



NFM2106/NFE2105

Mathematics

Integration (antiderivatives)

Definitions

Suppose $F(x)$ has a derivative $f(x)$, i.e.

$$F'(x) = f(x)$$

$F(x)$ is the **antiderivative** of $f(x)$

The **indefinite integral** of $f(x)$ is denoted by $\int f(x)dx$ and is given by

$$\int f(x)dx = F(x) + C$$

This is always determined up to an **arbitrary constant** C , because differentiating a constant gives zero.

Motivation

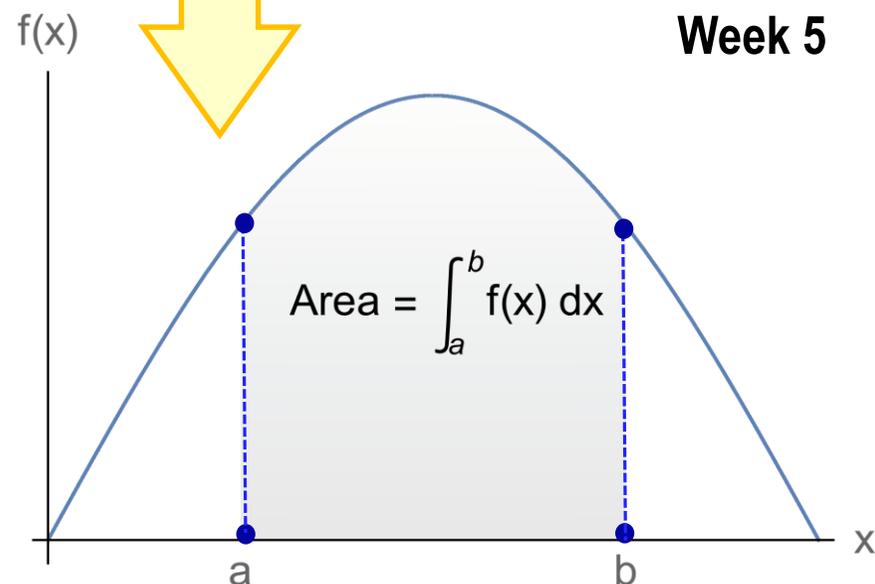
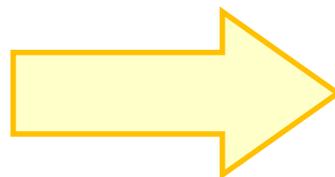
Why bother about the antiderivative of a function?

The **definite integral** of $f(x)$, with upper and lower limits b and a

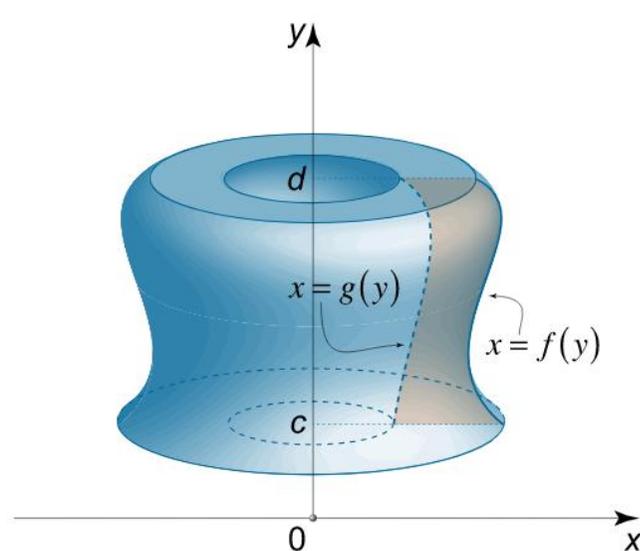
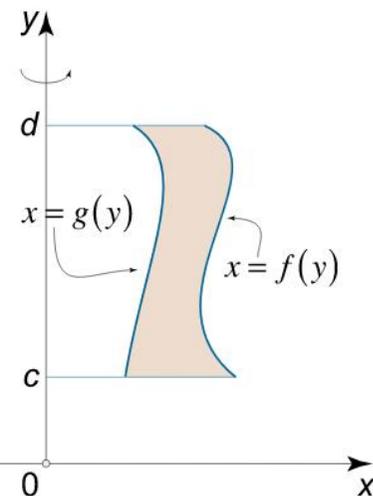
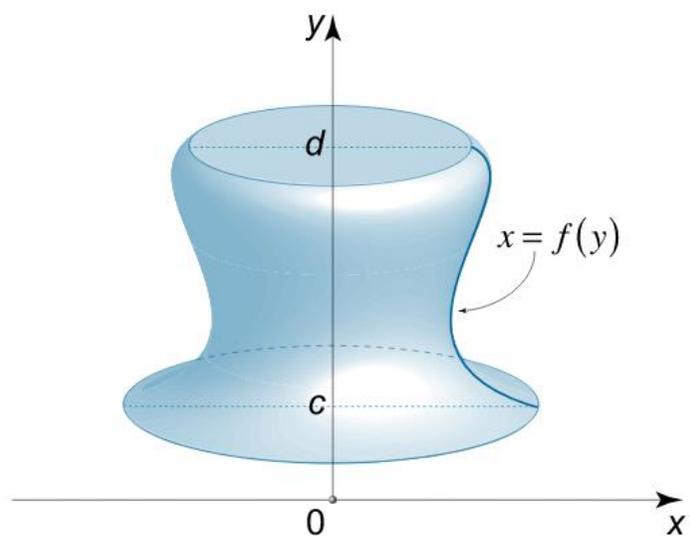
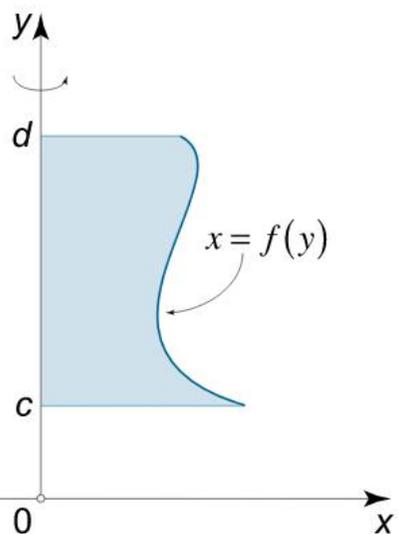
$$\int_a^b f(x) dx = F(b) - F(a)$$

Geometrical interpretation:

Area enclosed by the x -axis and $f(x)$, between $x = a$ and $x = b$

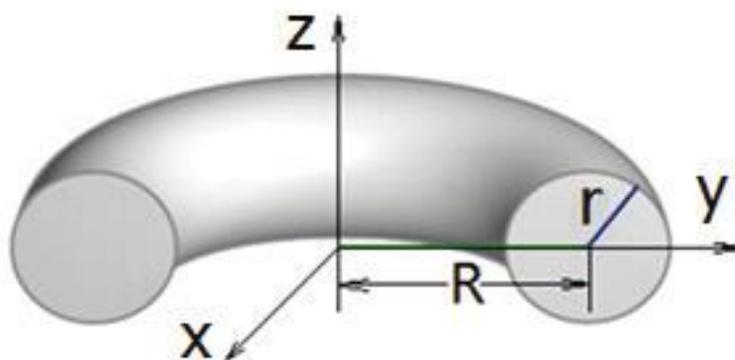
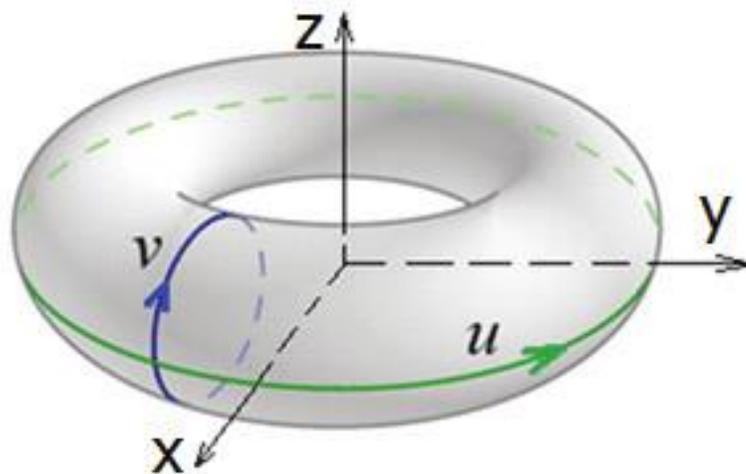


Applications: solids of revolutions





Application: volume /area of a torus





Tables of integrals

Please make sure that you are familiar with the table of integrals posted on Brightspace (you are welcome to use your own table, but please make sure it is not an oversimplified version).

If you have never used a table of integrals, please visit MathTutor:

<https://www.math tutor.ac.uk/integration/integrationusingatableofantiderivatives>

It is your responsibility to have the table available in your tutorial sessions.



Indefinite integrals of elementary functions

Powers (except $1/x$):

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

provided $n \neq -1$

This rule also works for **fractional powers**. For example:

$$\int \sqrt{x} dx = \int x^{1/2} dx = \frac{2}{3} x^{3/2} + C$$

Trigonometric:

$$\int \sin x dx = -\cos x + C$$

$$\int \cos x dx = \sin x + C$$

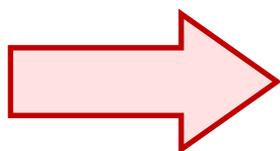
Indefinite integrals of elementary functions

Exponential:

$$\int e^x dx = e^x + C$$

$$\int e^{-x} dx = -e^{-x} + C$$

$$\frac{1}{x} = x^{-1}$$



$$\int \frac{dx}{x} = \ln x + C$$

Note how this is the only power that does not obey the general rule given on the previous page



Some basic integration rules

○ $\int [\alpha h(x)] dx = \alpha \left[\int h(x) dx \right]$ if α is a **constant**

(note, this is **not** true if α is a function of x)

○ $\int [h_1(x) + h_2(x)] dx = \int h_1(x) dx + \int h_2(x) dx$

These properties enable us to find the integrals of more complicated functions. They are called **linearity properties**.



Application: hyperbolic functions

We can integrate those by combining integrals of e^x and e^{-x}

Let us find $\int \sinh x \, dx$

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$

$$\int \sinh x \, dx = \int \frac{1}{2}(e^x - e^{-x}) \, dx = \frac{1}{2} \int e^x \, dx - \frac{1}{2} \int e^{-x} \, dx$$

$$= \frac{1}{2}e^x - \frac{1}{2}(-1)e^{-x} + C$$

$$= \frac{1}{2}(e^x + e^{-x}) + C = \cosh x + C$$

Note, **only one constant need be added at the end.** Each one of the two integrals generates a constant, but these can be added up into C

Likewise $\int \cosh x \, dx = \sinh x + C$



Integration by (direct) substitution

Can be used on integrals of the form

$$I = \int f(g(x)) \cdot g'(x) dx$$

$$\text{Let } t = g(x) \implies \frac{dt}{dx} = g'(x) \implies dt = g'(x) dx$$

Thus,

$$I = \int f(t) dt = F(t) + C = F(g(x)) + C$$

Example 4.1: $\int x e^{x^2} dx$

$$= \int \frac{1}{2} e^t dt = \frac{1}{2} \int e^t dt = \frac{1}{2} e^t = \frac{1}{2} e^{x^2} + C$$

Let $t = x^2$
Then $dt = 2x dx$

and
 $x dx = \frac{1}{2} dt$



Integration by (direct) substitution

Example 4.4: $I = \int \frac{x}{x^2 + 1} dx$

$$t = x^2 + 1 \implies dt = 2x dx$$
$$\implies x dx = \frac{1}{2} dt$$

$$I = \int \frac{1}{t} \cdot \frac{1}{2} dt = \frac{1}{2} \int \frac{dt}{t}$$
$$= \frac{1}{2} \ln t + C$$
$$= \frac{1}{2} \ln(x^2 + 1) + C$$

Example 4.5: $I = \int e^x \sqrt{1 + e^x} dx$

$$t = 1 + e^x \implies dt = e^x dx$$

$$I = \int t^{1/2} dt = \frac{2}{3} t^{3/2} + C$$
$$= \frac{2}{3} (1 + e^x)^{3/2} + C$$



Other examples

Example

Suppose now we wish to find the integral

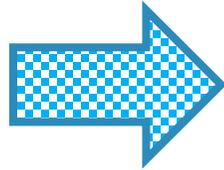
$$\int \cos(3x + 4) dx$$

with $u = 3x + 4$ and $\frac{du}{dx} = 3 \quad \longrightarrow \quad du = \left(\frac{du}{dx}\right) dx = 3 dx$

$$\begin{aligned} \int \cos(3x + 4) dx &= \int \frac{1}{3} \cos u du \\ &= \frac{1}{3} \int \cos u du \\ &= \frac{1}{3} \sin u + c. \end{aligned} \qquad \int \cos(3x + 4) dx = \frac{1}{3} \sin(3x + 4) + c.$$

The integration-by-parts formula (IPF)

Product Rule



$$\frac{d}{dx}[U(x)V(x)] = U(x)\frac{dV}{dx} + V(x)\frac{dU}{dx}$$

+

INTEGRATE:

$$U(x)V(x) = \int U(x)\frac{dV}{dx} + \int V(x)\frac{dU}{dx}$$

+

RE-ARRANGE:

$$\int U(x)\frac{dV}{dx} dx = U(x)V(x) - \int V(x)\frac{dU}{dx} dx$$

Simplified form:

$$\int U dV = UV - \int V dU$$



Example 4.6

$$\int U dV = UV - \int V dU$$

Calculate $\int x e^x dx$ using the above formula

We need to identify: U, V, dU, dV

$$\int x e^x dx$$

$U \leftarrow$ (points to x)
 $\rightarrow dV$ (points to $e^x dx$)

$$= x e^x - \int e^x dx$$

$$= x e^x - e^x + C \quad (C \in \mathbb{R})$$

let $U = x, \quad dV = e^x dx$

then $dU = dx, \quad V = e^x$



ASIDE

$$dU = U'(x) dx$$

$$V = \int dV$$

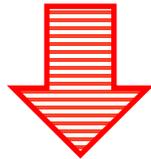
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General advice

$$\int U dV = UV - \int V dU$$

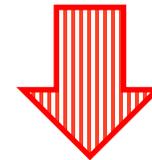
Look for a factor of the integrand that is **easily integrated**, and include **dx** with that factor to make up **dV**. Then **U** is the **remaining part** of the integrand (sometimes it might be necessary to take **dV=dx only**).



$$\text{integrand} = \text{polynomial} \times \left\{ \begin{array}{l} \text{exponential} \\ \text{sine} \\ \text{cosine} \end{array} \right\}$$

$$U = \text{polynomial}$$

$$dV = \text{the rest}$$



$$\text{integrand} = \left\{ \begin{array}{l} \text{log} \\ \text{inverse trig function} \\ \text{some other function} \\ \text{not readily integrable} \end{array} \right\}$$

$$U = \text{one of the above}$$

$$dV = \text{the rest}$$



$$\int U dV = UV - \int V dU$$

Example 4.7

Calculate $\int x^3 \log(x) dx$

let $U = \log(x)$, $dV = x^3 dx$
then $dU = [\log(x)]' dx$, $V = \frac{1}{4}x^4$

$$= \int \log(x) \cdot x^3 dx$$

$$= \frac{1}{4}x^4 \log(x) - \int \frac{1}{4}x^4 \cdot [\log(x)]' dx$$

$$= \frac{1}{4}x^4 \log(x) - \frac{1}{4} \int x^3 dx$$

$$= \frac{1}{4}x^4 \log(x) - \frac{x^4}{4 \cdot 4} + C$$

$$= \frac{1}{4}x^4 \left[\log(x) - \frac{1}{4} \right] + C$$

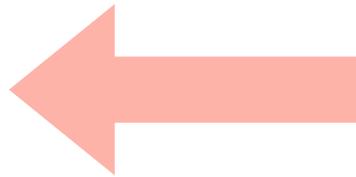
$$[\log(x)]' = \frac{1}{x}$$

$$\int U \, dV = UV - \int V \, dU$$

Repeated IPF

Example 4.8: Calculate $\int x^2 \sin(x) \, dx$

$$\int \underline{x^2} \underline{\sin(x)} \, dx$$



let $U = \underline{x^2}$, $dV = \underline{\sin(x)} \, dx$
 then $dU = 2x \, dx$, $V = -\cos(x)$



let $U = \underline{x}$, $dV = \underline{\cos(x)} \, dx$
 then $dU = 1 \cdot dx$, $V = \sin(x)$

$$= -x^2 \cos(x) + 2 \int \underline{x} \cdot \underline{\cos(x)} \, dx$$

$$= -x^2 \cos(x) + 2[x \sin(x) - \int \sin(x) \, dx]$$

$$= -x^2 \cos(x) + 2x \sin(x) + 2 \cos(x) + C$$

$$= (2 - x^2) \cos(x) + 2x \sin(x) + C$$



Partial fractions & integration (mathtutor)

Some integrands do not yield to integration by parts or by substitution, but may be expressible in terms of partial fractions.

Example 4.9: Evaluate $\int \frac{dx}{x(x^2 + 1)}$

Break this down into partial fractions: $\frac{1}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$

whereupon $A = 1, B = -1, C = 0$. So

$$\int \frac{dx}{x(x^2 + 1)} = \int \frac{dx}{x} - \int \frac{x dx}{x^2 + 1}$$

$$= \ln x - \frac{1}{2} \ln(1 + x^2) + C$$

these integrals are
in the Table





Integrals of rational functions (mathtutor)

Example

Suppose we want to find

$$\int \frac{x}{(2-x)(x+3)} dx.$$

$$\frac{x}{(2-x)(x+3)} = \frac{A}{(2-x)} + \frac{B}{(x+3)}$$

$$\frac{x}{(2-x)(x+3)} = \frac{2}{5(2-x)} - \frac{3}{5(x+3)}$$

$$\begin{aligned} \int \left(\frac{2}{5(2-x)} - \frac{3}{5(x+3)} \right) dx &= -\frac{2}{5} \int \frac{-1}{2-x} dx - \frac{3}{5} \int \frac{1}{x+3} dx \\ &= -\frac{2}{5} \ln |2-x| - \frac{3}{5} \ln |x+3| + c. \end{aligned}$$



Integrals of rational functions (mathtutor)

Example

Suppose we wish to find $\int \frac{x}{x^2 + x + 1} dx$

this is a challenging
example

$$\frac{1}{2} \left(\frac{2x}{x^2 + x + 1} \right) = \frac{1}{2} \left(\frac{2x + 1 - 1}{x^2 + x + 1} \right) = \frac{1}{2} \left(\frac{2x + 1}{x^2 + x + 1} - \frac{1}{x^2 + x + 1} \right)$$

$$\int \frac{x}{x^2 + x + 1} dx = \frac{1}{2} \ln |x^2 + x + 1| - \frac{1}{\sqrt{3}} \tan^{-1} \frac{2x + 1}{\sqrt{3}} + c$$



Learning Resources

<https://www.mathtutor.ac.uk/integration>

K.A. Stroud and D.J. Booth, Engineering Mathematics, 7th Edition,
(chapters on integration)