = (2\mu + 3\lambda)C_1 - \frac{4\mu C_2}{r^3}\,,$$

where C_1, C_2 are constants that need to be found by using the boundary conditions mentioned in the hypothesis.

(c) Show that the non-zero stresses in the shell are

$$S_{rr} = \frac{R_1^3 P_1 - R_2^3 P_2}{R_2^3 - R_1^3} - \frac{R_1^3 R_2^3}{r^3}\left(\frac{P_1 - P_2}{R_2^3 - R_1^3}\right)\,,$$

$$S_{\theta\theta} = S_{\varphi\varphi} = \frac{R_1^3 P_1 - R_2^3 P_2}{R_2^3 - R_1^3} + \frac{R_1^3 R_2^3}{2r^3}\left(\frac{P_1 - P_2}{R_2^3 - R_1^3}\right)\,.$$

(d) Finally, show that the result of the previous question can be recovered from the present solution by taking $P_1 = P$, $P_2 = 0$ and $R_2 \to \infty$.

10. A spherical shell of isotropic elastic material has internal radius $r = R_1$ and external radius $r = R_2$. The inner boundary is subjected to a prescribed radial displacement of magnitude γR_1 ($\gamma \in \mathbb{R}$), while the outer boundary is free of traction.

    Show that the radial displacement at radius $r \in [R_1, R_2]$ is given by

$$A\left(r^{-2} + \frac{4\mu r}{3\kappa R_2^3}\right)\,,$$

    where

$$A = \frac{3\kappa\gamma(R_1R_2)^3}{3\kappa R_2^3 + 4\mu R_1^3}\,.$$

11. A rigid sphere of radius $r = R_1$ is surrounded by a concentric spherical shell of internal radius R_1 and external radius $r = R_2$, which is subject to a uniform hydrostatic pressure P on its outer boundary.

    Assuming radial symmetry, show that the displacement in the shell is

$$-\frac{PR_2^3}{4\mu R_1^3 + 3\kappa R_2^3}\left(r - \frac{R_1^3}{r^2}\right)\,.$$

    Find the non-zero stress components in the shell.

12. Investigate the problem of a self-gravitating spherical shell of internal radius $r = R_1$ and external radius $r = R_2$. Note that the volume of material within radius r is

$$\frac{4}{3}\pi(r^3 - R_1^3)$$

and the radial component of the gravitational body force (per unit mass) is

$$-\frac{4}{3}\pi\rho_0 G\left(r-\frac{R_1^3}{r^2}\right)\,,$$

where ρ_0 is the mass density and G is the gravitational constant.

(a) By letting $\mathbf{u}=f(r)\mathbf{r}$, show that the equilibrium equation can be reduced to

$$r\frac{d^2f}{dr^2}+4\frac{df}{dr}=\gamma\left(r-\frac{R_1^3}{r^2}\right)\,,$$

where the constant $\gamma\in\mathbb{R}$ should be identified.

(b) Obtain the general solution in the form

$$f(r)=Ar^{-3}+B+\frac{\gamma}{10}r^2+\frac{\gamma R_1^3}{2r}\,,$$

where $A,\,B\in\mathbb{R}$.

(c) Determine the constants for the case in which both boundaries $r=R_j$ $(j=1,2)$ are free of tractions.