# NFM2106/NFE2105
## Mathematics

Integration (finding area)

# Outline/Learning outcomes

❖ In this unit we'll be concerned with **finding the areas** of various regions in the plane, which are bounded by the graphs of given functions and certain vertical and/or horizontal lines.

❖ This is an application of the material covered in the unit on **definite integration**.

❖ The general technique for finding the areas mentioned above is introduced and illustrated by a number of **examples** of varying degrees of difficulty.
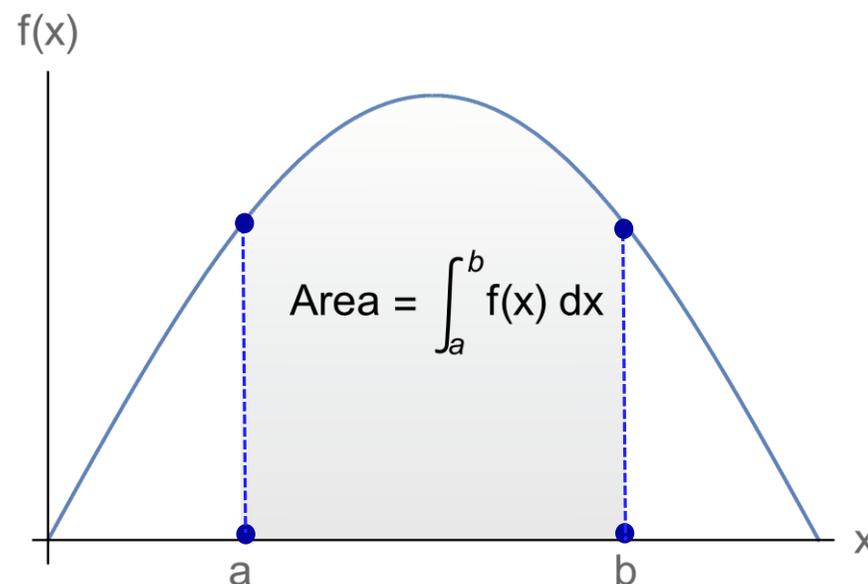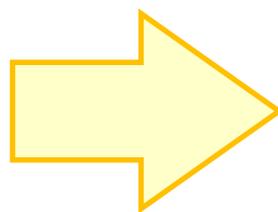
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# Reminder:

The **definite integral** of $f(x)$, with upper and lower limits $b$ and $a$, is

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

**Geometrical interpretation:**

*Area* enclosed by the $x$-axis and $f(x)$, between $x = a$ and $x = b$

$$\text{Area} = \int_a^b f(x)\, dx$$

f(x)

a          b          x

# Enclosed area: examples

To find the area between a curve, a line and the $x$-axis, you'll either have to **add** or **subtract** integrals to find the area you're after — it's always best to **draw a diagram** of the area.

**EXAMPLE:** **Find the area enclosed by the curve $y = x^2$, the line $y = 2 - x$ and the $x$-axis.**
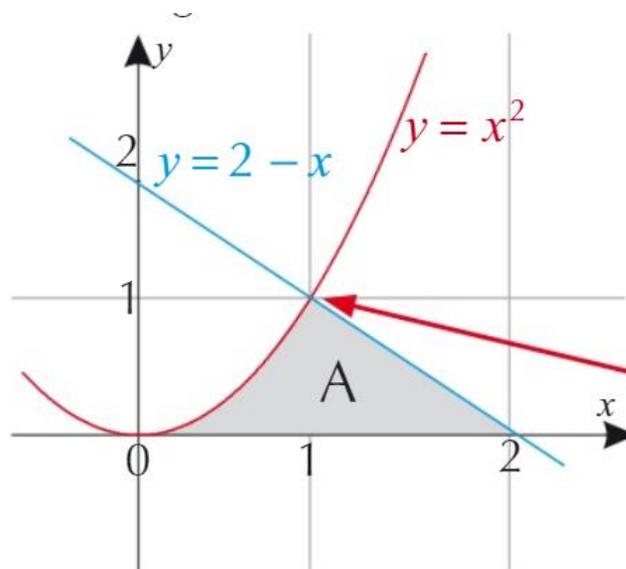
You have to find **area A** — but you'll need to **split** it into two smaller bits.

Find out where the curve and line meet by **solving** $x^2 = 2 - x$. They meet at $x = 1$ (they also meet at $x = -2$, but this isn't in A).

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**Tip:** You need to find the $x$-coordinate of the point of intersection between the line and the curve, as well as the $x$-intercept of each function, so that you know the limits to integrate between.

$$y = 2 - x$$

$$y = x^2$$

INTERSECTION POINT

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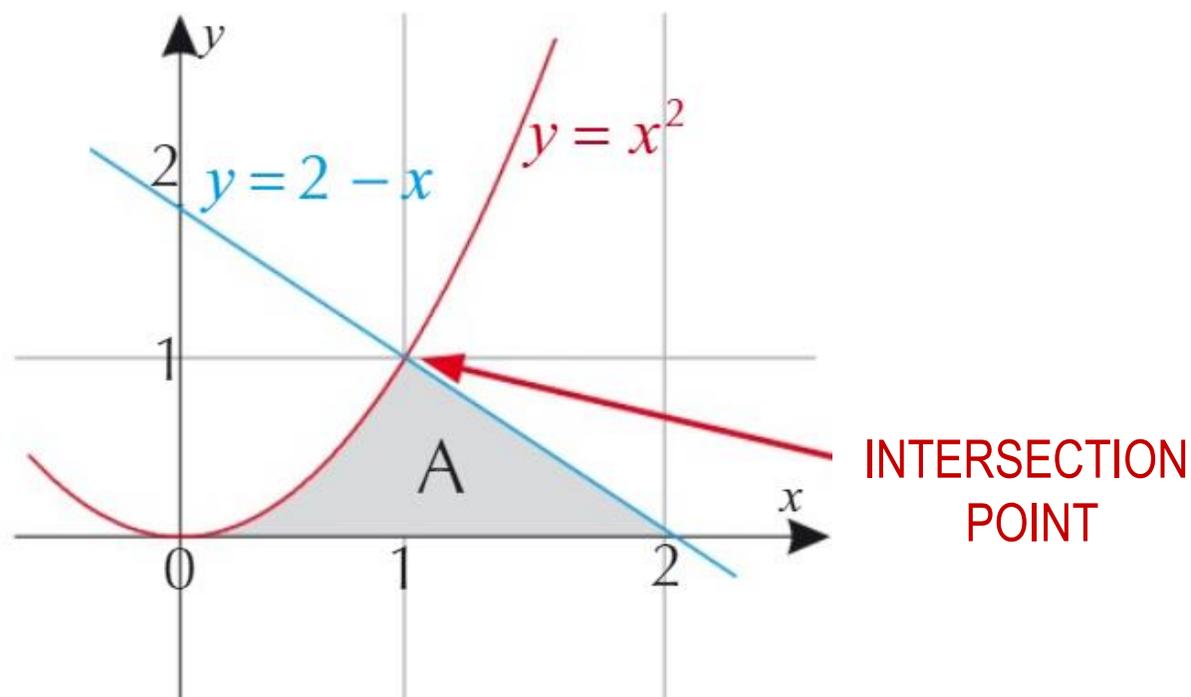
# Enclosed areas: examples

$A_1$ is the area under the curve $y = x^2$ between 0 and 1, so integrate between these limits to find the area:

$$\int_0^1 x^2 \, dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$$
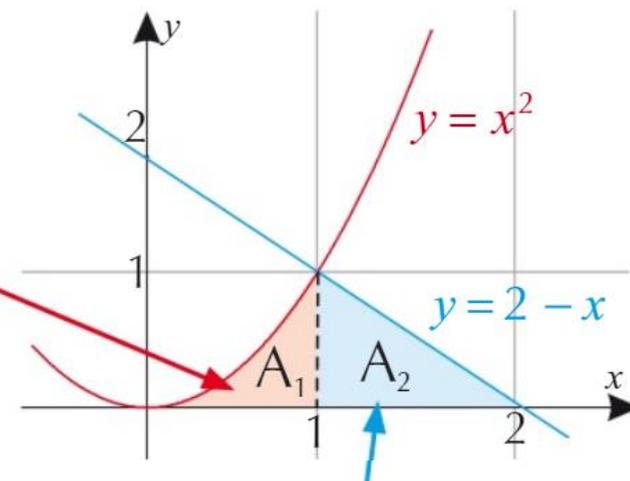
$A_2$ is the area under the line $y = 2 - x$ between 1 and 2, so integrate between these limits to find the area:

$$\int_1^2 (2 - x) \, dx = \left[2x - \frac{x^2}{2}\right]_1^2$$

$$= \left(2(2) - \frac{2^2}{2}\right) - \left(2(1) - \frac{1^2}{2}\right)$$

$$= 2 - \frac{3}{2} = \frac{1}{2}$$

**Add** the areas together to find the area A:

$$A = A_1 + A_2 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

# A general result:

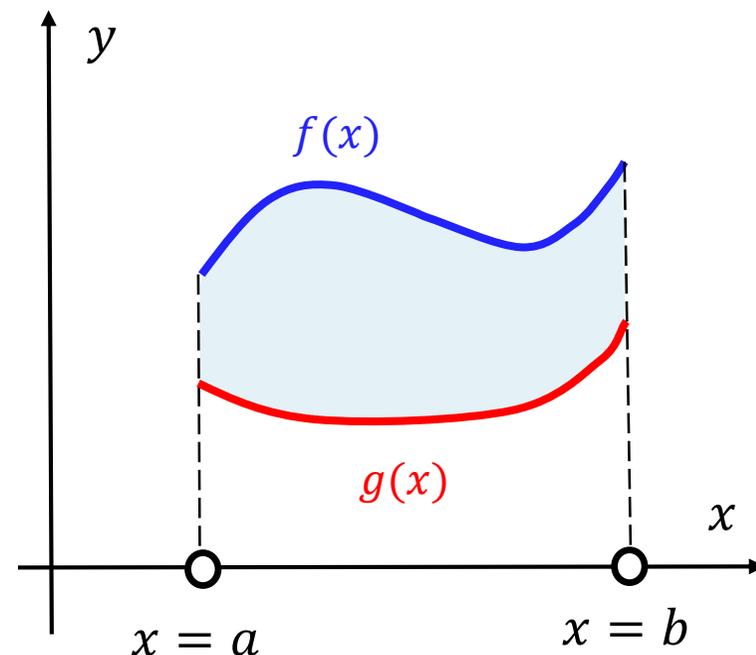Let $f, g$ be two continuous functions such that

$$f(x) \geq g(x) \quad \text{for all} \quad a \leq x \leq b$$

Then the **finite area** between the graphs of these two functions and the vertical lines $x = a$ and $x = b$ is given by
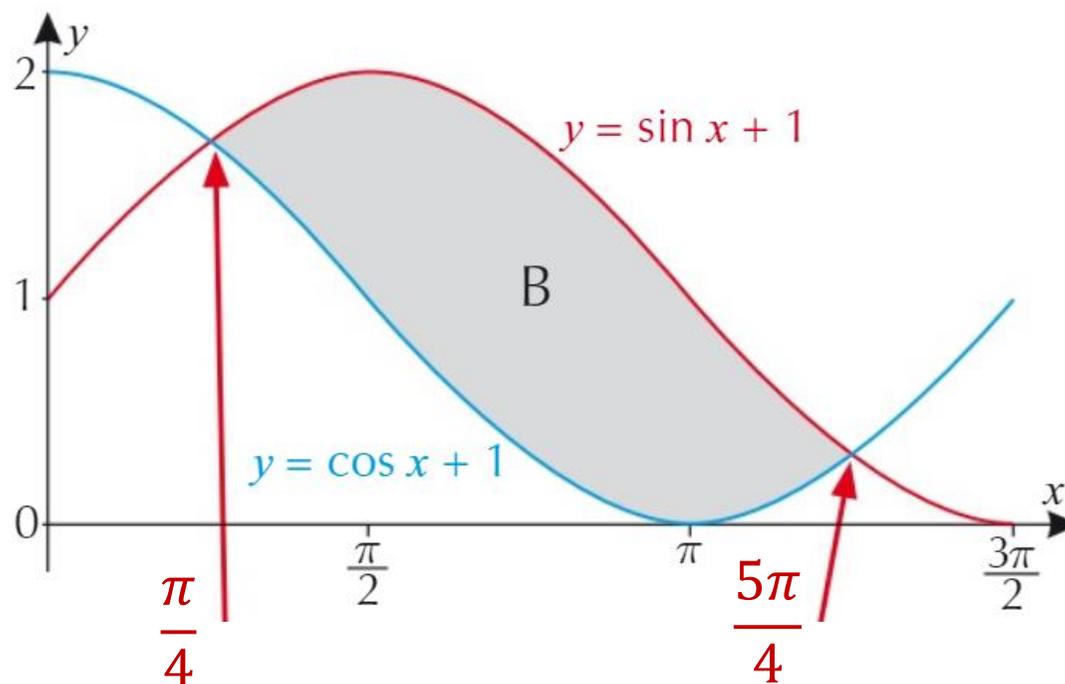
$$\int_a^b [f(x) - g(x)]dx$$

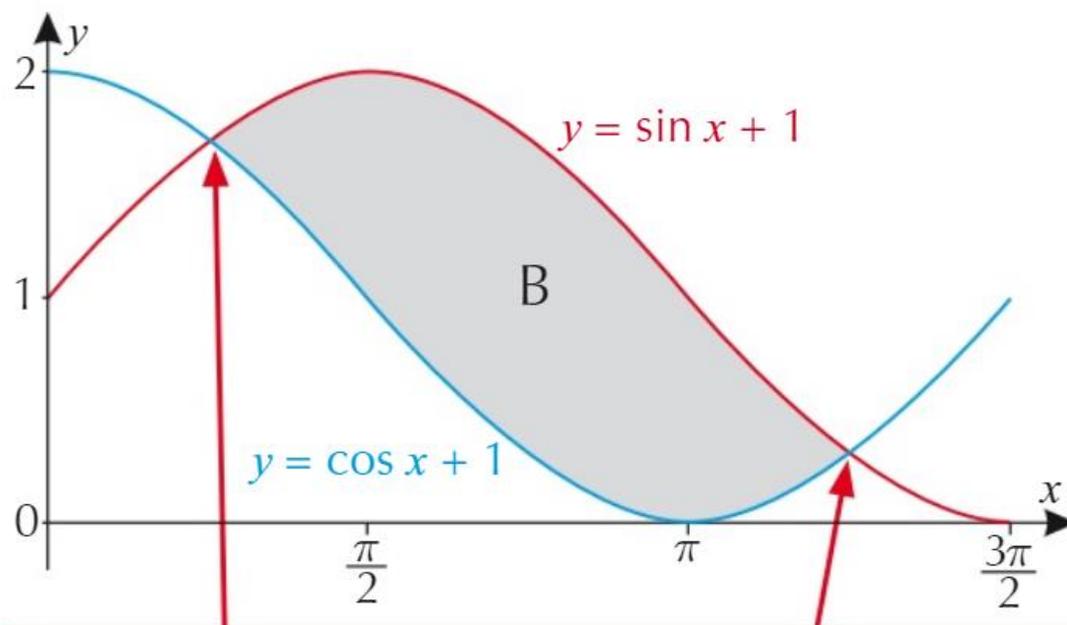$$\int_a^b \left(\begin{array}{c}upper \\ function\end{array}\right) - \left(\begin{array}{c}lower \\ function\end{array}\right) dx$$

Sometimes you'll need to find the area **enclosed** by the graphs of two functions — this usually means **subtracting** some area from another. Here is an example with two curves:

**EXAMPLE:**   **The diagram below shows the curves $y = \sin x + 1$ and $y = \cos x + 1$. Find the area of the shaded grey region.**
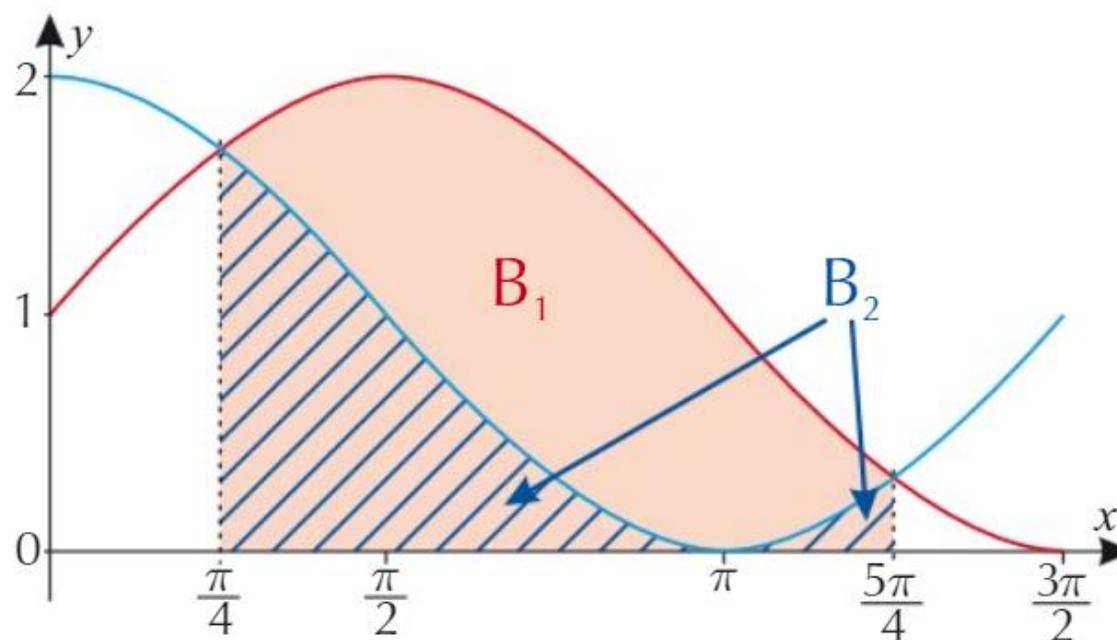
# Enclosed areas: examples



Find out where the graphs meet by **solving** $\sin x + 1 = \cos x + 1$

$\Rightarrow \sin x = \cos x \Rightarrow \dfrac{\sin x}{\cos x} = 1 \Rightarrow \tan x = 1$

$\Rightarrow x = \tan^{-1}(1) = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$. They meet at $x = \dfrac{\pi}{4}$ and $x = \dfrac{5\pi}{4}$.

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- You need to find two different integrals to work out the area of B:



- The area B is the area under the **red** curve between $\frac{\pi}{4}$ and $\frac{5\pi}{4}$ ($B_1$)
**minus** the area under the **blue** curve between $\frac{\pi}{4}$ and $\frac{5\pi}{4}$ ($B_2$).
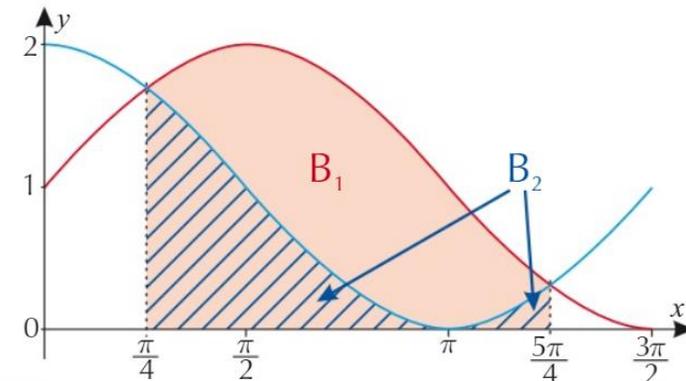
$$B_1 = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin x + 1 \, dx = [-\cos x + x]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} = \left(-\cos\left(\frac{5\pi}{4}\right) + \frac{5\pi}{4}\right) - \left(-\cos\left(\frac{\pi}{4}\right) + \frac{\pi}{4}\right)$$

$$= \left(-\left(-\frac{\sqrt{2}}{2}\right) + \frac{5\pi}{4}\right) - \left(-\frac{\sqrt{2}}{2} + \frac{\pi}{4}\right) = \boxed{\sqrt{2} + \pi}$$

$$B_2 = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \cos x + 1 \, dx = [\sin x + x]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} = \left(\sin\left(\frac{5\pi}{4}\right) + \frac{5\pi}{4}\right) - \left(\sin\left(\frac{\pi}{4}\right) + \frac{\pi}{4}\right)$$

$$= \left(-\frac{\sqrt{2}}{2} + \frac{5\pi}{4}\right) - \left(\frac{\sqrt{2}}{2} + \frac{\pi}{4}\right) = \boxed{-\sqrt{2} + \pi}$$



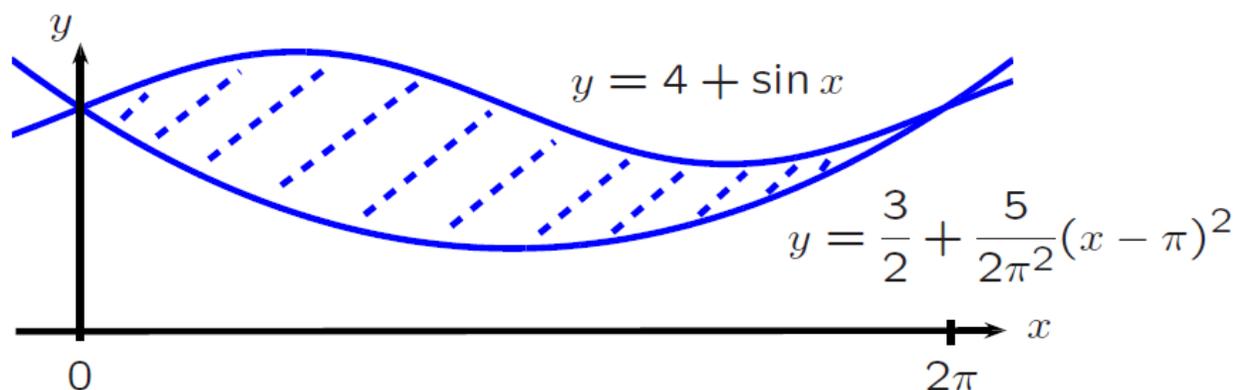- So the area $B = B_1 - B_2 = (\sqrt{2} + \pi) - (-\sqrt{2} + \pi) = \boxed{2\sqrt{2}}$

# A similar example:

Find the are between the curves

$$y = \sin x + 4 \qquad \text{and} \qquad y = \frac{3}{2} + \frac{5}{2\pi^2}(x - \pi)^2$$

for $0 \leq x \leq 2\pi$

Area required $=$ (area below $y = \sin(x) + 4$)

$- $ (area below $y = \dfrac{3}{2} + \dfrac{5}{2\pi^2}(x - \pi)^2$)

$$= \int_0^{2\pi} \sin(x) + 4 \, dx - \int_0^{2\pi} \frac{3}{2} + \frac{5}{2\pi^2}(x-\pi)^2 \, dx$$

$$= [-\cos x]_0^{2\pi} + [4x]_0^{2\pi} - \left[\frac{3x}{2}\right]_0^{2\pi} - \left[\frac{5}{2\pi^2}\frac{(x-\pi)^3}{3}\right]_0^{2\pi}$$

$$= [-1+1] + [8\pi - 0] - [3\pi - 0] - \frac{5}{6\pi^2}\left[\pi^3 - (-\pi^3)\right]$$
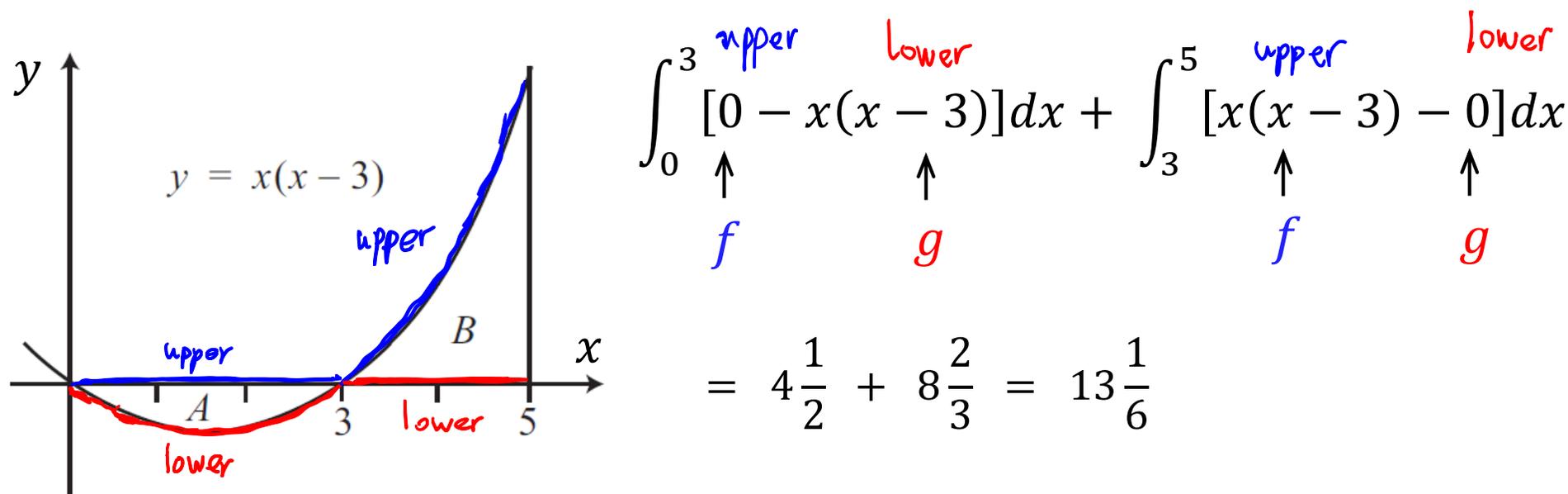
$$= \frac{10\pi}{3}$$

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# Observation

Sometimes you might need to add **and** subtract integrals to find the right area.
You'll often need to do this when the curve goes **below** the $x$-axis.
The integrations you need to do should be obvious if you draw a **picture**.

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# Here is such an example:

Find the area between the curve $y = x(x - 3)$
and the ordinates $x = 0$ and $x = 5$.

$$\int_0^3 \underset{f}{[0} - \underset{g}{x(x - 3)]}dx + \int_3^5 \underset{f}{[x(x - 3)} - \underset{g}{0]}dx$$

upper    lower                upper       lower

$$= 4\frac{1}{2} + 8\frac{2}{3} = 13\frac{1}{6}$$

$y$

$y = x(x - 3)$

upper

upper

$A$    $3$    lower    $5$
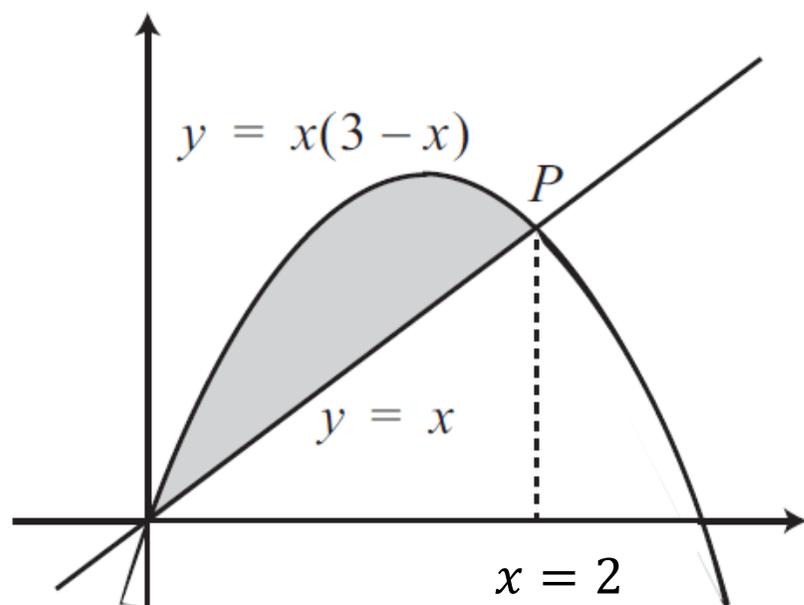
lower

$B$

$x$

**OBS.** The x-axis has equation y = 0

(see MathTutor)

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Calculate the area of the segment cut from the curve $y = x(3 - x)$ by the line $y = x$.



$$\int_0^2 [x(3 - x) - x]dx$$

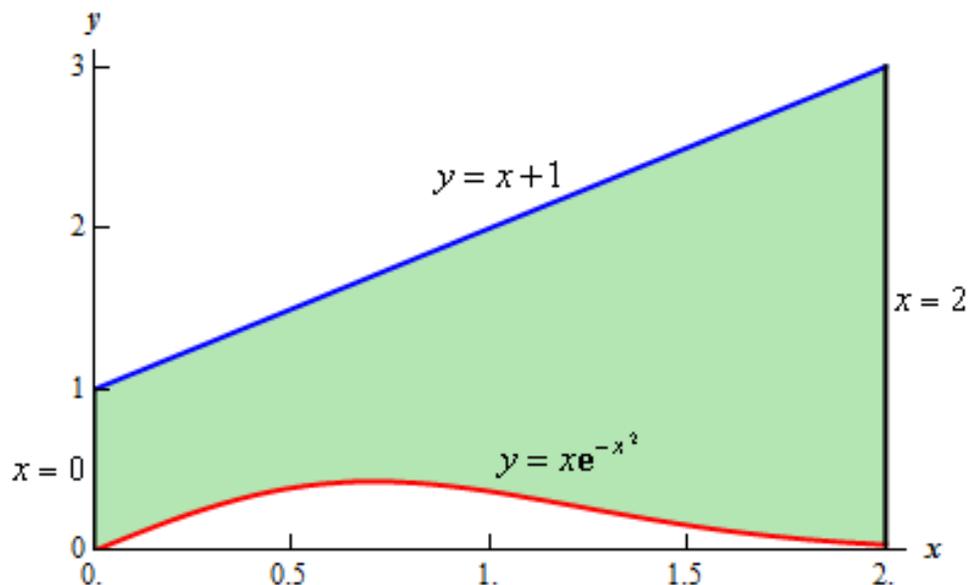$\uparrow$        $\uparrow$

$f$        $g$

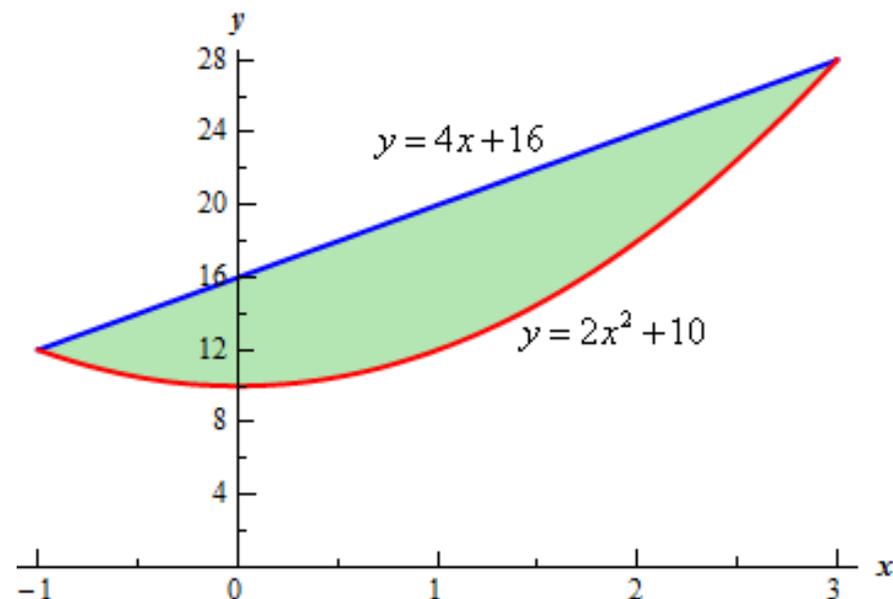$$= \frac{4}{3}$$

(see MathTutor)

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Ⓐ Determine the area of the finite region bounded by $y = xe^{-x^2}$, $y = x + 1$, $x = 2$ and the $y-$axis

$$\text{Answer} = \frac{7}{2} + \frac{e^{-4}}{2}$$



Ⓑ Determine the area of the finite region bounded by $y = 2x^2 + 10$ and $y = 4x + 16$

$$\text{Answer} = \frac{64}{3}$$

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Ⓐ
$$A = \int_a^b \left(\begin{array}{c}\text{upper}\\\text{function}\end{array}\right) - \left(\begin{array}{c}\text{lower}\\\text{function}\end{array}\right) dx$$

$$= \int_0^2 \left[(x+1) - xe^{-x^2}\right] dx = \int_0^2 \left(x+1 - xe^{-x^2}\right) dx$$

$$= \left[\frac{1}{2}x^2 + x + \frac{1}{2}e^{-x^2}\right]_0^2$$

$$= \left[\frac{1}{2}(2^2) + 2 + \frac{1}{2}e^{-4}\right] - \left[\frac{1}{2}(0^2) + (0) + \frac{1}{2}e^{-0}\right]$$

$$= \frac{7}{2} + \frac{e^{-4}}{2} \approx \boxed{3.5092}$$

Ⓑ  intersection points:

$$2x^2 + 10 = 4x + 16$$

$$2x^2 - 4x - 6 = 0 \implies 2(x+1)(x-3) = 0$$

The two curves intersect at $x = -1$ and $x = 3$. Hence,

$$A = \int_a^b \left(\begin{array}{c}\text{upper}\\\text{function}\end{array}\right) - \left(\begin{array}{c}\text{lower}\\\text{function}\end{array}\right) dx = \int_{-1}^3 \left[ (4x+16) - (2x^2+10) \right] dx$$

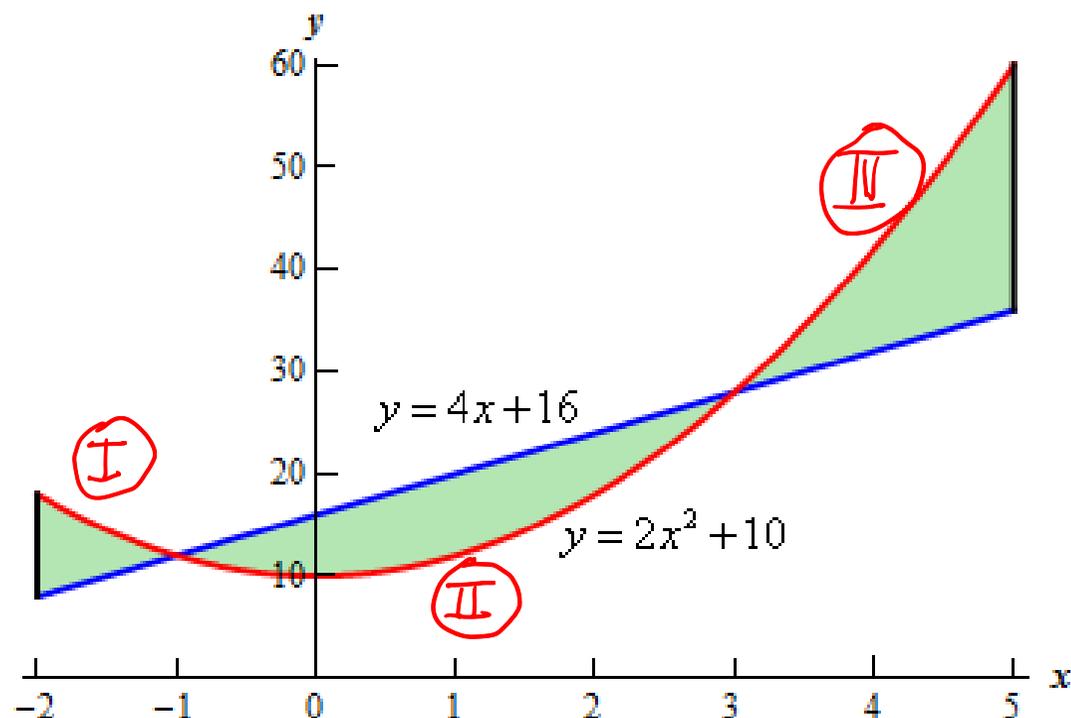$$= \int_{-1}^3 \left( 4x + 16 - 2x^2 - 10 \right) dx = \int_{-1}^3 \left( -2x^2 + 4x + 6 \right) dx$$

$$= \left[ \underbrace{-\frac{2}{3} x^3 + 2x^2 + 6x}_{\substack{\text{antiderivative}\\ \text{of the previous}\\ \text{integrand}}} \right]_{-1}^3 = \ldots = \boxed{\frac{64}{3}}$$

# A difficult example

Determine the area of the finite region bounded by $y = 2x^2 + 10$
$y = 4x + 16$, $x = -2$ and $x = 5$.

Answer = $\dfrac{142}{3}$

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OBS: The intersection points have already been found:

$$x = -1 \quad \text{and} \quad x = 3$$

$$A = \int_{-2}^{-1} (\ldots) dx + \int_{-1}^{3} (\ldots) dx + \int_{3}^{5} (\ldots) dx \qquad \leftarrow \quad \text{area} \, \text{I} + \text{area} \, \text{II} + \text{area} \, \text{III}$$

$$= \int_{-2}^{-1} \left[ (2x^2 + 10) - (4x + 16) \right] dx + \int_{-1}^{3} \left[ (4x + 16) - (2x^2 + 10) \right] dx + \int_{3}^{5} \left[ (2x^2 + 10) - (4x + 16) \right] dx$$

$$= \int_{-2}^{-1} \underbrace{(2x^2 - 4x - 6)}_{? \text{ antiderivative}} dx + \int_{-1}^{3} \underbrace{(-2x^2 + 4x + 6)}_{? \text{ antiderivative}} dx + \int_{3}^{5} \underbrace{(2x^2 - 4x - 6)}_{? \text{ antiderivative}} dx$$

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$$= \left[ \frac{2}{3}x^3 - 2x^2 - 6x \right]_{-2}^{-1} + \left[ -\frac{2}{3}x^3 + 2x^2 + 6x \right]_{-1}^{3} + \left[ \frac{2}{3}x^3 - 2x^2 - 6x \right]_{3}^{5}$$

$$= \cdots = \frac{14}{3} + \frac{64}{3} + \frac{64}{3} = \boxed{\frac{142}{3}}$$

# Learning Resources

REQUIRED:

**MathTutor:**

https://www.mathtutor.ac.uk/integration/findingareasbyintegration

OPTIONAL:

K.A. Stroud and D.J. Booth, Engineering Mathematics, 7th Edition,
**p.366-279; Optional questions: Test Exercise F.13: Q.5 (p.380); Further Problems F.13 (p.381): Q.5, 6** (a,b,…,e).

(The practice questions for this week are available in a separate document on Brightspace; the above **optional** questions are for those who want to do more)

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