UNIVERSITY
of
GLASGOW

Department of Mathematics

Level 4: Mathematical Modelling III

Background information

More than two hundred years ago the mechanics of solid bodies and the flow of fluids provided much of the impetus needed for the development of mathematics and its many specialised branches. It is therefore not surprising that nowadays many practical problems can be understood relatively easily by employing the "right" mathematical apparatus. The calculus of complex variables is one such representative example that is still hugely relevant to the study of both fluids and elastic solids.

Elasticity deals with reversible (non-permanent) motions and deformations experienced by solid bodies when subjected to various loading conditions. Rubber is an archetypal elastic material, but most metals behave elastically when subjected to sufficiently weak forces. The mathematical theory for describing the elasticity of metals is quite different from that exhibited by rubbers under large strains, which is essentially a nonlinear behaviour. Linear elasticity has always played a pivotal role in the design of buildings and in assessing the safety margin of many structural components (e.g., pipes, plates, walls, roofs, etc).

Fluid mechanics is concerned with the study of liquids & gases in motion or in equilibrium, and has wide applicability in our everyday life. Examples range from the design of aircraft and ships to weather forecasting and blood flow in arteries.

Aims of the course

To provide a *mathematical* introduction to a variety of physical phenomena involving fluid flow and small deformations of elastic solids.

Pre-requisites

This course is the natural continuation of *MM I & II*, which represent essential prerequisites for this course. The course will also require a basic knowledge of vector calculus, complex analysis, and simple methods for solving both partial and ordinary differential equations.

Recommended reading

It is important to keep in mind that the lecturer will **not** follow any particular textbook. Having said that, the following texts contain material similar to that discussed in this course:

1. W. S. Slaughter, *The Linearised Theory of Elasticity*, Birkhauser, 2002.

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2. R.J. Atkin and N. Fox, *An Introduction to the Theory of Elasticity*, Dover Publications, 2005.

3. D. J. Acheson, *Elementary Fluid Dynamics*, Oxford University Press, 1990.

Contents

1. Review of tensor calculus and continuum mechanics (using the dyadic notation).
2. Linearised kinematics (infinitesimal strains and infinitesimal rotations).
3. Linear constitutive laws and the elastic constants. Restrictions imposed by the well-posedness of the corresponding boundary-value problems.
4. Compatibility equations: the Beltrami-Michell system. Linearised displacement field equations: the Lamé-Navier system.
5. Plane stress, plane strain, anti-plane strain. Airy stress functions (e.g., stretching of a plate with a circular hole, bending of a beam by terminal couples or by a transverse force, stretching of a circular beam, etc).
6. Review of complex variables; conformal mappings.
7. Complex variables in linear elasticity.
8. Further antiplane problems: torsion of shafts with various cross-sections. Prandtl stress function.
9. Preliminary concepts and ideal fluids: streamlines, material derivatives, steady flows, 2D flows, Bernoulli's streamline theorem, vorticity, irrotational flows, circulation, vorticity equation.
10. Velocity potential, stream function, complex potential, method of images, Milne-Thompson circle theorem, irrotational flow past a circular cylinder, Blasius' theorem.
11. Elementary viscous flow: Navier-Stokes equations, Reynolds number, flow between two coaxial cylinders.
12. High Reynolds number flow and boundary layers (if time permits).
13. Energy methods and their use to find approximate solutions in linear elasticity (if time permits).

Learning objectives

On completion of this course the students will be expected to know and understand the main aspects of the theory and should

- be able to state the various linear constitutive laws for elastic materials;
- have an understanding of the role played by the constitutive behaviour in formulating various boundary-value problems (BVP);
- be able to derive the Lamé-Navier system of PDE's and use the compatibility conditions to obtain the Beltrami-Michell equations;

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- understand the mathematical arguments which lead to the uniqueness of the displacement or stress fields in linear elasticity;
- explain the role played by the Saint-Venant's Principle in setting up and solving BVP's;
- solve a variety of representative problems related to plane stress, plane strain or anti-plane strain situations;
- describe the torsion of beams of circular and polygonal cross-sections and explain the role of the Prandtl function in this context; solve simple torsion problems for a range of different cross-sections;
- be able to use effectively the complex variable formalism for solving linear elasticity and incompressible 2D flow problems;
- be able to define physical quantities and measures as detailed in the course;
- have a good understanding of the fundamental equations describing the motion of incompressible Newtonian fluids, their derivation, and some of the various tools available to study them in the context of simple problems;
- be able to apply their knowledge of solution techniques to solve classical problems, to appreciate the complications associated with more realistic problems, and to discuss the limitations of the mathematical techniques.

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