Some important equations involving ∇^2

Harmonic equation in polar coordinates: $\nabla^2\phi \equiv \dfrac{\partial^2\phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial\phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial\phi}{\partial\theta^2} = 0$

General solution:

$$\phi(r,\theta) = (A_{00} + A_{01}\log r) + \sum_{n=1}^{\infty}(A_{n0}r^n + A_{n1}r^{-n})\cos n\theta + \sum_{n=1}^{\infty}(B_{n0}r^n + B_{n1}r^{-n})\sin n\theta$$

The arbitrary constants A_{nj} $(j = 0, 1)$ are determined by using appropriate boundary conditions or other requirements consistent with the problem at hand (e.g., in the case when the domain over which the equation is integrated is the disk centred at the origin, we need to ensure that the solution is bounded as $r \to 0$ and thus $A_{01} = 0$, $A_{n1} = B_{n1} = 0$ for all $n \geq 1$).

Bi-harmonic equation in polar coordinates: $\nabla^4\phi \equiv \nabla^2(\nabla^2\phi) = \left(\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial}{\partial\theta^2}\right)^2 \phi = 0$

General solution:

$$\phi(r,\theta) = f_0(r) + \sum_{n=1}^{\infty} f_n(r)\cos n\theta + \sum_{n=1}^{\infty} g_n(r)\sin n\theta\,,$$

where

$$f_0(r) = A_{01}r^2 + A_{02}r^2\log r + A_{03}\log r + A_{04}\,,$$

$$f_1(r) = A_{11}r^3 + A_{12}r\log r + A_{13}r + A_{14}r^{-1}\,,$$

$$f_n(r) = A_{n1}r^{n+2} + A_{n2}r^{-n+2} + A_{n3}r^n + A_{n4}r^{-n}\,.$$

The functions $g_n(r)$ mentioned above have the same structure as the f_n's $(n \geq 1)$, with the only difference that the constants involved can be different.