NFM2106/NFE2105
Mathematics

Integration (numerical approximation)
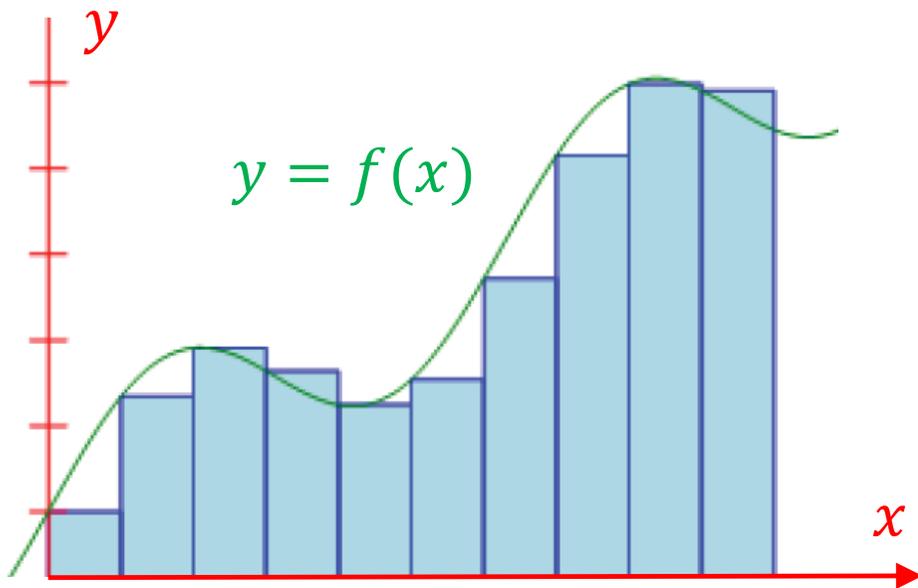
Outline/learning outcomes

❖ The **Trapezium** Rule

❖ **Simpson**'s Rule

❖ You'll learn to **approximate** definite integrals that are not easily evaluated by the standard techniques (substitution, parts, and partial fractions)
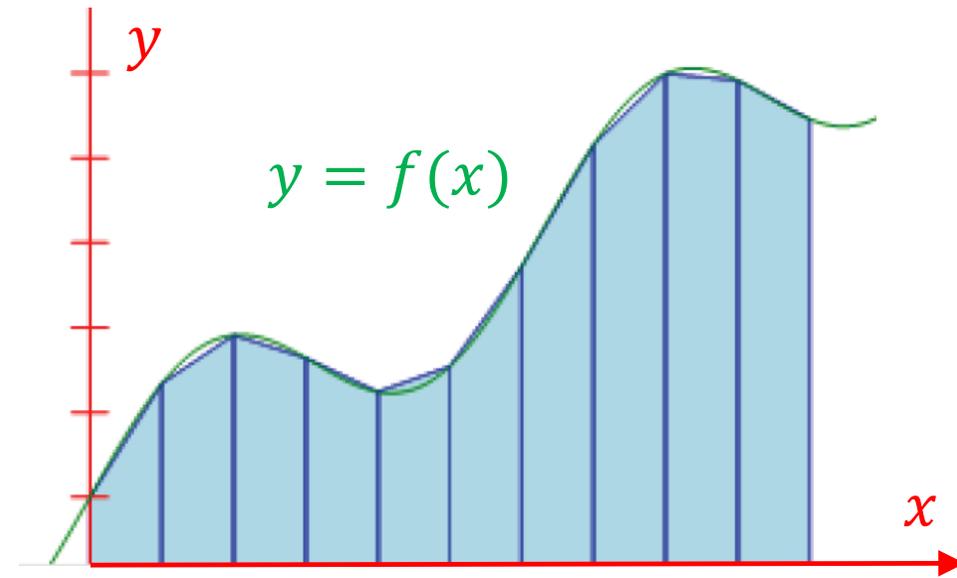
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General Idea: approximate area by simpler shapes

RECTANGLES:

y

$y = f(x)$

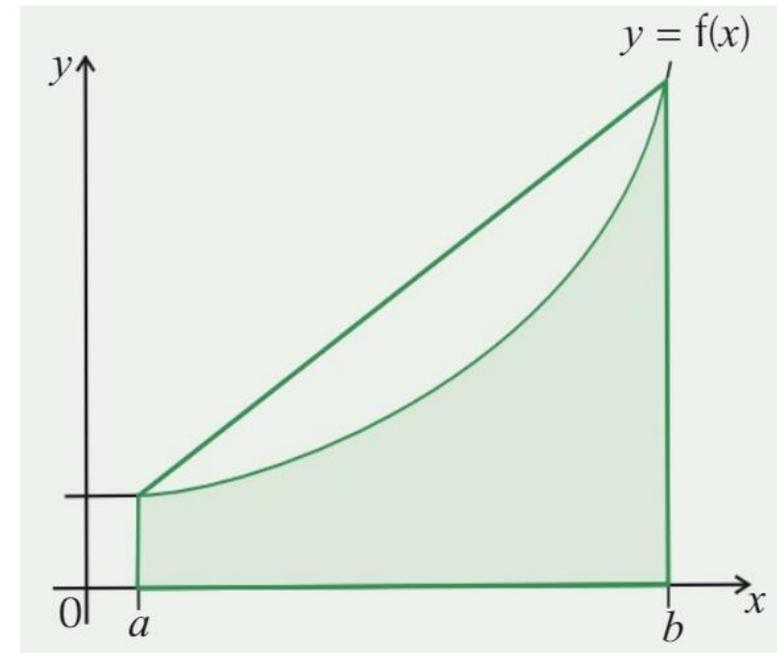x

TRAPEZIUMS:

y

$y = f(x)$

x

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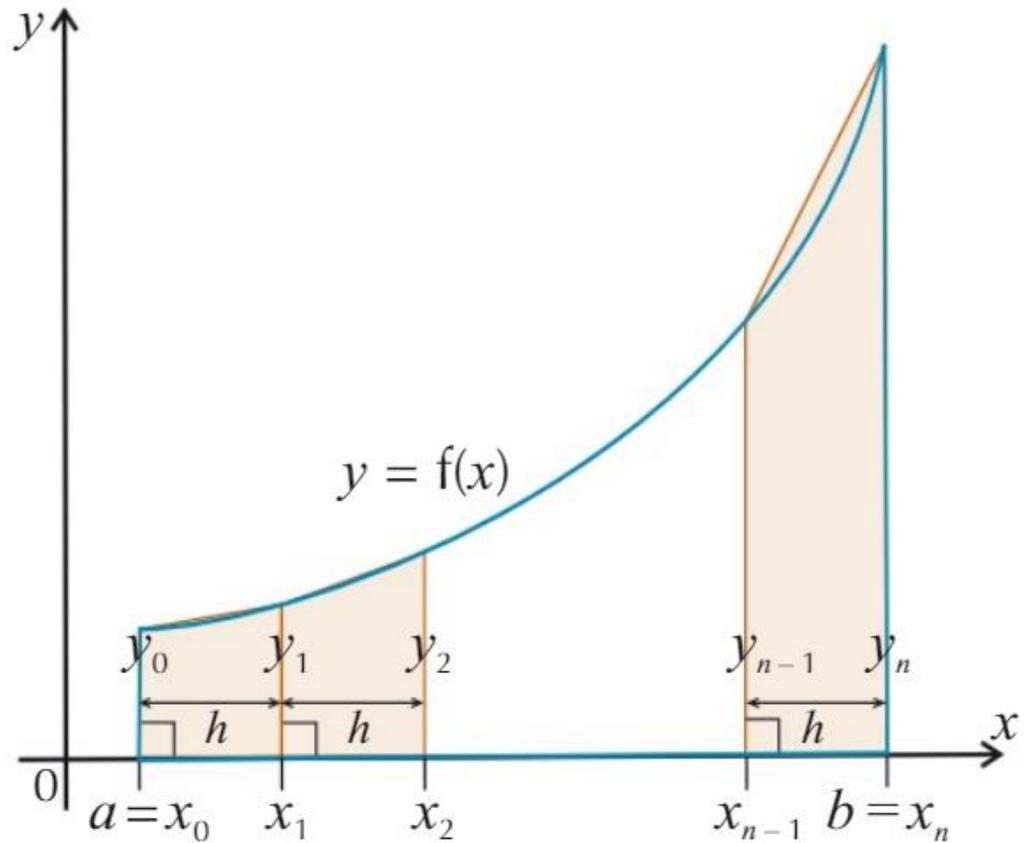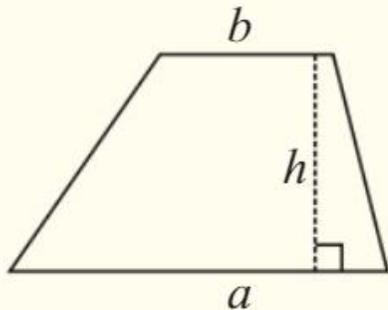
The Trapezium Rule

- The **area** under this curve between **a** and **b** can be approximated by the green **trapezium** shown.

- It has height $(b - a)$ and parallel sides of length $f(a)$ and $f(b)$.

- The area of the trapezium is an **approximation** of the integral $\int_a^b f(x)\, dx$.

- It's not a very good approximation, but if you split the area up into **more** trapeziums of equal width, the approximation will get more and more **accurate** because the **difference** between the trapeziums and the curve will get **smaller**.

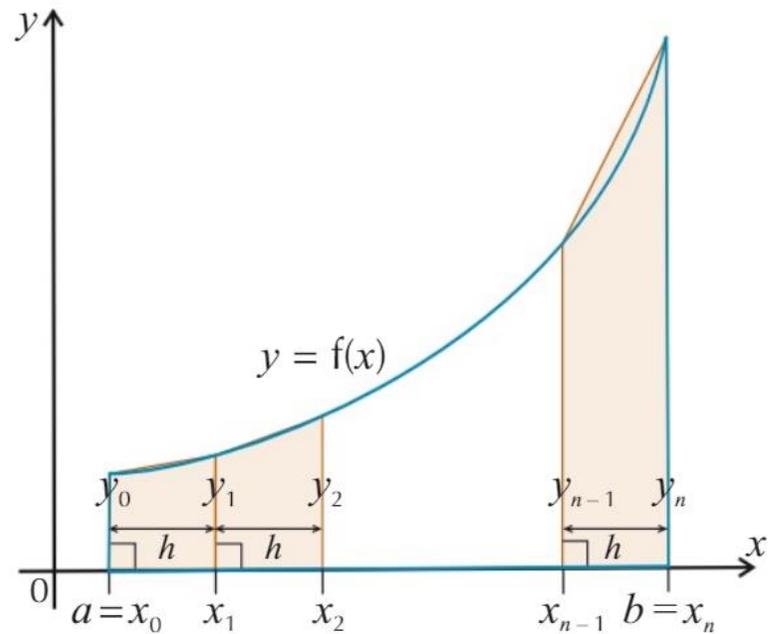# The Trapezium Rule

Tip: The area of a trapezium is given by the formula $\frac{h}{2}(a+b)$.

$y = f(x)$

$y_0 \quad y_1 \quad y_2 \quad \quad y_{n-1} \quad y_n$

$a = x_0 \quad x_1 \quad x_2 \quad \quad x_{n-1} \quad b = x_n$

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The Trapezium Rule

The **trapezium rule** for approximating $\int_a^b f(x) \, dx$ works like this:

- n is the **number** of strips i.e. trapeziums.

- h is the **width** of each strip — it's equal to $\dfrac{(b-a)}{n}$.

- The **x-values** go up in steps of h, starting with $x_0 = a$.

- The **y-values** are found by putting the x-values into the equation of the curve — so $y_1 = f(x_1)$. They give the **heights** of the sides of the trapeziums.
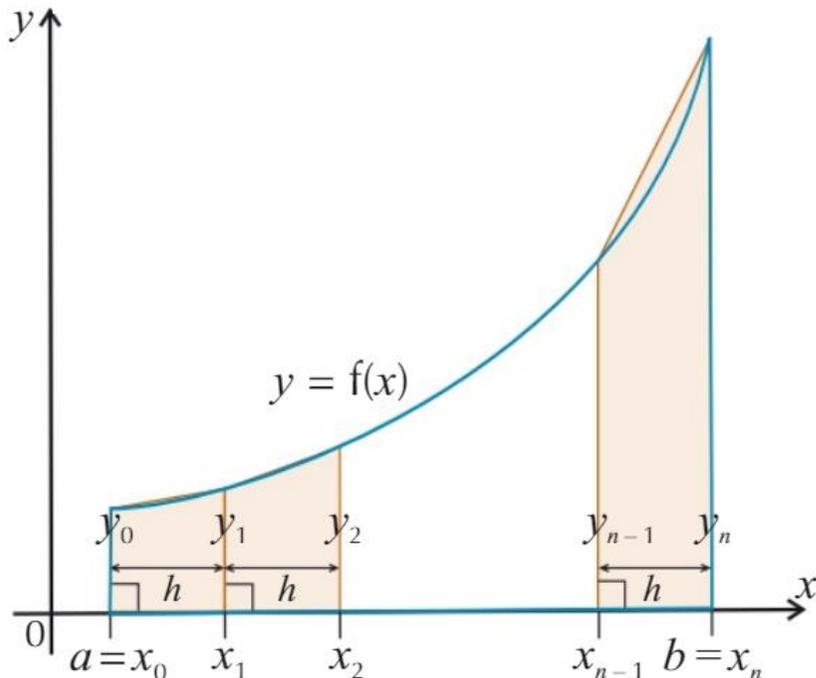
$y = f(x)$

- The **area** of each trapezium is $A = \dfrac{h}{2}(y_r + y_{r+1})$.

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The **area** of each trapezium is $A = \dfrac{h}{2}(y_r + y_{r+1})$.

The Trapezium Rule

Then an **approximation** for $\displaystyle\int_a^b f(x)\,dx$ is found by **adding** the **areas** of all the trapeziums:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}(y_0 + y_1) + \frac{h}{2}(y_1 + y_2) + \dots + \frac{h}{2}(y_{n-1} + y_n)$$

$$= \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + \dots + y_{n-1}) + y_n\right]$$

Tip: This just says 'Add the first and last heights $(y_0 + y_n)$ and add this to twice all the other heights added up — then multiply by $\dfrac{h}{2}$.'

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The Trapezium Rule:

$$\int_a^b f(x)\, dx \approx \frac{h}{2}[y_0 + 2(y_1 + y_2 + \dots + y_{n-1}) + y_n]$$

To approximate the integral $\int_a^b f(x)\, dx$:

- **Split** the interval up into a number of equal sized strips, n. You'll always be told what n is (it could be 4, 5 or even 6).

- Work out the **width** of each strip: $h = \dfrac{(b-a)}{n}$

- Make a **table** of x and y values:

x	$x_0 = a$	$x_1 = a + h$	$x_2 = a + 2h$	...	$x_n = b$
y	$y_0 = f(x_0)$	$y_1 = f(x_1)$	$y_2 = f(x_2)$	...	$y_n = f(x_n)$

- Put all the values into the **trapezium rule**:
  $$\int_a^b f(x)\, dx \approx \frac{h}{2}[y_0 + 2(y_1 + y_2 + \dots + y_{n-1}) + y_n]$$

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The Trapezium Rule:

EXAMPLE 1: Use the trapezium rule with 3 strips to find an approximate value for $\int_0^{1.5} \sqrt{x^2 + 2x}\ dx$.

- $n = 3$, $a = 0$ and $b = 1.5$, so the width of each strip is $h = \dfrac{1.5 - 0}{3} = 0.5$.

- This gives x-values of $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$ and $x_3 = 1.5$.

- Calculate the value of y for each of $x_0,\ x_1,\ x_2,\ x_3$:

x	$y = \sqrt{x^2 + 2x}$
$x_0 = 0$	$y_0 = 0$
$x_1 = 0.5$	$y_1 = \sqrt{1.25} = 1.118...$
$x_2 = 1$	$y_2 = \sqrt{3} = 1.732...$
$x_3 = 1.5$	$y_3 = \sqrt{5.25} = 2.291...$

Now use the formula to find the approximate value of the integral:

$$\int_0^{1.5} \sqrt{x^2 + 2x}\ dx \approx \frac{h}{2}[y_0 + 2(y_1 + y_2) + y_3]$$

$$= \frac{0.5}{2}[0 + 2(1.118... + 1.732...) + 2.291...]$$

$$= \frac{1}{4}[7.991...]$$

$$= 2.00\ (3\ \text{s.f.})$$

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The Trapezium Rule (observation)

The **approximation** that the trapezium rule gives will either be an **overestimate** (too big) or an **underestimate** (too small).

This will depend on the **shape** of the graph — a sketch can show whether the tops of the trapeziums lie **above** the curve or stay **below** it.

The estimate is **less** than the real area.

The estimate is **more** than the real area.
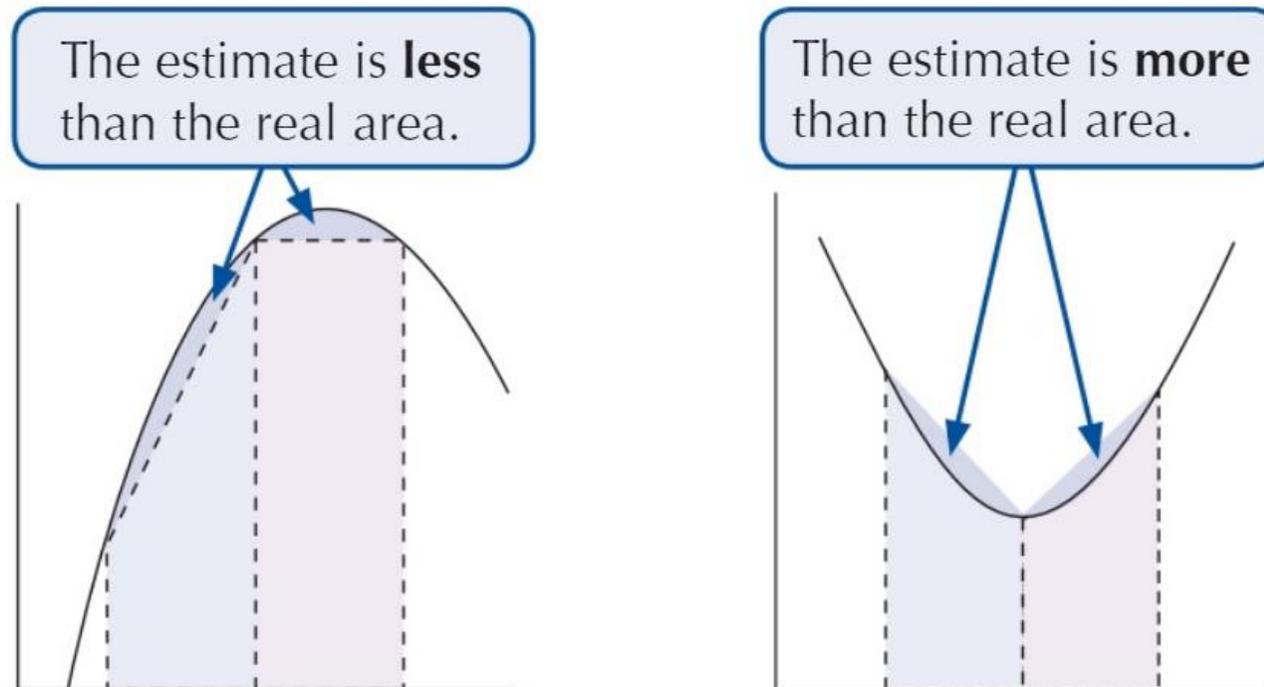
The Trapezium Rule:

EXAMPLE 2: Use the trapezium rule to approximate $\int_0^4 \dfrac{6x^2}{x^3+2}\, dx$ to 3 d.p. using:

a) $n = 2$ b) $n = 4$

- For 2 strips, the width of each strip is $h = \dfrac{4-0}{2} = 2$, so the x-values are 0, 2 and 4.

- Calculate the corresponding y-values:

- Calculate the corresponding y-values:

x	$y = \dfrac{6x^2}{x^3+2}$
$x_0 = 0$	$y_0 = 0$
$x_1 = 2$	$y_1 = 2.4$
$x_2 = 4$	$y_2 = 1.4545$ (4 d.p.)

- Putting these values into the formula gives:

$$\int_0^4 \frac{6x^2}{x^3+2}\, dx \approx \frac{h}{2}[y_0 + 2y_1 + y_2]$$

$$= \frac{2}{2}[0 + 2(2.4) + 1.4545]$$

$$= 6.255 \text{ (3 d.p.)}$$

The Trapezium Rule

b) $n = 4$

- For 4 strips, the width of each strip is $h = \frac{4-0}{4} = 1$, so the x-values are 0, 1, 2, 3 and 4.

- Calculate the corresponding y-values:

x	$y = \dfrac{6x^2}{x^3 + 2}$
$x_0 = 0$	$y_0 = 0$
$x_1 = 1$	$y_1 = 2$
$x_2 = 2$	$y_2 = 2.4$
$x_3 = 3$	$y_3 = 1.8621$ (4 d.p.)
$x_4 = 4$	$y_4 = 1.4545$ (4 d.p.)

- Putting these values into the formula gives:
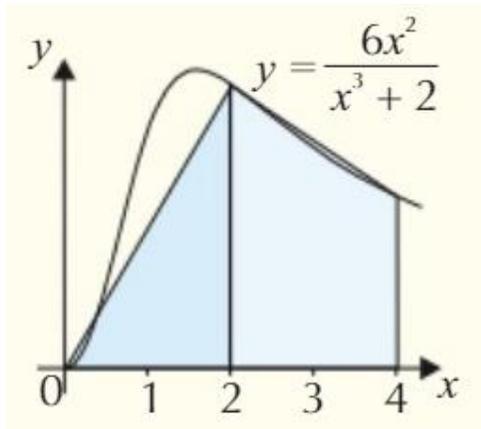
$$\int_0^4 \frac{6x^2}{x^3 + 2}\, dx \approx \frac{h}{2}[y_0 + 2(y_1 + y_2 + y_3) + y_4]$$

$$= \frac{1}{2}[0 + 2(2 + 2.4 + 1.8621) + 1.4545]$$

$$= \frac{1}{2}[13.9787] = 6.989 \ (3 \ \text{d.p.})$$
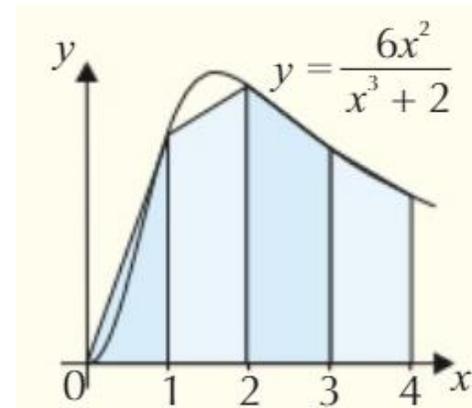
Example 2 (observation)

Increasing the number of strips increases the **accuracy** because the gaps between curve and line are smaller
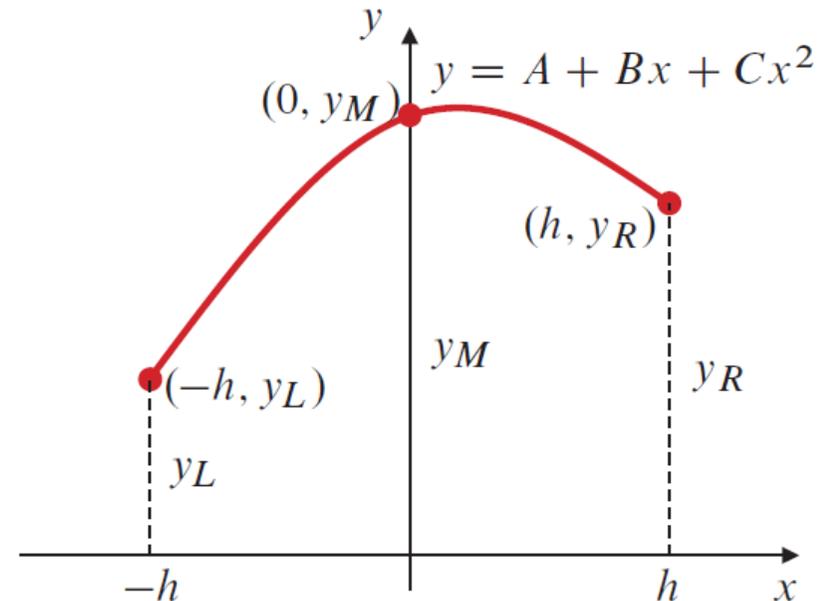
$$n = 2 \qquad\qquad n = 4$$

Simpson's Rule

$$\left. \begin{array}{l} y_L = A - Bh + Ch^2 \\ y_M = A \\ y_R = A + Bh + Ch^2 \end{array} \right\} \quad \Rightarrow \quad A = y_M \quad \text{and} \quad 2Ch^2 = y_L - 2y_M + y_R$$

$$\int_{-h}^{h} (A + Bx + Cx^2)\, dx = \left(Ax + \frac{B}{2}x^2 + \frac{C}{3}x^3 \right) \Big|_{-h}^{h} = 2Ah + \frac{2}{3}Ch^3$$
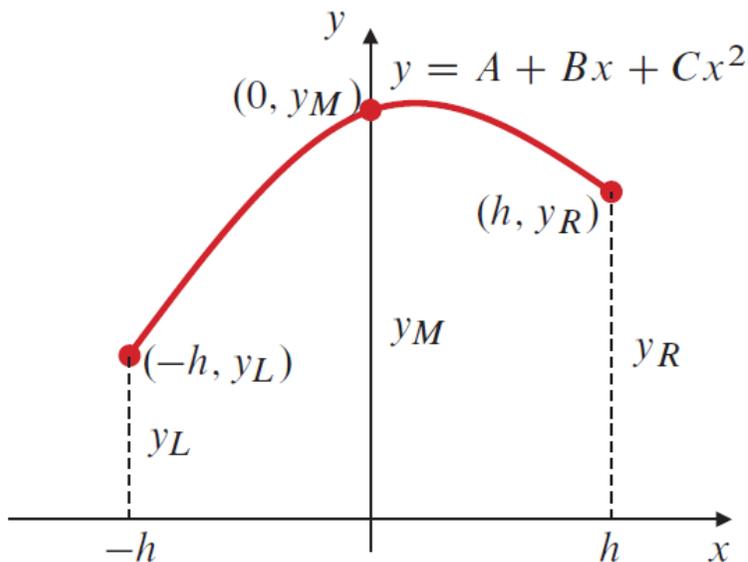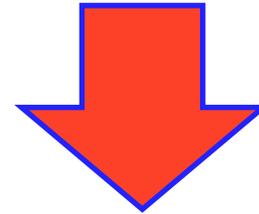
$$= h\left(2y_M + \frac{1}{3}(y_L - 2y_M + y_R) \right)$$

$$= \frac{h}{3}(y_L + 4y_M + y_R).$$

$y = A + Bx + Cx^2$

$(0, y_M)$

(h, y_R)

y_M

y_R

$(-h, y_L)$

y_L

$-h$

h

Simpson's Rule

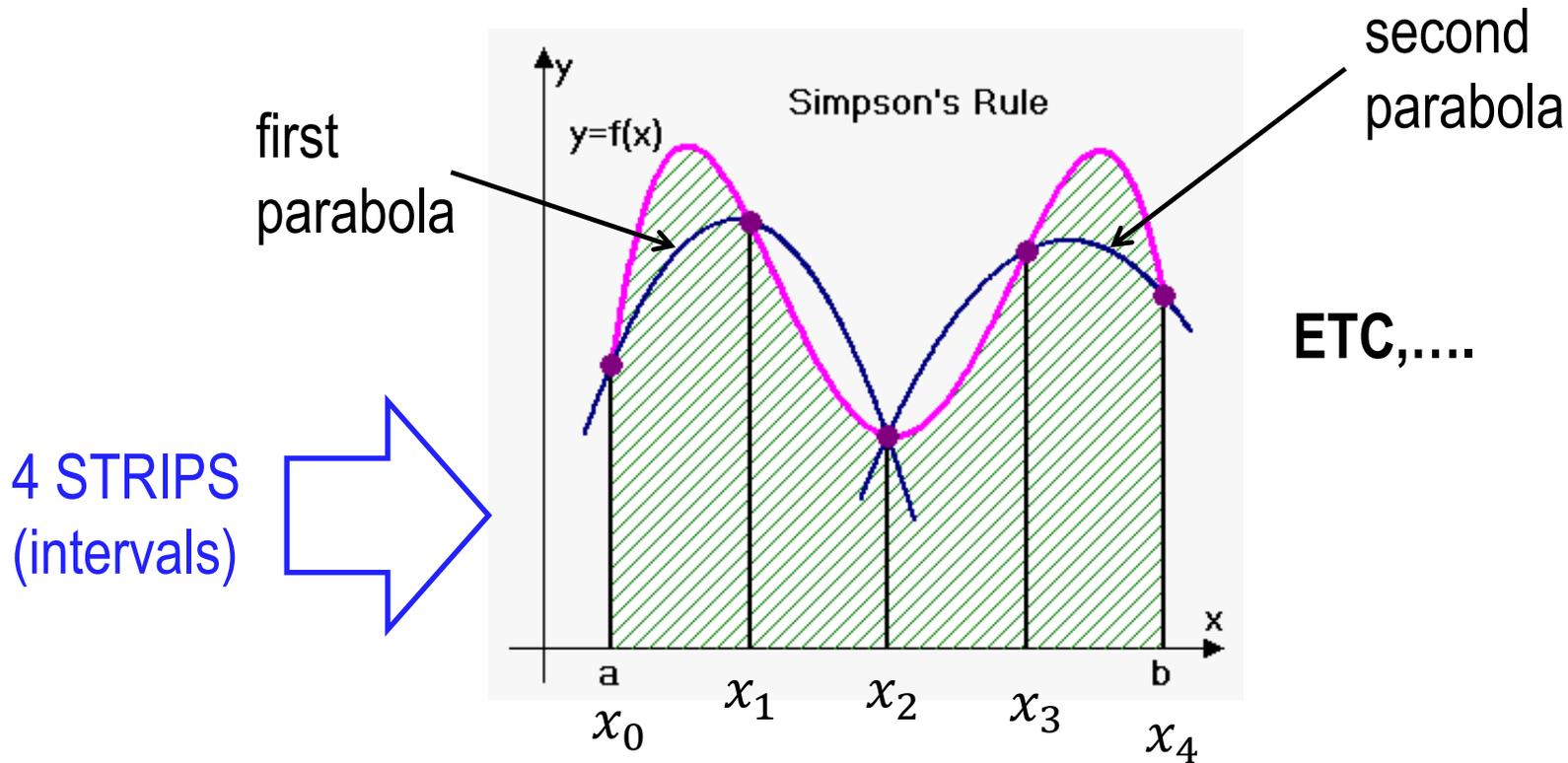$$\int_{-h}^{h} (A + Bx + Cx^2)\, dx = \frac{h}{3}(y_L + 4y_M + y_R)$$

Thus, the **area** of the **plane region** bounded by a parabolic arc, the interval of length 2h, and the left and right vertical lines is equal to (h/3) times the sum of the heights of the region at the left and right edges and four times the height at the middle
(It is **independent** of the position of the y-axis)

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A key point:

first
parabola

second
parabola

ETC,....

4 STRIPS
(intervals)

$y = f(x)$

Simpson's Rule

x_0 x_1 x_2 x_3 x_4

a   b

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Simpson's Rule

$$x_0 = a, \quad x_1 = a + h, \quad x_2 = a + 2h, \quad \ldots, \quad x_n = a + nh = b$$

$n = $ **EVEN**

$$\int_{x_0}^{x_2} f(x)\,dx \approx \frac{h}{3}(y_0 + 4y_1 + y_2)$$

$$\int_{x_2}^{x_4} f(x)\,dx \approx \frac{h}{3}(y_2 + 4y_3 + y_4)$$

$$\vdots$$

$$\int_{x_{n-2}}^{x_n} f(x)\,dx \approx \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

$n/2$ TERMS

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Simpson's Rule

$$\int_a^b f(x)\,dx \approx S_n$$

$$= \frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right)$$

$$= \frac{h}{3}\left(\sum y_{\text{“ends”}} + 4\sum y_{\text{“odds”}} + 2\sum y_{\text{“evens”}}\right).$$

Example 1

Use Simpson's rule with n=4 intervals (strips) to evaluate $\int_1^3 \frac{2}{\sqrt{x}}\,dx$ correct to 3 decimal places.

$$h = \frac{3-1}{4} = 0.5$$

$$\int_1^3 \frac{2}{\sqrt{x}}\,dx \approx \frac{1}{3}(0.5)\left[(2.0000+1.1547)\right.$$

$$+\,4(1.6330+1.2649)+2(1.4142)]$$

$$= \frac{1}{3}(0.5)[3.1547+11.5916$$

$$+\,2.8284]$$

$$= \mathbf{2.929}, \text{ correct to 3 decimal places.}$$

x	$\dfrac{2}{\sqrt{x}}$
1.00	2.0000
1.25	1.7889
1.50	1.6330
1.75	1.5119
2.00	1.4142
2.25	1.3333
2.50	1.2649
2.75	1.2060
3.00	1.1547

0 → 1.00
1 → 1.50
2 → 2.00
3 → 2.50
4 → 3.00

Example 2

$$\int_0^{\frac{\pi}{3}} \sqrt{\left(1 - \frac{1}{3}\sin^2\theta\right)}\, d\theta$$

Evaluate this correct to 3 decimal places using **Simpson's rule** with 6 intervals (strips)

$$\int_0^{\frac{\pi}{3}} \sqrt{\left(1 - \frac{1}{3}\sin^2\theta\right)}\, d\theta$$

$$\approx \frac{1}{3}\left(\frac{\pi}{18}\right)[(1.0000 + 0.8660) + 4(0.9950$$

$$+ 0.9574 + 0.8969)$$

$$+ 2(0.9803 + 0.9286)]$$

$$= \frac{1}{3}\left(\frac{\pi}{18}\right)[1.8660 + 11.3972 + 3.8178]$$

$$= \mathbf{0.994}, \text{ correct to 3 decimal places.}$$

θ	0	$\frac{\pi}{18}$	$\frac{\pi}{9}$	$\frac{\pi}{6}$	$\frac{2\pi}{9}$	$\frac{5\pi}{18}$	$\frac{\pi}{3}$
		(or 10°)	(or 20°)	(or 30°)	(or 40°)	(or 50°)	(or 60°)
$\sqrt{\left(1 - \frac{1}{3}\sin^2\theta\right)}$	1.0000	0.9950	0.9803	0.9574	0.9286	0.8969	0.8660

$$h = \frac{\frac{\pi}{3} - 0}{6} = \frac{\pi}{18}$$

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Notes