UNIVERSITY of GLASGOW

Mathematical Modelling III

Examples #1

Review of vectors & tensors

1. (a) If $\mathbf{a}$ and $\mathbf{b}$ are orthogonal constant vectors, solve the equation

$$\mathbf{X} \wedge \mathbf{a} = \mathbf{b}\,. \tag{1}$$

   (b) Given four constant vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ such that $\mathbf{a} \cdot \mathbf{c} \neq 0$, solve the equation

$$\mathbf{X} \wedge \mathbf{a} + (\mathbf{X} \cdot \mathbf{b})\mathbf{c} = \mathbf{d}\,,$$

   by reducing it to (1).

2. Show that the general solution of the equation

$$\alpha\mathbf{X} + \mathbf{X} \wedge \mathbf{a} = \mathbf{b}\,,$$

where $\alpha \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b}$ are given constant vectors, is provided by the formula

$$\mathbf{X} = \frac{\alpha^2\mathbf{b} + (\mathbf{b} \cdot \mathbf{a})\mathbf{a} + \alpha(\mathbf{a} \wedge \mathbf{b})}{\alpha(\alpha^2 + |\mathbf{a}|^2)}\,.$$

[Hint: none of the above questions requires anything advanced; just use the fact that the vector product of two parallel vectors is zero, and/or the dot product of two orthogonal vectors is also zero, etc.]

3. Use basic properties of *triple scalar products* to show that

$$(\mathbf{a} \wedge \mathbf{b}) \cdot (\mathbf{c} \wedge \mathbf{d}) = \begin{vmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{a} \cdot \mathbf{d} \\ \mathbf{b} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{d} \end{vmatrix}\,.$$

Hence, or otherwise, establish the important identity

$$\epsilon_{ijk}\epsilon_{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}\,. \tag{2}$$

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4. Show that
$$(\mathbf{b} \wedge \mathbf{c}) \wedge (\mathbf{c} \wedge \mathbf{a}) = [\mathbf{a}, \mathbf{b}, \mathbf{c}]\mathbf{c}\,,$$
and deduce that
$$[\mathbf{b} \wedge \mathbf{c},\, \mathbf{c} \wedge \mathbf{a},\, \mathbf{a} \wedge \mathbf{b}] = [\mathbf{a}, \mathbf{b}, \mathbf{c}]^2\,.$$
[Hint: use the identity $\mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$, etc]

5. Prove the following gradient formulae
$$\text{(i)} \qquad \nabla \otimes (\phi \mathbf{u}) = (\nabla \phi) \otimes \mathbf{u} + \phi(\nabla \otimes \mathbf{u})\,,$$
$$\text{(ii)} \qquad \nabla(\mathbf{u} \cdot \mathbf{v}) = (\nabla \otimes \mathbf{u}) \cdot \mathbf{v} + (\nabla \otimes \mathbf{v}) \cdot \mathbf{u}\,,$$
$$\text{(iii)} \qquad \nabla \otimes (\mathbf{T} \cdot \mathbf{u}) = (\nabla \otimes \mathbf{T}) \cdot \mathbf{u} + (\nabla \otimes \mathbf{u}) \cdot \mathbf{T}^T\,,$$
$$\text{(iv)} \qquad \nabla(\phi\varphi) = (\nabla \phi)\varphi + \phi(\nabla \varphi)\,,$$
where ϕ, φ are a scalar field, $\mathbf{u}$, $\mathbf{v}$ are vector fields, and $\mathbf{T}$ is a second-order tensor field.
[Hint: use Cartesian coordinates to evaluate the component form of the left- and right-hand sides, and then compare the results, etc.]

6. Using the above formulae obtain the following results, where $\mathbf{r}$ is the position vector with respect to the origin, $r = |\mathbf{r}|$, and $\mathbf{b}$ is a constant unit vector,
$$\nabla(r^n) = nr^{n-2}\mathbf{r}\,,$$
$$\nabla \otimes (r^n \mathbf{r}) = r^n \mathbf{I} + nr^{n-2}\mathbf{r} \otimes \mathbf{r}\,,$$
$$\nabla \otimes (r^n \mathbf{b}) = nr^{n-2}\mathbf{r} \otimes \mathbf{b}\,,$$
$$\nabla \otimes (r^n \mathbf{r} \cdot \mathbf{b}) = r^n \mathbf{b} + nr^{n-2}\mathbf{r} \otimes \mathbf{r} \cdot \mathbf{b}\,,$$
$$\nabla \otimes (r^n \mathbf{r} \otimes \mathbf{r} \cdot \mathbf{b}) = r^n\left[\mathbf{I} \otimes \mathbf{r} \cdot \mathbf{b} + \mathbf{b} \otimes \mathbf{r}\right] + nr^{n-2}\mathbf{r} \otimes \mathbf{r} \otimes \mathbf{r} \cdot \mathbf{b}\,.$$

7. Prove the following divergence formulae
$$\text{(i)} \qquad \nabla \cdot (\phi \mathbf{u}) = (\nabla \phi) \cdot \mathbf{u} + \phi(\nabla \cdot \mathbf{u})\,,$$
$$\text{(ii)} \qquad \nabla \cdot (\mathbf{u} \otimes \mathbf{v}) = (\nabla \cdot \mathbf{u})\mathbf{v} + \mathbf{u} \cdot (\nabla \otimes \mathbf{v})\,,$$
$$\text{(iii)} \qquad \nabla \cdot (\mathbf{T} \cdot \mathbf{u}) = (\nabla \cdot \mathbf{T}) \cdot \mathbf{u} + \mathbf{T}:(\nabla \otimes \mathbf{u})\,,$$
$$\text{(iv)} \qquad \nabla \cdot (\mathbf{u} \wedge \mathbf{v}) = (\nabla \wedge \mathbf{u}) \cdot \mathbf{v} - \mathbf{u} \cdot (\nabla \wedge \mathbf{v})\,,$$
where ϕ is a scalar field, $\mathbf{u}$, $\mathbf{v}$ are vector fields, and $\mathbf{T}$ is a second-order tensor field.
[Hint: same as above, etc.]

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8. Practice the divergence formulae by establishing formulae

$$\boldsymbol{\nabla}\cdot(r^n\mathbf{r}) = (n+3)r^n\,,$$

$$\boldsymbol{\nabla}\cdot(r^n\mathbf{r}\otimes\mathbf{r}) = (n+4)r^n\mathbf{r}\,,$$

$$\boldsymbol{\nabla}\cdot(r^n\mathbf{I}\otimes\mathbf{r}\cdot\mathbf{b}) = r^n\mathbf{b} + nr^{n-2}\mathbf{r}\otimes\mathbf{r}\cdot\mathbf{b}\,,$$

$$\boldsymbol{\nabla}\cdot[r^n(\mathbf{r}\otimes\mathbf{b}+\mathbf{b}\otimes\mathbf{r})] = (n+4)r^n\mathbf{b} + nr^{n-2}\mathbf{r}\otimes\mathbf{r}\cdot\mathbf{b}\,,$$

$$\boldsymbol{\nabla}\cdot(r^n\mathbf{r}\otimes\mathbf{r}\otimes\cdot\mathbf{b}) = (n+5)r^n\mathbf{r}\otimes\mathbf{r}\cdot\mathbf{b}\,.$$

9. Prove the following curl formulae

(i) $$\boldsymbol{\nabla}\wedge(\phi\mathbf{u}) = (\boldsymbol{\nabla}\phi)\wedge\mathbf{u} + \phi(\boldsymbol{\nabla}\wedge\mathbf{u})\,,$$

(ii) $$\boldsymbol{\nabla}\wedge(\mathbf{u}\otimes\mathbf{v}) = (\boldsymbol{\nabla}\wedge\mathbf{u})\otimes\mathbf{v} - \mathbf{u}\wedge(\boldsymbol{\nabla}\otimes\mathbf{v})\,,$$

(iii) $$\boldsymbol{\nabla}\wedge(\mathbf{u}\wedge\mathbf{v}) = \mathbf{u}(\boldsymbol{\nabla}\cdot\mathbf{v}) + \mathbf{v}\cdot(\boldsymbol{\nabla}\otimes\mathbf{u}) - \mathbf{u}\cdot(\boldsymbol{\nabla}\otimes\mathbf{v}) - \mathbf{v}(\boldsymbol{\nabla}\cdot\mathbf{u})\,,$$

(iv) $$\boldsymbol{\nabla}\wedge(\boldsymbol{\nabla}\wedge\mathbf{u}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) - \nabla^2\mathbf{u}\,,$$

(v) $$\boldsymbol{\nabla}\wedge(\boldsymbol{\nabla}\phi) = \mathbf{0}\,,$$

where $\mathbf{u}$, $\mathbf{v}$ are vector fields and ϕ is a scalar field.

10. Show that if $\mathbf{u}$ is a vector field and $\mathbf{I}$ the usual second-order identity tensor, then

(a) $$\mathbf{I}\wedge(\boldsymbol{\nabla}\wedge\mathbf{u}) = \mathbf{u}\otimes\boldsymbol{\nabla} - \boldsymbol{\nabla}\otimes\mathbf{u}\,,$$

(b) $$\boldsymbol{\nabla}\wedge(\mathbf{I}\wedge\mathbf{u}) = \mathbf{u}\otimes\boldsymbol{\nabla} - \mathbf{I}(\boldsymbol{\nabla}\cdot\mathbf{u})\,,$$

(c) $$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\wedge\mathbf{u}) = 0\,,$$

(d) $$\boldsymbol{\nabla}\wedge(\boldsymbol{\nabla}\otimes\mathbf{u}) = \mathbf{0}\,.$$

[Hint: use the definition of the curl and the vector product in conjunction with formula (2).]

Hence, or otherwise, show that if

$$\mathbf{E} = \frac{1}{2}(\mathbf{u}\otimes\boldsymbol{\nabla} + \boldsymbol{\nabla}\otimes\mathbf{u})$$

then $\boldsymbol{\nabla}\wedge(\mathbf{E}\wedge\boldsymbol{\nabla}) = \mathbf{0}$.

11. The *double vector product* of two tensors is defined by

$$(\mathbf{a}\otimes\mathbf{b})\times\times(\mathbf{c}\otimes\mathbf{d}) \equiv (\mathbf{a}\wedge\mathbf{c})\otimes(\mathbf{b}\wedge\mathbf{d}).$$

Show that if $\mathbf{A}$ is a symmetric second-order tensor field, then

$$\mathbf{I}\times\times[\boldsymbol{\nabla}\wedge(\mathbf{A}\wedge\boldsymbol{\nabla})] = \boldsymbol{\nabla}\otimes(\boldsymbol{\nabla}\cdot\mathbf{A}) + (\boldsymbol{\nabla}\cdot\mathbf{A})\otimes\boldsymbol{\nabla} - \boldsymbol{\nabla}\otimes(\boldsymbol{\nabla}|\mathbf{A}|) - \nabla^2\mathbf{A}\,,$$

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where $|\mathbf{A}| \equiv \mathrm{tr}(\mathbf{A})$ is the first principal invariant of $\mathbf{A}$.

[Warning: This is a fairly difficult question.]

12. Let $\mathbf{u}$ be a vector field and $\mathbf{A} = \mathbf{I} + k(\mathbf{v} \otimes \mathbf{v})$, where $k \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^2$ are constants. Show that

$$\boldsymbol{\nabla} \cdot (\mathbf{A} \cdot \mathbf{u}) = \boldsymbol{\nabla} \cdot \mathbf{u} + k\mathbf{v} \cdot (\mathbf{u} \otimes \boldsymbol{\nabla}) \cdot \mathbf{v}\,,$$

$$\boldsymbol{\nabla} \cdot (\mathbf{A} \wedge \mathbf{u}) = \boldsymbol{\nabla} \wedge \mathbf{u} - k\mathbf{v} \cdot (\mathbf{u} \otimes \boldsymbol{\nabla}) \wedge \mathbf{v}\,.$$

13. (a) If f is a scalar field, then

$$\nabla f = \frac{\partial f}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\mathbf{e}_\theta + \frac{\partial f}{\partial z}\,\mathbf{e}_z\,;$$

(b) The gradient of the vector field $\mathbf{u}$ is given by

$$\begin{aligned}\mathbf{u} \otimes \boldsymbol{\nabla} = {} & \frac{\partial u_r}{\partial r}\mathbf{e}_r \otimes \mathbf{e}_r + \frac{1}{r}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right)\mathbf{e}_r \otimes \mathbf{e}_\theta + \frac{\partial u_r}{\partial z}\mathbf{e}_r \otimes \mathbf{e}_z \\ & + \frac{\partial u_\theta}{\partial r}\mathbf{e}_\theta \otimes \mathbf{e}_r + \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \frac{\partial u_\theta}{\partial z}\mathbf{e}_\theta \otimes \mathbf{e}_z \\ & + \frac{\partial u_z}{\partial r}\mathbf{e}_z \otimes \mathbf{e}_r + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\mathbf{e}_z \otimes \mathbf{e}_\theta + \frac{\partial u_z}{\partial z}\mathbf{e}_z \otimes \mathbf{e}_z\,.\end{aligned}$$

14. Show that the divergence of a second-order tensor field $\mathbf{T}$ wih respect to $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is a vector field of the form,

$$\nabla \cdot \mathbf{T} = (\nabla \cdot \mathbf{T})_r\,\mathbf{e}_r + (\nabla \cdot \mathbf{T})_\theta\,\mathbf{e}_\theta + (\nabla \cdot \mathbf{T})_z\,\mathbf{e}_z\,,$$

whose components are given by

$$\begin{aligned}(\nabla \cdot \mathbf{T})_r &= \frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{\partial T_{rz}}{\partial z} + \frac{1}{r}(T_{rr} - T_{\theta\theta})\,, \\ (\nabla \cdot \mathbf{T})_\theta &= \frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{\theta z}}{\partial z} + \frac{1}{r}(T_{r\theta} + T_{\theta r})\,, \\ (\nabla \cdot \mathbf{T})_z &= \frac{\partial T_{zr}}{\partial r} + \frac{1}{r}\frac{\partial T_{z\theta}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} + \frac{1}{r}T_{zr}\,.\end{aligned}$$

15. The Laplacian of a field (scalar, vector, tensor) is defined to be the divergence of its gradient, and it is usually denoted by $\nabla^2 \equiv \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$ or Δ. Show that for a *scalar field* f expressed in cylindrical coordinates the following formula holds

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}\,.$$

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For a vector field $\mathbf{u}$ the counterpart of the above formula is

$$\nabla^2\mathbf{u} = (\nabla^2\mathbf{u})_r\,\mathbf{e}_r + (\nabla^2\mathbf{u})_\theta\,\mathbf{e}_\theta + (\nabla^2\mathbf{u})_z\,\mathbf{e}_z\,,$$

where

$$(\nabla^2\mathbf{u})_r = \nabla^2 u_r - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta} - \frac{1}{r^2}u_r\,,$$
$$(\nabla^2\mathbf{u})_\theta = \nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{1}{r^2}u_\theta\,,$$
$$(\nabla^2\mathbf{u})_z = \nabla^2 u_z\,.$$

16. For a cylindrical coordinate system (r,θ,z) obtain the component form for the curl,

$$\boldsymbol{\nabla}\wedge\mathbf{u} = (\boldsymbol{\nabla}\wedge\mathbf{u})_r\,\mathbf{e}_r + (\boldsymbol{\nabla}\wedge\mathbf{u})_\theta\,\mathbf{e}_\theta + (\boldsymbol{\nabla}\wedge\mathbf{u})_z\,\mathbf{e}_z\,,$$

where

$$(\boldsymbol{\nabla}\wedge\mathbf{u})_r = \frac{1}{r}\left(\frac{\partial u_z}{\partial\theta} - r\frac{\partial u_\theta}{\partial z}\right)\,,$$
$$(\boldsymbol{\nabla}\wedge\mathbf{u})_\theta = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\,,$$
$$(\boldsymbol{\nabla}\wedge\mathbf{u})_z = \frac{1}{r}\left[\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial\theta}\right]\,.$$

17. The components of a vector field $\mathbf{v}$ with regard to spherical polar coordinates (r,θ,φ) are (u,v,w). Write down the expression for the components of $\boldsymbol{\nabla}\wedge\mathbf{v}$ in terms of u, v, and w. If

$$u = v = \frac{\partial w}{\partial\varphi} \equiv 0 \qquad \text{and} \qquad \boldsymbol{\nabla}\wedge(\boldsymbol{\nabla}\wedge\mathbf{v}) = \mathbf{0},$$

show that

$$r^2\frac{\partial^2 w}{\partial r^2} + 2r\frac{\partial w}{\partial r} + \frac{\partial}{\partial\theta}\left(\frac{\partial w}{\partial\theta} + w\cot\theta\right) = 0\,.$$

By using a solution with separable variables of the form $f(r)\sin\theta$, find the general expression for $f(r)$.

18. (a) Consider the **bi-harmonic equation** in cylindrical polar coordinates (r,θ,z),

$$\nabla^4\Phi = 0\,,$$

where $\nabla^4(\bullet) = \nabla^2(\nabla^2(\bullet))$ is the **bi-Laplacian**,

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\,,$$

and $\Phi = \Phi(r,\theta)$.

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(b) Check that the following functions can be used as particular solutions of the above equation,

$$C\theta\,, \qquad Cr^2\theta\,, \qquad Cr\theta\sin\theta\,, \qquad Cr\theta\cos\theta\,, \qquad (C\in\mathbb{R})\,.$$

Furthermore, verify that $\Phi(r,\theta)=f_n(r)\cos n\theta$ and $\Phi(r,\theta)=f_n(r)\sin n\theta$ are enjoying the same property, where

$$f_0(r)=a_0r^2+b_0r^2\log r+c_0+d_0\log r\,,$$
$$f_1(r)=a_1r^3+b_1r+c_1r\log r+d_1r^{-1}\,,$$
$$f_n(r)=a_nr^{n+2}+b_nr^n+c_nr^{-n+2}+d_nr^{-n}\,, \qquad (n>1)\,,$$

and a_n, b_n, c_n, $d_n\in\mathbb{R}$.

19. Let $\Omega\subset\mathbb{R}^2$ be a bounded domain and $\partial\Omega$ its boundary curve. Show that if u, $v:\Omega\longrightarrow\mathbb{R}$ are two sufficiently smooth functions (and $\partial\Omega$ is assumed to be regular) then

$$\int_\Omega\left(u\nabla^2v+\nabla u\cdot\nabla v\right)\mathrm{d}A=\oint_{\partial\Omega}u\frac{\partial v}{\partial n}\,\mathrm{d}S\,,$$

where

$$\frac{\partial v}{\partial n}:=\mathbf{n}\cdot\nabla v$$

represents the *normal derivative* of v.

[This FUNDAMENTAL result is konwn as **Green's Theorem**. It can be written in a number of different ways -- we shall have the opportunity to see this later on.]

20. Let $g\equiv g(X_3)$ be a given function and $\beta\in\mathbb{R}$ a constant. If $\mathbf{r}=X_1\mathbf{e}_1+X_2\mathbf{e}_2+X_3\mathbf{e}_3$ is the usual position vector in $\mathbb{R}^3$, define the tensor field $\mathbf{F}$ by

$$\mathbf{F}=\boldsymbol{\nabla}\otimes\left[g(X_3)\mathbf{e}_3-\frac{X_3^2}{2}\mathbf{r}\right]+\mathbf{e}_3\otimes\boldsymbol{\nabla}\left[\frac{1}{2}(\beta X_3^2+X_3r^2)\right]\,,$$

where $r^2=\mathbf{r}\cdot\mathbf{r}$.

(a) Show that

$$\mathbf{u}=\left[g(X_3)+\frac{1}{2}\left(\beta X_3^2+X_3r^2\right)\right]\mathbf{e}_3-\frac{X_3^2}{2}\mathbf{r}$$

is a particular solution of the equation (in $\mathbf{u}$),

$$\mathbf{u}\otimes\boldsymbol{\nabla}+\boldsymbol{\nabla}\otimes\mathbf{u}=2\mathbf{F}\,. \qquad (3)$$

(b) Explain why any rigid deformation can be added to this solution to yield yet another solution of (3).

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21. Consider the tensor field

$$\mathbf{F} = -KX_1\left[(1+\alpha)\mathbf{e}_3 \otimes \mathbf{e}_3 - \alpha\mathbf{I}\right],$$

where $K, \alpha \in \mathbb{R}$ are given constants.

(a) Show that this tensor field can be cast in the form

$$\mathbf{F} = -K\boldsymbol{\nabla} \otimes \mathbf{a} + K\mathbf{e}_1 \otimes \boldsymbol{\nabla}\phi, \qquad (4)$$

where $\mathbf{a} \equiv \mathbf{a}(\mathbf{r})$ and $\phi \equiv \phi(\mathbf{r})$ are vector and, respectively, scalar fields that you should identify.

(b) Check that

$$\mathbf{u} = K\left\{\frac{1}{2}\left[X_3^2 + \alpha(X_1^2 - X_2^2)\right]\mathbf{e}_1 + \alpha X_1 X_2 \mathbf{e}_2 - X_1 X_3 \mathbf{e}_3\right\}$$

is a solution of the equation

$$\mathbf{u} \otimes \boldsymbol{\nabla} + \boldsymbol{\nabla} \otimes \mathbf{u} = 2\mathbf{F}.$$

(c) By using (4) deduce that the particular solution of this equation stated above could have been obtained by inspection. The same assertion applies to `Q.20`.

`[Note: these last two questions are relevant to what we shall study later on.]`

22. Consider the fourth-order tensors $\mathbb{C}$ and $\mathbb{S}$ whose components are defined by the following formulae

$$C_{klmn} = \lambda\,\delta_{kl}\delta_{mn} + \mu(\delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm}),$$

$$S_{klmn} = -\frac{\lambda}{2\mu(3\lambda + 2\mu)}\,\delta_{kl}\delta_{mn} + \frac{1}{4\mu}\,(\delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm}),$$

where λ, μ are real constants, $(3\lambda + 2\mu)\mu \neq 0$.

Show that $\mathbb{C}$ is the inverse of $\mathbb{S}$.

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