



# **NFM2106/NFE2105**

## **Mathematics**

Integration (definite integrals)

# Overview/Learning outcomes

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- ❖ **Definite integrals**
- ❖ **Fundamental Theorem of Calculus**
- ❖ **Area** below a curve and the x-axis
- ❖ Definite integrals evaluated by using: substitution OR integration by parts

Be able to **evaluate definite integrals**

Be able to **find the area** between a curve and the x-axis using definite integration

## Recall (from last week):

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Suppose  $F(x)$  has a derivative  $f(x)$ , i.e.

$$F'(x) = f(x)$$

$F(x)$  is the **antiderivative** of  $f(x)$

The **indefinite integral** of  $f(x)$  is denoted by  $\int f(x)dx$  and is given by

$$\int f(x)dx = F(x) + C$$

This is always determined up to an **arbitrary constant**  $C$ , because differentiating a constant gives zero.

# Fundamental Theorem of Calculus

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The **definite integral** of  $f(x)$ , with upper and lower limits  $b$  and  $a$ , satisfies

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

# Examples

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Evaluate  $\int_1^3 (x^2 + 2) dx$ .

- Find the integral in the normal way — but put the integrated function in **square brackets** and rewrite the **limits** on the right-hand side.

$$\int_1^3 (x^2 + 2) dx = \left[ \frac{x^3}{3} + 2x \right]_1^3$$

Notice that there's no constant of integration.

# (Cont'd)

$$\int_1^3 (x^2 + 2) dx = \left[ \frac{x^3}{3} + 2x \right]_1^3$$

UPPER LIMIT

LOWER LIMIT

- Put in the limits:

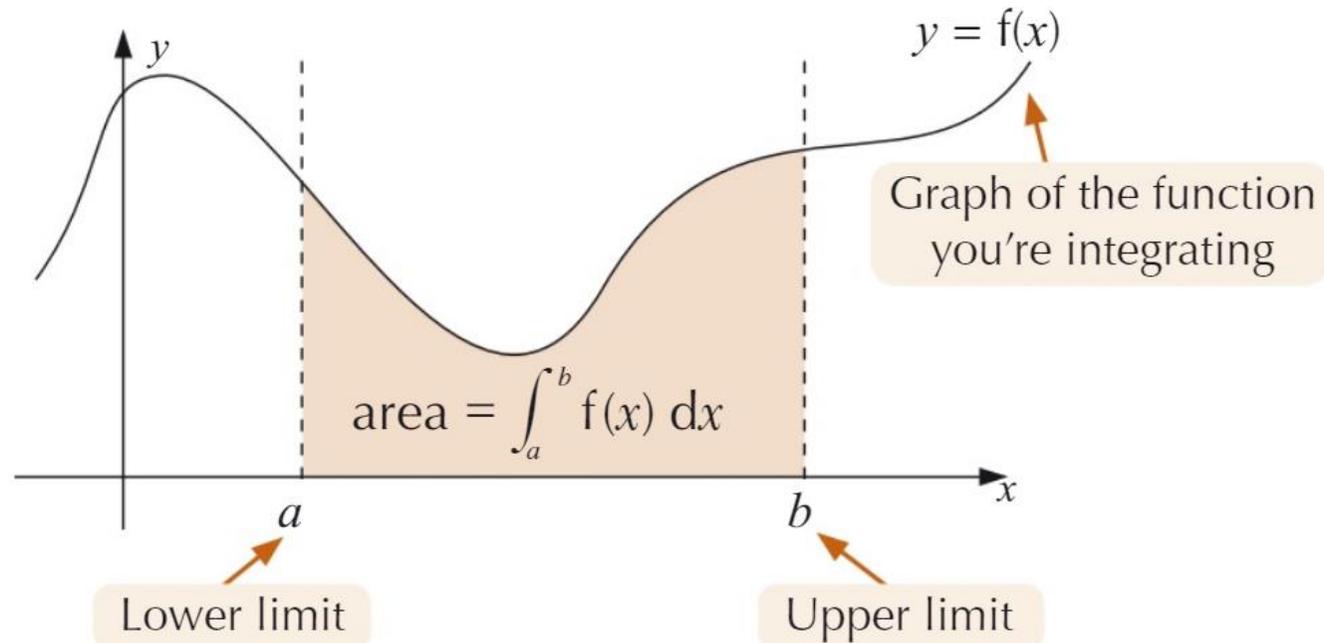
Put the upper limit into the integral...

...then subtract the value of the integral at the lower limit.

$$\begin{aligned} \left[ \frac{x^3}{3} + 2x \right]_1^3 &= \left( \frac{3^3}{3} + 6 \right) - \left( \frac{1^3}{3} + 2 \right) \\ &= 15 - \frac{7}{3} = \frac{38}{3} \end{aligned}$$

# The area under a curve

The value of a **definite integral** represents the **area** between the  $x$ -axis and the graph of the function you're integrating between the two limits.



# The area under a curve/example

Find the area between the graph of  $y = x^2$ , the  $x$ -axis and the lines  $x = -1$  and  $x = 2$ .

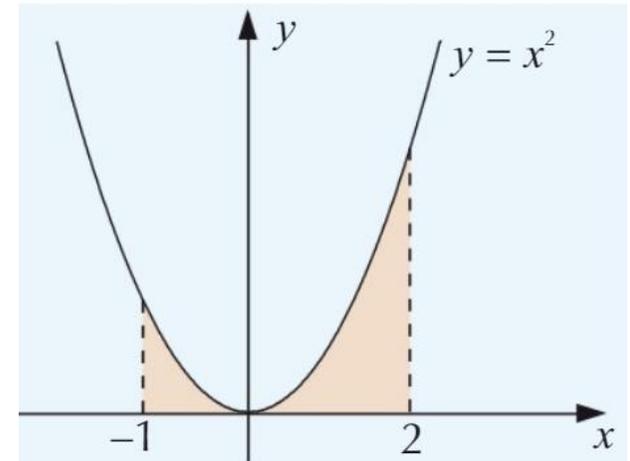
You just need to integrate the function  $f(x) = x^2$  between  $-1$  and  $2$  with respect to  $x$ .

The limits of integration are  $-1$  and  $2$ .

$$f(x) = x^2$$

Put in the limits

$$\int_{-1}^2 x^2 dx = \left[ \frac{x^3}{3} \right]_{-1}^2 = \left( \frac{2^3}{3} \right) - \left( \frac{(-1)^3}{3} \right) = \frac{8}{3} + \frac{1}{3} = \frac{9}{3} = 3$$



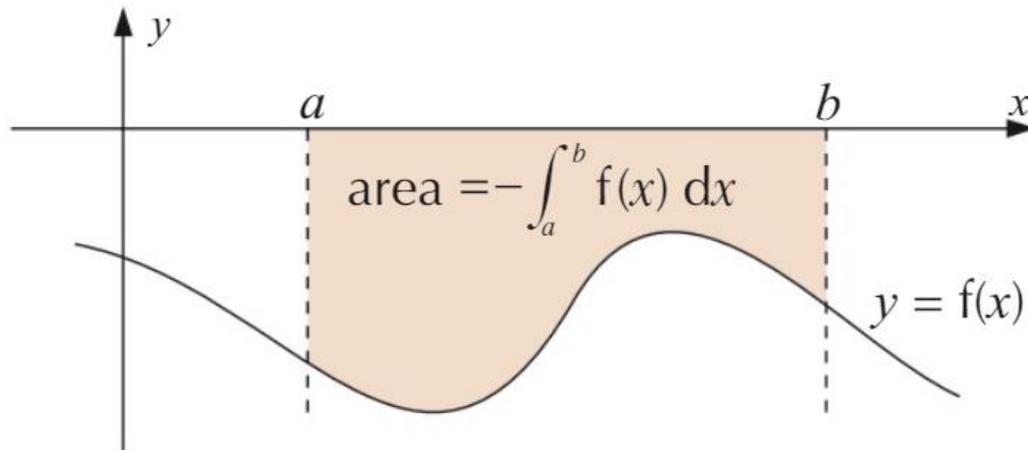
So the area is 3

# The area under a curve

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If you integrate a function to find an area that lies **below** the  $x$ -axis, it'll give a **negative** value.

If you need to find an area like this, you'll need to make your answer **positive** at the end as you can't have **negative area**.

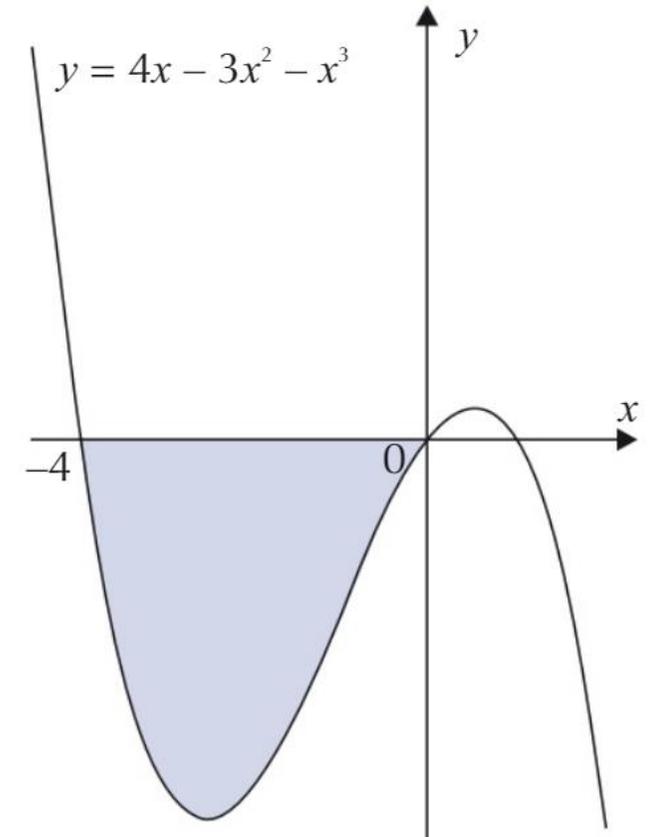


**Tip:** It's important to note that you're actually finding the area between the curve and the  **$x$ -axis**, not the area under the curve (the area below a curve that lies under the  $x$ -axis will be infinite).

# Example

Find the area between the graph of  $y = 4x - 3x^2 - x^3$  and the  $x$ -axis between  $x = -4$  and  $x = 0$ .

- You can see from the sketch of the graph that the area you're trying to find lies **below** the  $x$ -axis.
- So all you have to do is integrate the curve between the given limits and then make the area **positive** at the end.

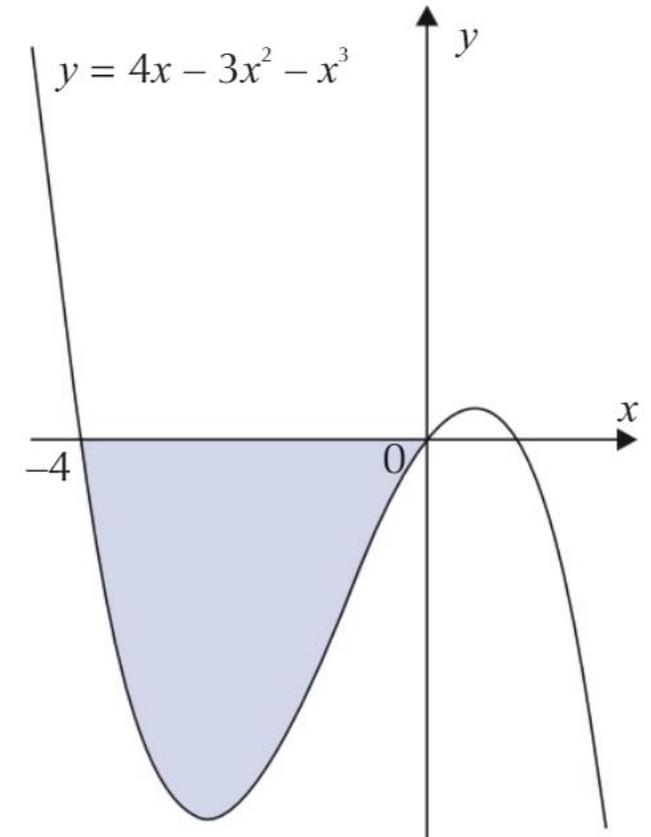


$$\int_{-4}^0 (4x - 3x^2 - x^3) dx = \left[ 2x^2 - x^3 - \frac{x^4}{4} \right]_{-4}^0$$

# The area under a curve

$$\begin{aligned}\int_{-4}^0 (4x - 3x^2 - x^3) dx &= \left[ 2x^2 - x^3 - \frac{x^4}{4} \right]_{-4}^0 \\ &= (0) - \left( 2(-4)^2 - (-4)^3 - \frac{(-4)^4}{4} \right) \\ &= 0 - (32 + 64 - 64) \\ &= -32\end{aligned}$$

- So the area between the curve and the  $x$ -axis between  $x = -4$  and  $x = 0$  is 32.



# Tips

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If you need to find the area for a portion of a curve which lies both **above** and **below** the  $x$ -axis, you'll need to find the areas above and below **separately** and add them up at the end so that the negative and positive integrals don't **cancel each other out**.

In the unit **Integration (finding area)** you'll find an **alternative way** of dealing with evaluating areas that are **both above and below** the  $x$ -axis. That approach is **more general** and can be used for finding the finite area bounded by the graphs of two functions.

# NEXT

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We are going to look at how the **integration techniques from last week** can be used to evaluate definite integrals.

# Last week: integration by substitution

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Can be used on integrals of the form

$$I = \int f(g(x)) \cdot g'(x) dx$$

$$\text{Let } t = g(x) \implies \frac{dt}{dx} = g'(x) \implies dt = g'(x) dx$$

Thus,

$$I = \int f(t) dt = F(t) + C = F(g(x)) + C$$

**Example:**  $\int x e^{x^2} dx$

$$= \int \frac{1}{2} e^t dt = \frac{1}{2} \int e^t dt = \frac{1}{2} e^t = \frac{1}{2} e^{x^2} + C$$

Let  $t = x^2$   
Then  $dt = 2x dx$

and  
 $x dx = \frac{1}{2} dt$

# “Substitution” for definite integrals

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$$\int_a^b f(g(x)) \cdot g'(x) dx = \int_{g(a)}^{g(b)} f(t) dt$$
$$= [F(t)]_{g(a)}^{g(b)} = \underline{F(g(b)) - F(g(a))}$$

The only difference is that in this case we need to calculate  $g(a)$  and  $g(b)$  and then evaluate the underlined expression. Study this example:

$$\int_1^3 \frac{x}{x^2 + 1} dx = \left[ \frac{1}{2} \ln(t) \right]_2^{10} = \frac{1}{2} (\ln 10 - \ln 2) = \ln \sqrt{5}$$

## Another example:

$$\text{Find } \int_0^{\pi/4} \tan x \, dx = \int_0^{\pi/4} \frac{\sin x}{\cos x} \, dx$$

Let  $t = \cos x$

$$\Rightarrow dt = -\sin x \, dx$$

Looking at the limits:

$$x = 0 \quad \Rightarrow \quad t = \cos 0 = 1$$

$$x = \pi/4 \quad \Rightarrow \quad t = \cos(\pi/4) = 1/\sqrt{2}$$

$$\int_0^{\pi/4} \frac{\sin x}{\cos x} \, dx = \int_0^{1/\sqrt{2}} \frac{1}{t} \times (-dt) = - \int_1^{1/\sqrt{2}} \frac{dt}{t} = -\ln\left(\frac{1}{\sqrt{2}}\right) + \ln(1)$$

$$= -\ln\left(\frac{1}{\sqrt{2}}\right)$$

# Integration by parts (for definite integrals)

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$$\int_a^b U dV = [UV]_a^b - \int_a^b V dU, \quad (\blacksquare)$$

where we have used the short-hand notation

$$[UV]_a^b = U(b)V(b) - U(a)V(a)$$

We can also write  $(\blacksquare)$  as 
$$\int_a^b U(x)V'(x)dx = [UV]_a^b - \int_a^b V(x)U'(x)dx$$

# Example:

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Evaluate  $\int_0^{\frac{\pi}{2}} 2\theta \sin \theta \, d\theta$

Let  $u = 2\theta$ , from which,  $\frac{du}{d\theta} = 2$ , i.e.  $du = 2 \, d\theta$  and let  $dv = \sin \theta \, d\theta$ , from which,

$$v = \int \sin \theta \, d\theta = -\cos \theta$$

Substituting into  $\int u \, dv = uv - \int v \, du$  gives:

$$\begin{aligned} \int 2\theta \sin \theta \, d\theta &= (2\theta)(-\cos \theta) - \int (-\cos \theta)(2 \, d\theta) \\ &= -2\theta \cos \theta + 2 \int \cos \theta \, d\theta \\ &= -2\theta \cos \theta + 2 \sin \theta + c \end{aligned}$$

Hence  $\int_0^{\frac{\pi}{2}} 2\theta \sin \theta \, d\theta$

$$\begin{aligned} &= [-2\theta \cos \theta + 2 \sin \theta]_0^{\frac{\pi}{2}} \\ &= \left[ -2 \left( \frac{\pi}{2} \right) \cos \frac{\pi}{2} + 2 \sin \frac{\pi}{2} \right] - [0 + 2 \sin 0] \\ &= (-0 + 2) - (0 + 0) = 2 \end{aligned}$$

## Another example:

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Evaluate  $\int_0^1 5xe^{4x} dx$ , correct to 3 significant figures

Let  $u = 5x$ , from which  $\frac{du}{dx} = 5$ , i.e.  $du = 5 dx$  and

let  $dv = e^{4x} dx$ , from which,  $v = \int e^{4x} dx = \frac{1}{4}e^{4x}$

Substituting into  $\int u dv = uv - \int v du$  gives:

$$\begin{aligned}\int 5xe^{4x} dx &= (5x) \left( \frac{e^{4x}}{4} \right) - \int \left( \frac{e^{4x}}{4} \right) (5 dx) = \frac{5}{4}xe^{4x} - \frac{5}{4} \int e^{4x} dx \\ &= \frac{5}{4}xe^{4x} - \frac{5}{4} \left( \frac{e^{4x}}{4} \right) + c \\ &= \frac{5}{4}e^{4x} \left( x - \frac{1}{4} \right) + c\end{aligned}$$

## (Cont'd)

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$$\begin{aligned}\text{Hence } \int_0^1 5xe^{4x} dx &= \left[ \frac{5}{4}e^{4x} \left( x - \frac{1}{4} \right) \right]_0^1 \\ &= \left[ \frac{5}{4}e^4 \left( 1 - \frac{1}{4} \right) \right] - \left[ \frac{5}{4}e^0 \left( 0 - \frac{1}{4} \right) \right] \\ &= \left( \frac{15}{16}e^4 \right) - \left( -\frac{5}{16} \right) \\ &= 51.186 + 0.313 = 51.499 = \mathbf{51.5}, \\ &\quad \text{correct to 3 significant figures.}\end{aligned}$$