



NFM2106/NFE2105

Mathematics

Complex Numbers (Part 2)



Outline/Learning outcomes

- ❖ In the second part of this unit we are going to be concerned with various aspects related to complex numbers (please make sure you check the first part of the notes on complex numbers)
- ❖ Polar form of complex numbers; principal argument
- ❖ Integral powers of complex numbers
- ❖ Solutions of the equation $w^n = z$ (the n n -th roots of a non-zero complex number z)

Modulus

The **modulus** of a complex number $z = x + iy$, is the real number

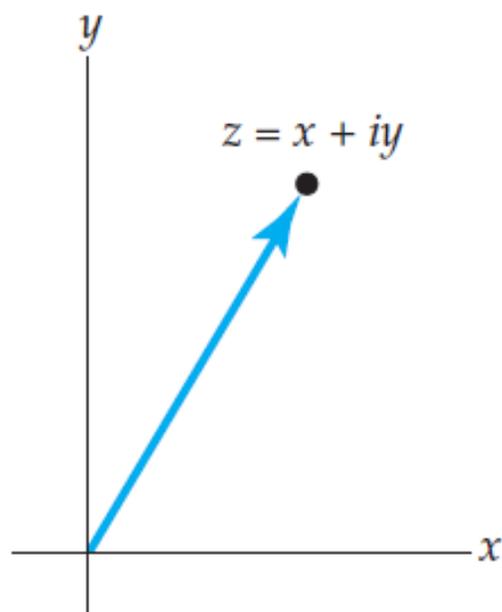
$$|z| = \sqrt{x^2 + y^2}. \quad (1)$$

Properties:

$$|z|^2 = z\bar{z}$$

$$|z_1 z_2| = |z_1| |z_2|$$

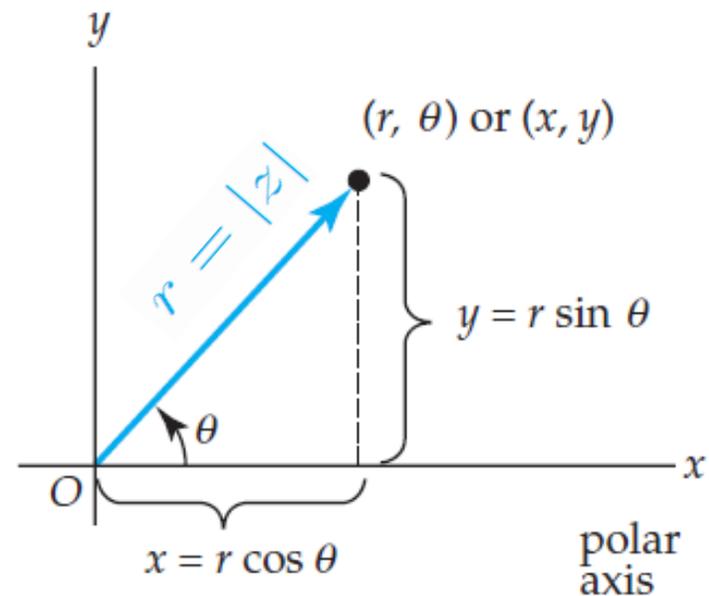
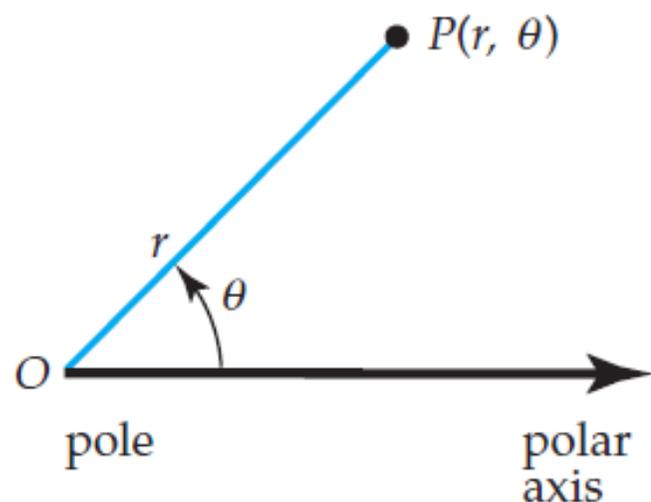
$$\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$$



EXAMPLE 1 Modulus of a Complex Number

If $z = 2 - 3i$, then from (1) we find the modulus of the number to be $|z| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$. If $z = -9i$, then (1) gives $|-9i| = \sqrt{(-9)^2} = 9$.

Polar form



$$z = r (\cos \theta + i \sin \theta).$$

$$z = r \angle \theta$$

θ is called an **argument** of z

$$\theta = \arg(z)$$



arg(z)

If θ_0 is an argument of z , then the angles $\theta_0 \pm 2\pi, \theta_0 \pm 4\pi, \dots$ are also arguments of z .

In practice, we find θ by using the trig equation $\tan \theta = y/x$

Because the 'tan' function is π –periodic, some care must be taken when solving this equation.

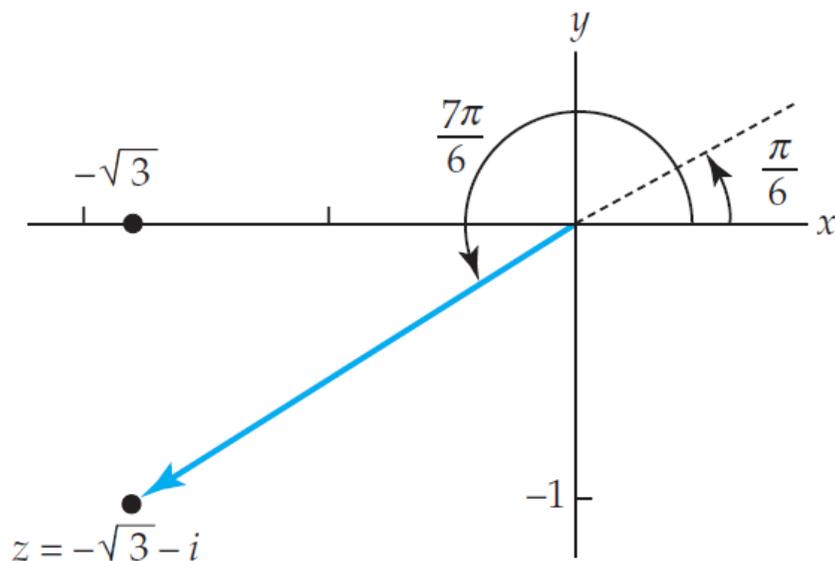
A calculator will give only angles that satisfy $-\pi/2 < \tan^{-1}(y/x) < \pi/2$

i.e., angles in the first and fourth quadrant. We MUST choose θ consistent with the quadrant in which our complex number z is located; this may require adding or subtracting π to $\tan^{-1}(y/x)$ when appropriate.

Polar form: example

Express $-\sqrt{3} - i$ in polar form.

Solution With $x = -\sqrt{3}$ and $y = -1$ we obtain $r = |z| = \sqrt{(-\sqrt{3})^2 + (-1)^2} = 2$. Now $y/x = -1/(-\sqrt{3}) = 1/\sqrt{3}$, and so a calculator gives $\tan^{-1}(1/\sqrt{3}) = \pi/6$, which is an angle whose terminal side is in the first quadrant. But since the point $(-\sqrt{3}, -1)$ lies in the third quadrant, we take the solution of $\tan \theta = -1/(-\sqrt{3}) = 1/\sqrt{3}$ to be $\theta = \arg(z) = \pi/6 + \pi = 7\pi/6$.

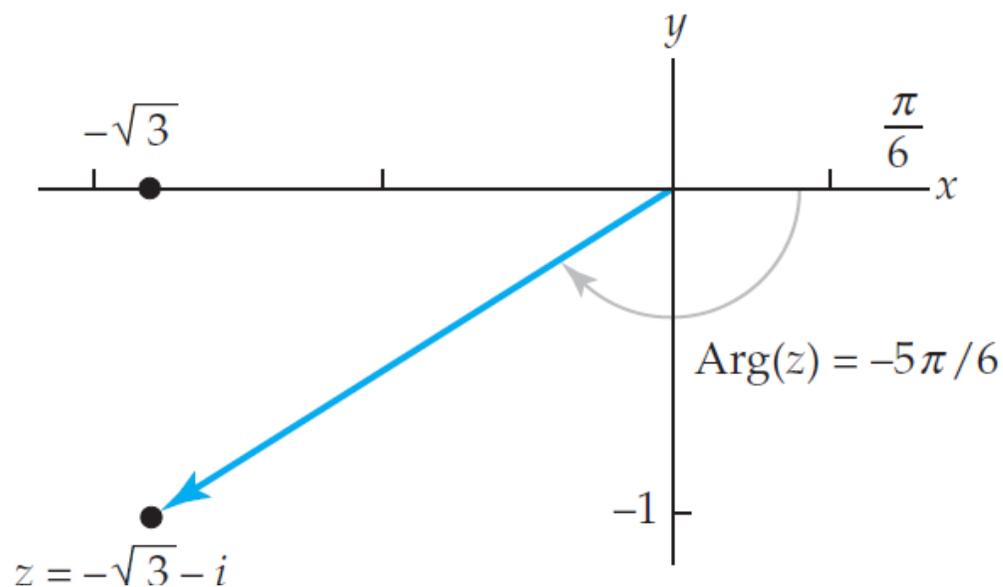


$$z = 2 \left(\cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6} \right)$$

Principal argument

Principal Argument The symbol $\arg(z)$ actually represents a set of values, but the argument θ of a complex number that lies in the interval $-\pi < \theta \leq \pi$ is called the **principal value** of $\arg(z)$ or the **principal argument** of z . The principal argument of z is *unique* and is represented by the symbol $\text{Arg}(z)$, that is,

$$-\pi < \text{Arg}(z) \leq \pi$$



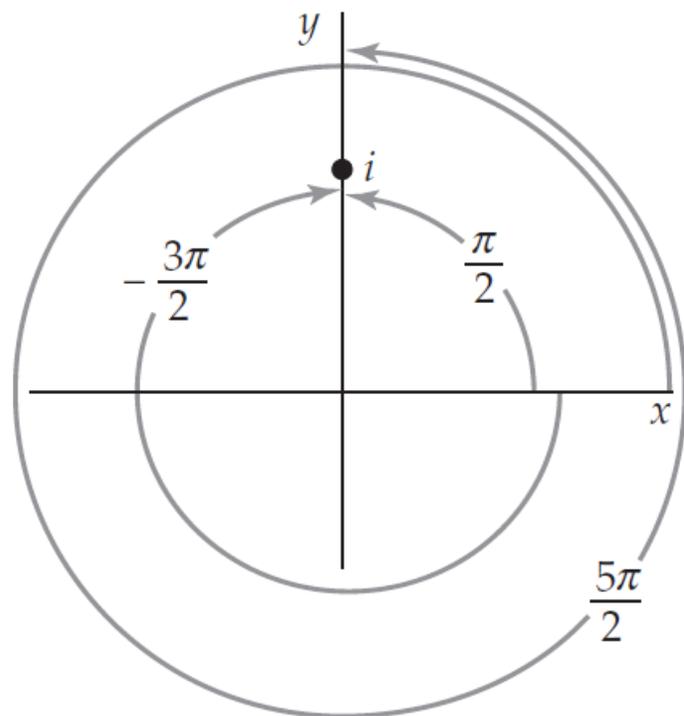
$$\text{Arg}(z) = \pi/6 - \pi = -5\pi/6$$

$$z = 2 \left[\cos \left(-\frac{5\pi}{6} \right) + i \sin \left(-\frac{5\pi}{6} \right) \right]$$

Principal argument (cont'd)

In general, $\arg(z)$ and $\text{Arg}(z)$ are related by

$$\arg(z) = \text{Arg}(z) + 2n\pi, \quad n = 0, \pm 1, \pm 2, \dots$$



For example, $\arg(i) = \frac{\pi}{2} + 2n\pi$.

For the choices $n = 0$ and $n = -1$

$$\arg(i) = \text{Arg}(i) = \pi/2$$

$$\arg(i) = -3\pi/2$$



Multiplication and division

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$$

$$z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$$

$$z_1 z_2 = r_1 r_2 [\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos (\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2)].$$

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$



Multiplication and division (example)

We have just seen that for $z_1 = i$ and $z_2 = -\sqrt{3} - i$ that $\text{Arg}(z_1) = \pi/2$ and $\text{Arg}(z_2) = -5\pi/6$, respectively.

$$z_1 z_2 = i(-\sqrt{3} - i) = 1 - \sqrt{3}i$$

$$\frac{z_1}{z_2} = \frac{i}{-\sqrt{3} - i} = -\frac{1}{4} - \frac{\sqrt{3}}{4}i$$

$$\arg(z_1 z_2) = \frac{\pi}{2} + \left(-\frac{5\pi}{6}\right) = -\frac{\pi}{3}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2} - \left(-\frac{5\pi}{6}\right) = \frac{4\pi}{3}$$



Integer powers of z

$$z^n = r^n (\cos n\theta + i \sin n\theta)$$

EXAMPLE:

Compute z^3 for $z = -\sqrt{3} - i$

$$z = 2[\cos(7\pi/6) + i \sin(7\pi/6)]$$

$$r = 2, \theta = 7\pi/6, \text{ and } n = 3$$

$$(-\sqrt{3} - i)^3 = 2^3 \left[\cos\left(3 \frac{7\pi}{6}\right) + i \sin\left(3 \frac{7\pi}{6}\right) \right] = 8 \left[\cos \frac{7\pi}{2} + i \sin \frac{7\pi}{2} \right] = -8i$$

since $\cos(7\pi/2) = 0$ and $\sin(7\pi/2) = -1$



Roots $w^n = z$

Suppose $z = r(\cos \theta + i \sin \theta)$ and $w = \rho(\cos \phi + i \sin \phi)$

$$\rho^n (\cos n\phi + i \sin n\phi) = r(\cos \theta + i \sin \theta)$$

$$\rho^n = r$$

$$\rho = \sqrt[n]{r}$$

$$\cos n\phi + i \sin n\phi = \cos \theta + i \sin \theta$$

$$\cos n\phi = \cos \theta$$

$$\sin n\phi = \sin \theta$$

$$n\phi = \theta + 2k\pi$$

$$\phi = \frac{\theta + 2k\pi}{n}$$



Roots $w^n = z$

The n n th roots of a nonzero complex number

$$z = r(\cos \theta + i \sin \theta)$$

are given by

$$w_k = \sqrt[n]{r} \left[\cos \left(\frac{\theta + 2k\pi}{n} \right) + i \sin \left(\frac{\theta + 2k\pi}{n} \right) \right]$$

where $k = 0, 1, 2, \dots, n - 1$



EXAMPLE 1:

Find the three cube roots of $z = i$

Solution we are basically solving the equation $w^3 = i$

$$z = \cos(\pi/2) + i \sin(\pi/2) \quad r = 1, \theta = \arg(i) = \pi/2 \quad n = 3$$

$$w_k = \sqrt[3]{1} \left[\cos \left(\frac{\pi/2 + 2k\pi}{3} \right) + i \sin \left(\frac{\pi/2 + 2k\pi}{3} \right) \right], \quad k = 0, 1, 2$$

$$k = 0, \quad w_0 = \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$k = 1, \quad w_1 = \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$k = 2, \quad w_2 = \cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} = -i.$$



EXAMPLE 2:

Find the four fourth roots of $z = 1 + i$

Solution In this case, $r = \sqrt{2}$ and $\theta = \arg(z) = \pi/4$.

$$w_k = \sqrt[4]{2} \left[\cos \left(\frac{\pi/4 + 2k\pi}{4} \right) + i \sin \left(\frac{\pi/4 + 2k\pi}{4} \right) \right], \quad k = 0, 1, 2, 3.$$

$$k = 0, \quad w_0 = \sqrt[4]{2} \left[\cos \frac{\pi}{16} + i \sin \frac{\pi}{16} \right] \approx \mathbf{1.06955 + 0.212748i}$$

$$k = 1, \quad w_1 = \sqrt[4]{2} \left[\cos \frac{9\pi}{16} + i \sin \frac{9\pi}{16} \right] \approx \mathbf{-0.212748 + 1.06955i}$$

$$k = 2, \quad w_2 = \sqrt[4]{2} \left[\cos \frac{17\pi}{16} + i \sin \frac{17\pi}{16} \right] \approx \mathbf{-1.06955 - 0.212748i}$$

$$k = 3, \quad w_3 = \sqrt[4]{2} \left[\cos \frac{25\pi}{16} + i \sin \frac{25\pi}{16} \right] \approx \mathbf{0.212748 - 1.06955i}$$