

NFE2105/NFM2106: Mathematics

Week 10 – ODEs & Laplace transforms

School of Computing & Engineering

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HUDDERSFIELD

10.1 Introduction

Often LTs offer a convenient method of finding solutions to IVPs, both homogeneous and otherwise.

As always, when using LT tables it is often necessary to manipulate the format somewhat first. The main results are given in your table of LTs, which is available on Brightspace (under `Lecture Notes` → `Week 10`).

Before looking at a few representative examples it is necessary to introduce some standard notations and a couple of **key formulae**.

10.2 Notation

The first derivative is $\frac{dy}{dt}$, which can also be written as y' or $y'(t)$.

The second derivative is $\frac{d^2y}{dt^2}$, which can also be written as y'' or $y''(t)$.

Initial conditions are those which apply at time $t = 0$; recall that for first-order ODEs, an IVP has just one initial condition, while for second-order ODEs we need two initial conditions.

$y(0)$ denotes the value of $y(t)$ when $t = 0$.

$y'(0)$ denotes the value of $y'(t) \equiv \frac{dy}{dt}$ when $t = 0$.

10.3 Laplace transforms of derivatives

The Laplace transform of the first derivative is:

$$\mathcal{L} \left[\frac{dy}{dt} \right] = sY(s) - y(0)$$

The Laplace transform of the second derivative is:

$$\mathcal{L} \left[\frac{d^2y}{dt^2} \right] = s^2Y(s) - sy(0) - y'(0)$$

10.4 Solution strategy:

- STEP 1:** Take the LT of all the terms on both sides of the ODE, and use the formulae on the previous page to “get rid” of the derivatives.
- STEP 2:** Input the initial conditions.
- STEP 3:** Re-arrange the algebraic equation obtained to make $Y(s)$ the subject, and then solve for this function.
- STEP 4:** Compute the inverse Laplace transform of $Y(s)$ found at the previous step in order to get the solution $y = y(t)$ of the original IVP. This step is carried out with the help of the LTs table in conjunction with the linearity property of the inverse of the LT.

10.5 Examples:

(a) Solve $\frac{dy}{dt} - 4y = t$, given that $y(0) = 1$.

(b) Solve $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 5y = e^{2t}$, subject to $y(0) = y'(0) = 0$.

(c) Solve $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 10$, with $y(0) = 0$ and $y'(0) = 2$.

(d) Solve $\frac{d^2y}{dt^2} + 4y = 12t$, given that $y = 0$ and $y' = 9$ at $t = 0$.

10.5 Solutions:

(a) Taking the LT of all terms in the ODE + using the linearity of the LT:

$$\mathcal{L}[y'(t)] - 4\mathcal{L}[y(t)] = \mathcal{L}[t] \Rightarrow [sY(s) - y(0)] - 4Y(s) = \frac{1}{s^2}$$

Input initial conditions: $[sY(s) - 1] - 4Y(s) = \frac{1}{s^2}$

Make $Y(s)$ the subject: $Y(s) = \frac{s^2 + 1}{s^2(s - 4)}$

Next, use partial fractions to manipulate $Y(s)$ so that you can find its inverse Laplace transform by using the info available in the LTs table.

Details on next page.

10.5 Solutions (cont'd):

$$(a) Y(s) = \frac{s^2 + 1}{s^2(s - 4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s - 4}, \quad A, B, C = \text{to be found.}$$

Eventually, after the usual manipulations: $A = -\frac{1}{16}$, $B = -\frac{1}{4}$, $C = \frac{17}{16}$.

$$\text{Thus, } Y(s) = -\frac{1}{16s} - \frac{1}{4s^2} + \frac{17}{16} \left(\frac{1}{s - 4} \right).$$

Inverting:

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[Y(s)] \\ &= -\frac{1}{16} \mathcal{L}^{-1} \left[\frac{1}{s} \right] - \frac{1}{4} \mathcal{L}^{-1} \left[\frac{1}{s^2} \right] + \frac{17}{16} \mathcal{L}^{-1} \left[\frac{1}{s - 4} \right] \\ &= -\frac{1}{16} - \frac{t}{4} + \frac{17}{16} e^{4t}. \end{aligned}$$

10.5 Solutions:

(b) Taking the LT of all terms in the ODE + using the linearity of the LT:

$$\begin{aligned}\mathcal{L}[y''(t)] - 4\mathcal{L}[y'(t)] + 5\mathcal{L}[y(t)] &= \mathcal{L}[e^{2t}] \Rightarrow \\ [s^2 Y(s) - sy(0) - y'(0)] - 4[sY(s) - y(0)] + 5Y(s) &= \frac{1}{s-2}\end{aligned}$$

$$\text{Input initial conditions: } s^2 Y(s) - 4sY(s) + 5Y(s) = \frac{1}{s-2}$$

$$\text{Make } Y(s) \text{ the subject: } Y(s) = \frac{1}{(s-2)(s^2 - 4s + 5)}$$

Next, use partial fractions to manipulate $Y(s)$ so that you can find its inverse Laplace transform by using the info available in the LTs table.

Details on next page.

10.5 Solutions (cont'd):

$$(b) Y(s) = \frac{1}{(s-2)(s^2-4s+5)} = \frac{A}{s-2} + \frac{Bs+C}{s^2-4s+5}, \quad A, B, C = ?$$

Eventually, after the usual manipulations: $A = 1, B = -1, C = 2$.

$$\text{Thus, } Y(s) = \frac{1}{s-2} - \frac{s-2}{(s-2)^2 + 1^2}.$$

Inverting:

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[Y(s)] \\ &= \mathcal{L}^{-1}\left[\frac{1}{s-2}\right] - \mathcal{L}^{-1}\left[\frac{s-2}{(s-2)^2 + 1^2}\right] \\ &= e^{2t} - e^{2t} \cos(t). \end{aligned}$$

10.5 Solutions:

(c) Taking the LT of all terms in the ODE + using the linearity of the LT:

$$\mathcal{L}[y''(t)] + 5\mathcal{L}[y'(t)] + 6\mathcal{L}[y(t)] = \mathcal{L}[10] \Rightarrow$$
$$[s^2 Y(s) - sy(0) - y'(0)] + 5[sY(s) - y(0)] + 6Y(s) = \frac{10}{s}$$

Input initial conditions: $[s^2 Y(s) - 2] + 5sY(s) + 6Y(s) = \frac{10}{s}$

Make $Y(s)$ the subject: $Y(s) = \frac{2(s+5)}{s(s+2)(s+3)}$

Next, use partial fractions to manipulate $Y(s)$ so that you can find its inverse Laplace transform by using the info available in the LTs table.

Details on next page.

10.5 Solutions (cont'd):

$$(c) Y(s) = \frac{2(s+5)}{s(s+2)(s+3)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3}, \quad A, B, C = ?$$

Eventually, after the usual manipulations: $A = \frac{5}{3}$, $B = -3$, $C = \frac{4}{3}$.

$$\text{Thus, } Y(s) = \frac{5}{3s} - \frac{3}{s+2} + \frac{4}{3} \left(\frac{1}{s+3} \right).$$

Inverting:

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[Y(s)] \\ &= \frac{5}{3} \mathcal{L}^{-1} \left[\frac{1}{s} \right] - 3 \mathcal{L}^{-1} \left[\frac{1}{s+2} \right] + \frac{4}{3} \mathcal{L}^{-1} \left[\frac{1}{s+3} \right] \\ &= \frac{5}{3} - 3e^{-2t} + \frac{4}{3}e^{-3t}. \end{aligned}$$

10.5 Solutions:

(d) Taking the LT of all terms in the ODE + using the linearity of the LT:

$$\mathcal{L}[y''(t)] + 4\mathcal{L}[y(t)] = \mathcal{L}[12t] \Rightarrow$$
$$[s^2 Y(s) - sy(0) - y'(0)] + 4Y(s) = \frac{12}{s^2}$$

Input initial conditions: $[s^2 Y(s) - 9] + 4Y(s) = \frac{12}{s^2}$

Make $Y(s)$ the subject: $Y(s) = \frac{3(3s^2 + 4)}{s^2(s^2 + 4)}$

Next, use partial fractions to manipulate $Y(s)$ so that you can find its inverse Laplace transform by using the info available in the LTs table.

Details on next page.

10.5 Solutions (cont'd):

$$(d) Y(s) = \frac{3(3s^2 + 4)}{s^2(s^2 + 4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 + 4}, \quad A, B, C, D = ?$$

Eventually, after the usual manipulations: $A = 0$, $B = 3$, $C = 0$, $D = 6$.

$$\text{Thus, } Y(s) = \frac{3}{s^2} + \frac{6}{s^2 + 4}.$$

Inverting:

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[Y(s)] \\ &= 3\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] + \frac{6}{2}\mathcal{L}^{-1}\left[\frac{2}{s^2 + 2^2}\right] \\ &= 3t + 3\sin(2t). \end{aligned}$$