NFE2105/NFM2106: Mathematics
Week 10 – Introduction to Laplace transforms

School of Computing & Engineering

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10.1 Introduction

This is an advanced mathematical technique used for solving
initial-value problems (IVPs) in which the independent variable is
time t, and where the initial conditions at time $t = 0$ are given.

For example, in an electrical circuit the voltage and current at time
$t = 0$ might be given (i.e., the instant of time when the circuit is
switched on).

The Laplace-transform technique allows us to be able to deal with
differential equations containing discontinuous functions, i.e. functions
that undergo an abrupt change in value at some particular values of
time.

10.2 Definition

The formal definition of the **Laplace transform** is given below:

$$\mathcal{L}[f(t)] \equiv F(s) := \int_0^\infty e^{-st} f(t)\, dt$$

ASIDE:

Integrals like the one above, in which the upper integration limit is $+\infty$ are called **improper integrals**. By definition, such integrals are calculated according to the following rule:

$$\int_0^\infty f(x)\, dx := \lim_{L \to \infty} \int_0^L f(x)\, dx$$

The theory of such integrals is beyond the scope of this module.

10.3 The linearity property

Rather than use the above definition that relies on integrals, **we'll make use of tables**. It will be necessary therefore to manipulate our functions to match the format of one of the tabulated functions. We shall also make use of the so-called **linearity property** of Laplace transforms:

If c_1 and c_2 are constants, and $f(t)$ and $g(t)$ any functions of t, then

$$\mathcal{L}[c_1 f(t) + c_2 g(t)] = c_1 \mathcal{L}[f(t)] + c_2 \mathcal{L}[g(t)]$$

A particular case of this important formula is:

$$\mathcal{L}[c_1 f(t)] = c_1 \mathcal{L}[f(t)]$$

Examples:

Find the Laplace transforms of:

- $f_1(t) = 4t^2$
- $f_2(t) = 2$
- $f_3(t) = 3t + e^{-2t} + 3\sin(6t)$

Solutions:

$$\mathcal{L}[f_1(t)] = \mathcal{L}[4t^2] = 4\mathcal{L}[t^2] = 4\left(\frac{2!}{s^{2+1}}\right) = \frac{8}{s^3}$$

$$\mathcal{L}[f_2(t)] = \mathcal{L}[2] = \frac{2}{s}$$

$$\mathcal{L}[f_3(t)] = 3\mathcal{L}[t] + \mathcal{L}[e^{-2t}] + 3\mathcal{L}[\sin(6t)] = \frac{3}{s^2} + \frac{1}{s+2} + 3\left(\frac{6}{s^2 + 6^2}\right)$$

$$= \frac{3}{s^2} + \frac{1}{s+2} + \frac{18}{s^2 + 36}$$

10.5 Observations

The Laplace transform of the product of two functions is **not** the product of their Laplace transforms; that is, if $f(t)$ and $g(t)$ are two functions, then

$$\mathcal{L}[f(t)g(t)] \neq \mathcal{L}[f(t)]\mathcal{L}[g(t)]$$

This means that we **cannot** find the Laplace transform of the product of two functions of t by finding their Laplace transforms individually, and then multiplying them together.

10.6 First Shift Theorem:

If the Laplace transform of $f(t)$ is $F(s)$, then

$$\mathcal{L}[e^{at}f(t)] = F(s-a)$$

This results shows that **some products** of functions can be found in a very simple way. More specifically, if we have the product of an exponential function in t and any other function in t, we can use the so-called *First Shift Theorem* stated above.

The Laplace-transform table on Brightspace gives the results for two such special cases, but you won't be using the *First Shift Theorem* otherwise this year.

10.7 Inverse Laplace Transforms:

Given a function $F(s)$ we can find the function $f(t)$ such that the Laplace transform of this latter function matches the former one, i.e. $\mathcal{L}[f(t)] = F(s)$.

As we are going to use this approach when applying the Laplace transform to the solution of various IVPs, we are going to assume that we have $Y(s)$ and we are interested in $y(t)$ such that $\mathcal{L}[y(t)] = Y(s)$.

We write

$$y(t) = \mathcal{L}^{-1}[Y(s)]$$

where $\mathcal{L}^{-1}$ represents the **inverse Laplace transform**.

10.7 Inverses (cont'd):

To find the inverse Laplace transforms we can read the table "backwards". Complicated functions need to be manipulated algebraically to bring them into the form of either a single term or a sum of terms, each of which is a constant multiple of a term (or terms) which appears (or appear) in our table of Laplace transforms. The manipulations will frequently involve finding partial fractions, as well as the use of the **linearity property** (for $\mathcal{L}^{-1}$):

$$\mathcal{L}^{-1}\left[c_1\,Y_1(s) + c_2\,Y_2(s)\right] = c_1\mathcal{L}^{-1}[Y_1(s)] + c_2\mathcal{L}^{-1}[Y_2(s)]$$

10.7 Examples:

(a) $Y(s) = \dfrac{7}{s}$

(b) $Y(s) = \dfrac{4}{s^6} - \dfrac{7}{s-9} + \dfrac{2s+3}{s^2+5}$

Formulae from the **table**:

$$\mathcal{L}^{-1}\left[\frac{n!}{s^{n+1}}\right] = t^n$$

Answers:

(a) $y(t) = 7\mathcal{L}^{-1}\left[\dfrac{1}{s}\right] = 7(t^0) = 7$

(b) $\mathcal{L}^{-1}[Y(s)]$ is equal to:

$$4\mathcal{L}^{-1}\left[\frac{1}{s^6}\right] - 7\mathcal{L}^{-1}\left[\frac{1}{s-9}\right]$$

$$+\, 2\mathcal{L}^{-1}\left[\frac{s}{s^2+(\sqrt{5})^2}\right] + \frac{3}{\sqrt{5}}\mathcal{L}^{-1}\left[\frac{\sqrt{5}}{s^2+(\sqrt{5})^2}\right]$$

$$\mathcal{L}^{-1}\left[\frac{1}{s+k}\right] = e^{-kt}$$

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos(at)$$

$$\mathcal{L}^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin(at)$$

10.7 Examples (cont'd):

(b) $\mathcal{L}^{-1}[Y(s)]$ is equal to:

Formulae from the **table**:

$$4\mathcal{L}^{-1}\left[\frac{1}{s^6}\right] - 7\mathcal{L}^{-1}\left[\frac{1}{s-9}\right]$$

$$+ 2\mathcal{L}^{-1}\left[\frac{s}{s^2 + (\sqrt{5})^2}\right]$$

$$+ \frac{3}{\sqrt{5}}\mathcal{L}^{-1}\left[\frac{\sqrt{5}}{s^2 + (\sqrt{5})^2}\right]$$

$$= \frac{t^5}{30} - 7e^{9t} + 2\cos(t\sqrt{5}) + \frac{3}{\sqrt{5}}\sin(t\sqrt{5})$$

$$\mathcal{L}^{-1}\left[\frac{n!}{s^{n+1}}\right] = t^n$$

$$\mathcal{L}^{-1}\left[\frac{1}{s+k}\right] = e^{-kt}$$

$$\mathcal{L}^{-1}\left[\frac{s}{s^2 + a^2}\right] = \cos(at)$$

$$\mathcal{L}^{-1}\left[\frac{a}{s^2 + a^2}\right] = \sin(at)$$

10.7 Partial Fractions

Find the inverse Laplace transform of $Y(s) = \dfrac{3s + 3}{(s - 1)(s + 2)}$.

Look for the partial-fraction decomposition of this function, i.e.

$$\frac{3s + 3}{(s - 1)(s + 2)} = \frac{A}{s - 1} + \frac{B}{s + 2}$$

Multiply both sides by $(s - 1)(s + 2)$ to eliminate the denominators

$$3s + 3 = (s + 2)A + (s - 1)B,$$

Eventually we find $A = 2$ and $B = 1$. Thus,

$$Y(s) = \frac{2}{s - 1} + \frac{1}{s + 2} \quad \Rightarrow \quad y(t) = 2\mathcal{L}^{-1}\left[\frac{1}{s - 1}\right] + \mathcal{L}^{-1}\left[\frac{1}{s + 2}\right]$$

$$\text{or } y(t) = 2e^t + e^{-2t}$$

10.7 Completing the square

If $Y(s)$ is a fraction with an **irreducible quadratic** denominator consisting of three terms, you must complete the square of the first two terms (i.e., the s^2 and s terms). This is achieved by adding on **the square of half of the coefficient of s to the expression consisting of the s^2 and s terms**. Before doing this you must ensure that the coefficient of s^2 is 1; you'll also have to subtract what you added in order to ensure that the value of the original expression has not changed.

More details on the next few pages.

10.7 Completing the square

Here are more details on **completing the square:**

The expression $s^2 + 2s + 1$ is considered a perfect square since it can be re-written as the square of $(s + 1)$. That is, $s^2 + 2s + 1 = (s + 1)^2$. The expression $s^2 - 4s + 4$ is also considered a perfect square since $s^2 - 4s + 4 = (s - 2)^2$.

The expression $s^2 + 6s$ is **not** a perfect square.. However, by adding 9 to it, the expression becomes a perfect square: $s^2 + 6s + 9 = (s + 3)^2$.

Question: How do we tackle the general case: $s^2 + Ps$?
(where P is some concrete numerical constant)

10.7 Completing the square

STEP 1: Take one-half of P, the coefficient of s, obtaining $P/2$

STEP 2: Square $P/2$, obtaining $P^2/4$

STEP 3: Add $P^2/4$ to $s^2 + Ps$, obtaining

$$s^2 + Ps + \frac{P^2}{4} = \left(s + \frac{P}{2} \right)^2$$

STEP 4: We now have a perfect square; remember to subtract what you added, i.e.

$$s^2 + Ps = \left(s + \frac{P}{2} \right)^2 - \frac{P^2}{4}$$

10.7 Completing the square

Examples:

$$\begin{aligned}
s^2 + 6s + 10 &= (s^2 + 6s) + 10 \\
&= (s^2 + 6s + 9) - 9 + 10 \\
&= (s + 3)^2 + 1
\end{aligned}$$

$$\begin{aligned}
s^2 + 8s + 3 &= (s^2 + 8s) + 3 \\
&= (s^2 + 8s + 16) - 16 + 3 \\
&= (s + 4)^2 - 13
\end{aligned}$$

$$\begin{aligned}
s^2 - 4s + 29 &= (s^2 - 4s) + 29 \\
&= (s^2 - 4s + 4) - 4 + 29 \\
&= (s - 2)^2 + 25
\end{aligned}$$

10.7 Further examples of LT inverses:

Examples:

Find the inverse Laplace transforms of

(a) $Y(s) = \dfrac{s+3}{s^2 + 6s + 10}$

(b) $Y(s) = \dfrac{3}{s^2 - 4s + 29}$

(c) $Y(s) = \dfrac{2s - 7}{(s-2)^2 + 5^2}$ by expressing the RHS in the form:

$$c_1 \left\{ \frac{s-2}{(s-2)^2 + 5^2} \right\} + c_2 \left\{ \frac{5}{(s-2)^2 + 5^2} \right\}$$

10.7 Further examples of LT inverses:

Solutions (inverse Laplace transforms):

(a) $\dfrac{s+3}{s^2+6s+10} = \dfrac{s+3}{(s+3)^2+1} = \dfrac{s+3}{(s+3)^2+1^2}$

Use the table ($k=3$ and $a=1$) to find that the inverse of this corresponds to $y(t) = e^{-3t}\cos(t)$.

(b) $\dfrac{3}{s^2-4s+29} = \dfrac{3}{(s-2)^2+5^2} = \dfrac{3}{5}\left[\dfrac{5}{(s-2)^2+5^2}\right]$

Use the table ($k=-2$ and $a=5$) to find that the inverse corresponds to $y(t) = \dfrac{3}{5}e^{2t}\sin(5t)$.

$$\mathcal{L}^{-1}\left[\frac{a}{(s+k)^2+a^2}\right] = e^{-kt}\sin(at)$$

10.7 Further examples of LT inverses:

Solutions (inverse Laplace transforms):

For (*c*):

$$\frac{2s - 7}{(s - 2)^2 + 5^2} = 2\left\{\frac{s - 2}{(s - 2)^2 + 5^2}\right\} - \frac{3}{5}\left\{\frac{5}{(s - 2)^2 + 5^2}\right\}.$$

Using the table (see the formulae below), we find

$$y(t) = 2e^{2t}\cos(5t) - \frac{3}{5}e^{2t}\sin(5t).$$

$$\mathcal{L}^{-1}\left[\frac{a}{(s + k)^2 + a^2}\right] = e^{-kt}\sin(at)$$

$$\mathcal{L}^{-1}\left[\frac{s + k}{(s + k)^2 + a^2}\right] = e^{-kt}\cos(at)$$