



NFM2106/NFE2105

Mathematics

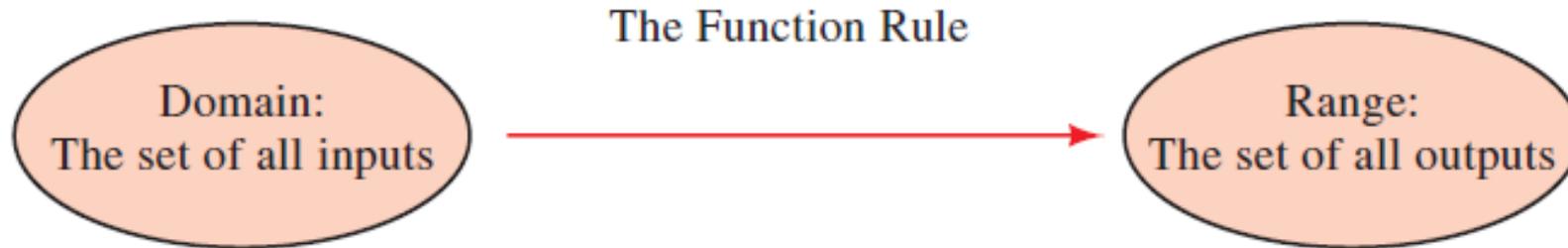
Further Trigonometry
(inverse trig functions + trig eqns.)



Outline

- Inverse functions (see Week 1 as well)
- Inverse Trigonometric functions
- Trigonometric equations

Functions (see Week 1)



EXAMPLE: State the domain and range for the function

$$y = 7.5x, \quad 0 \leq x \leq 40$$

ANSWER:

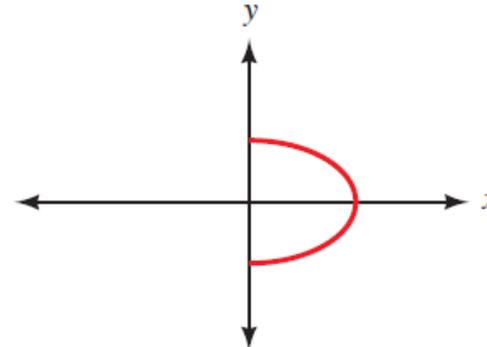
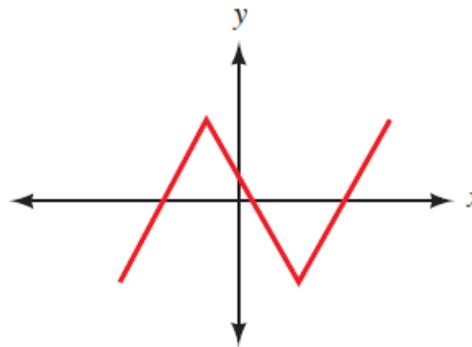
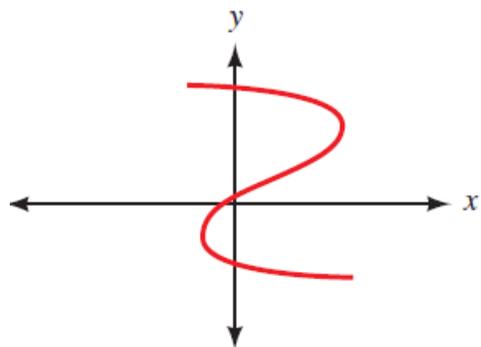
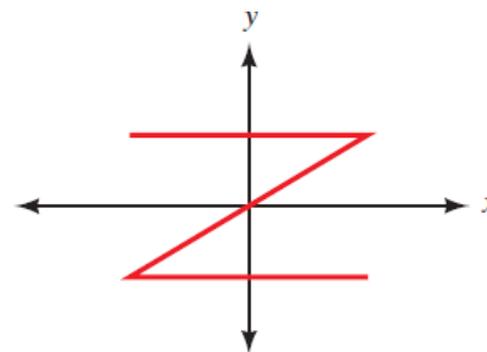
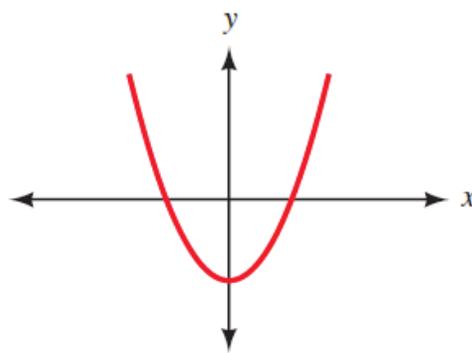
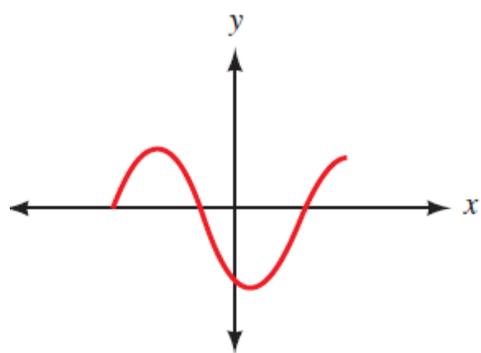
Domain = $\{x \mid 0 \leq x \leq 40\}$
Range = $\{y \mid 0 \leq y \leq 300\}$



Functions (Vertical Line Test)

VERTICAL LINE TEST

If a vertical line crosses the graph of a relation in more than one place, the relation cannot be a function. If no vertical line can be found that crosses a graph in more than one place, then the graph is the graph of a function.



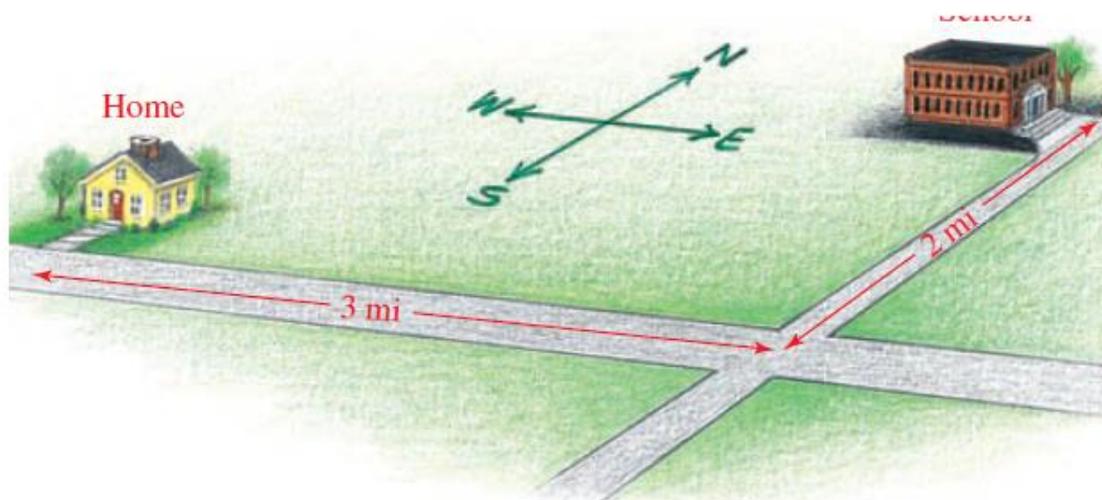


The Inverse of a Function

To get to his school, Fred drives 3 miles east and then turns left and drives 2 miles north.

When he leaves school, he drives the same two segments but in reverse order and the opposite direction; that is, he drives 2 miles south, turns right, and drives 3 miles west. When he arrives home from school, he is right where he started.

His **route home** “undoes” his **route to school**, leaving him where he began.



The Inverse of a Function

The relationship between a function and its **inverse function** is similar to the relationship between Fred's route from home to school and his route from school to home.

We can obtain the equation of the inverse of the function $y = f(x)$ by **interchanging** the role of x and y in the equation of f

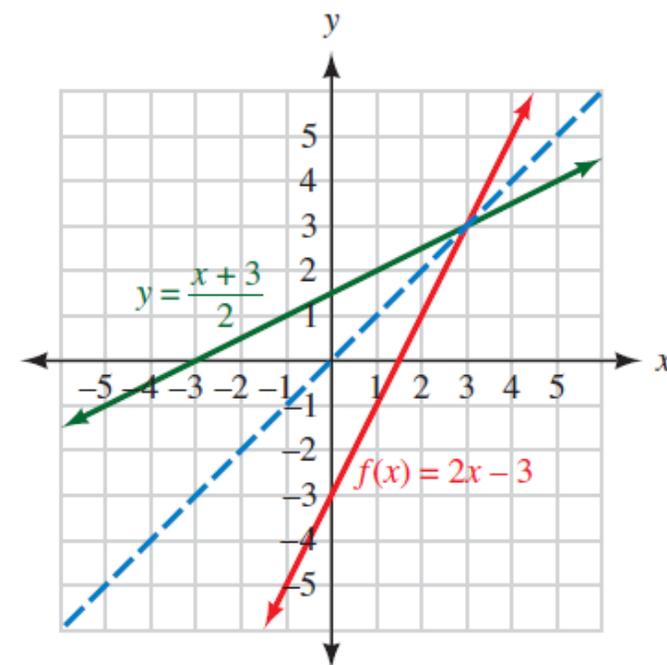
$$f(x) = 2x - 3 \qquad y = 2x - 3$$

$$x = 2y - 3$$

$$x + 3 = 2y$$

$$\frac{x + 3}{2} = y$$

$$y = \frac{x + 3}{2}$$





One-to-One Functions

DEFINITION

A function is a *one-to-one function* if every element in the range comes from exactly one element in the domain.

Horizontal Line Test: If a horizontal line crosses the graph of a function in more than one place, then the function is not a one-to-one function because the points at which the horizontal line crosses the graph will be points with the same y-coordinates, but different x-coordinates.

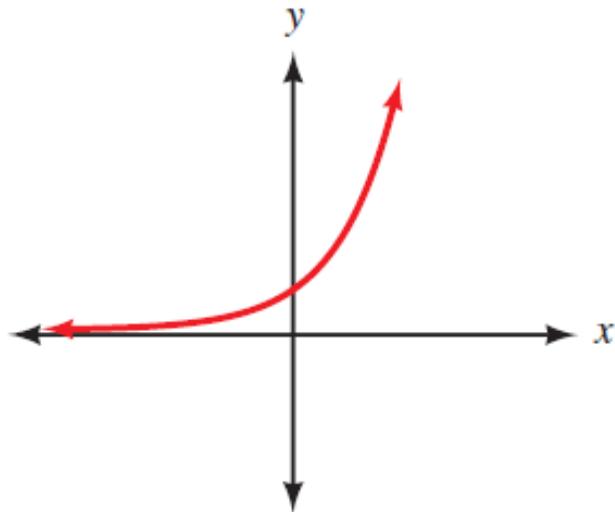
INVERSE FUNCTION NOTATION

If $y = f(x)$ is a one-to-one function, then the inverse of f is also a function and can be denoted by $y = f^{-1}(x)$.

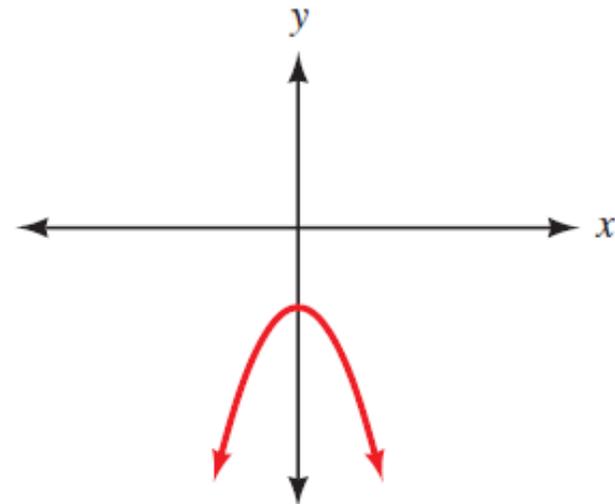


Horizontal Line Test

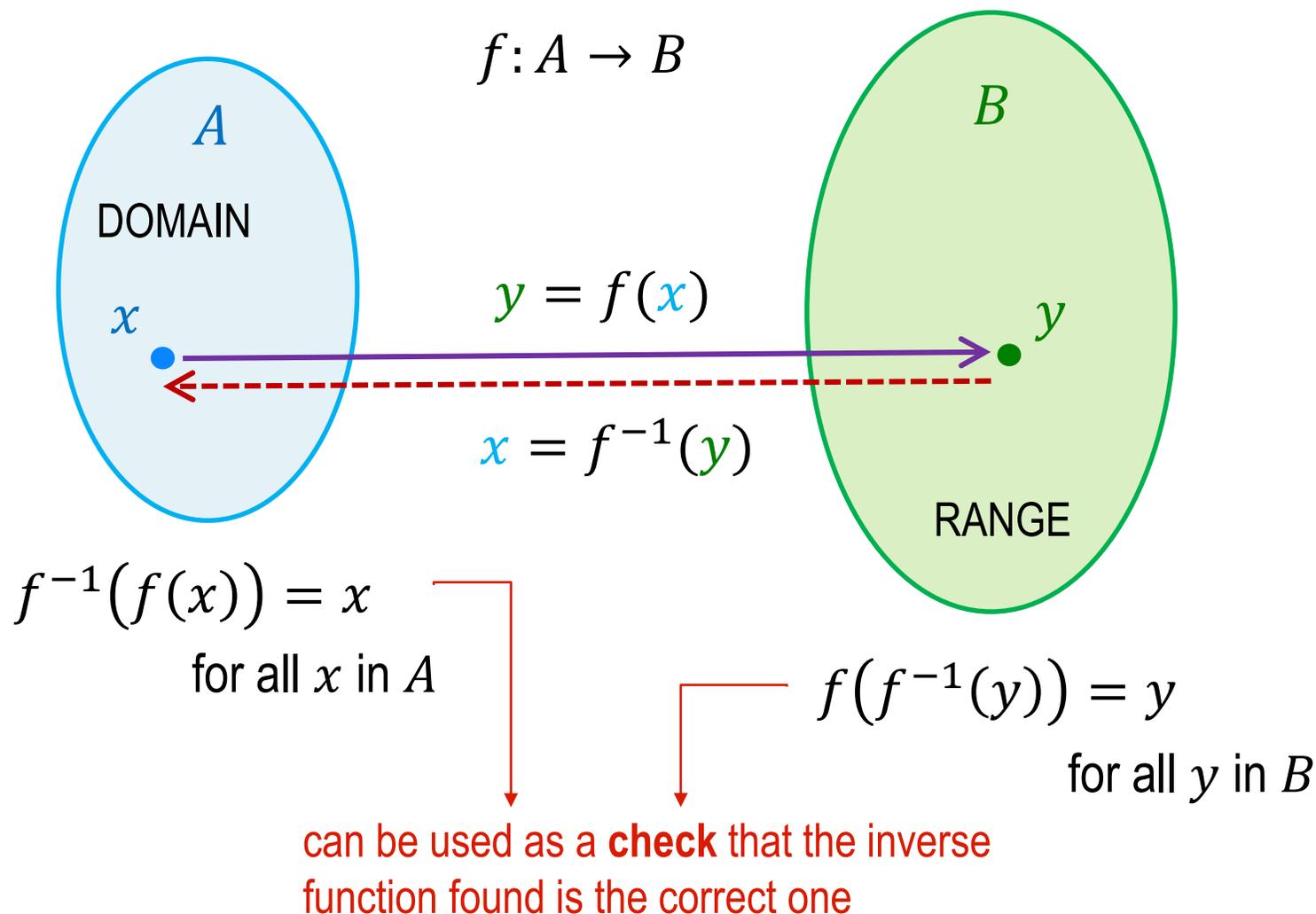
a.



b.



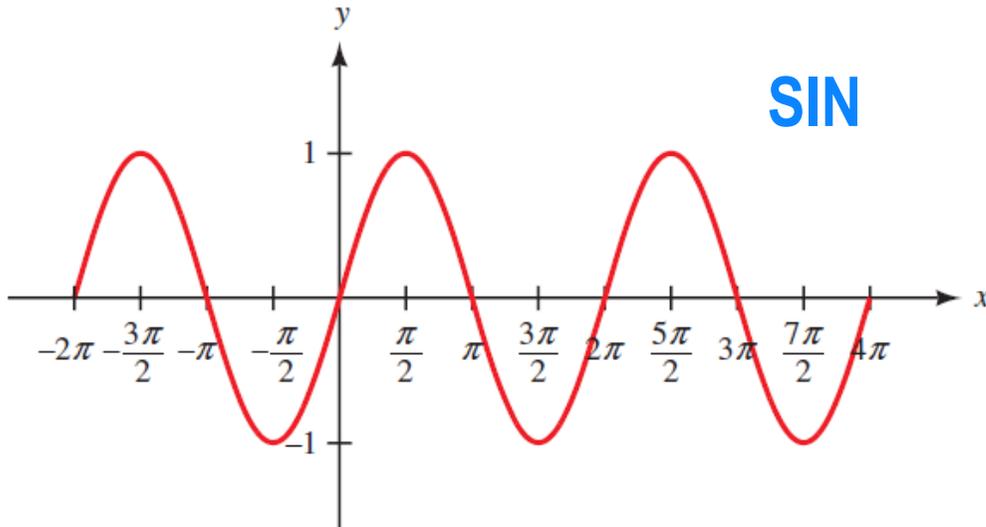
The “cancellation” identities



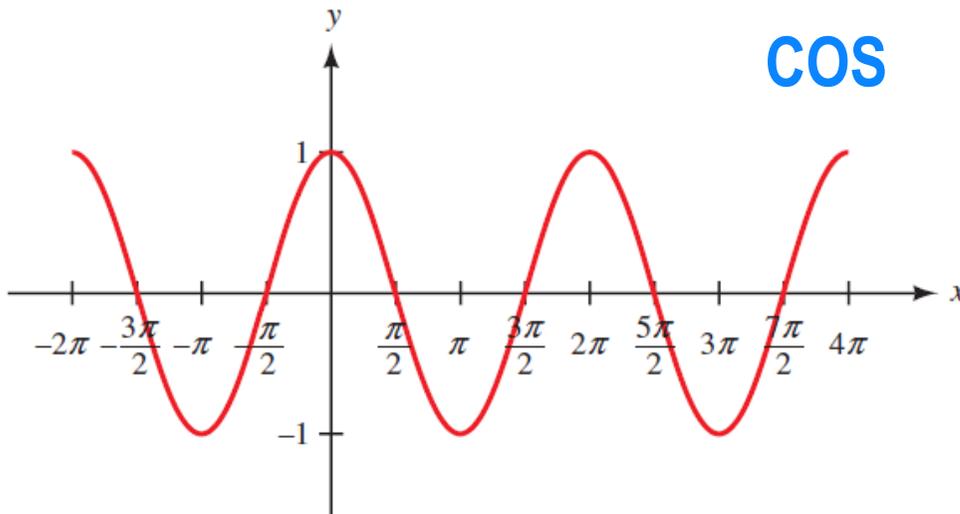


The SIN and COS graphs

SIN



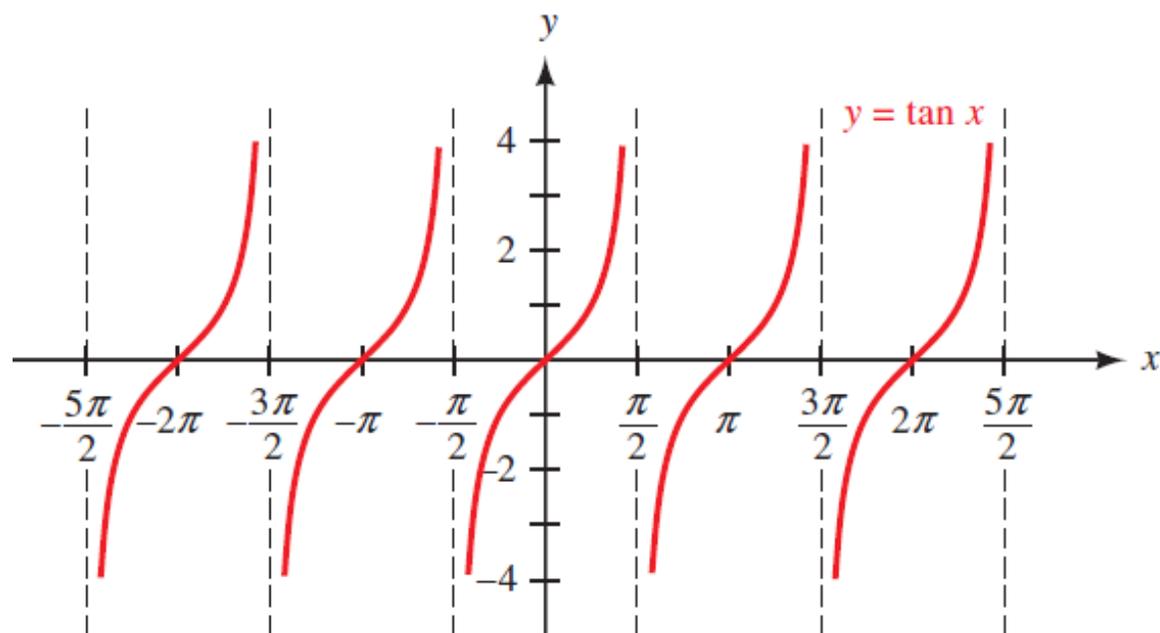
COS



do **NOT** satisfy
the horizontal line test



The TAN graph



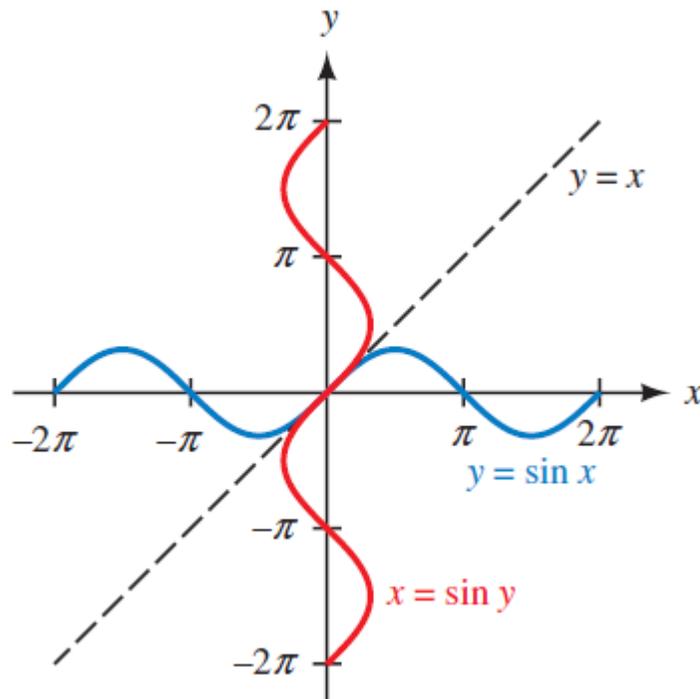
does **NOT** satisfy
the horizontal line test

The inverse SINE “rule”

To find the inverse “rule” for $y = \sin(x)$, we interchange x and y :

$$x = \sin(y)$$

This is the equation of the inverse sine “rule”.



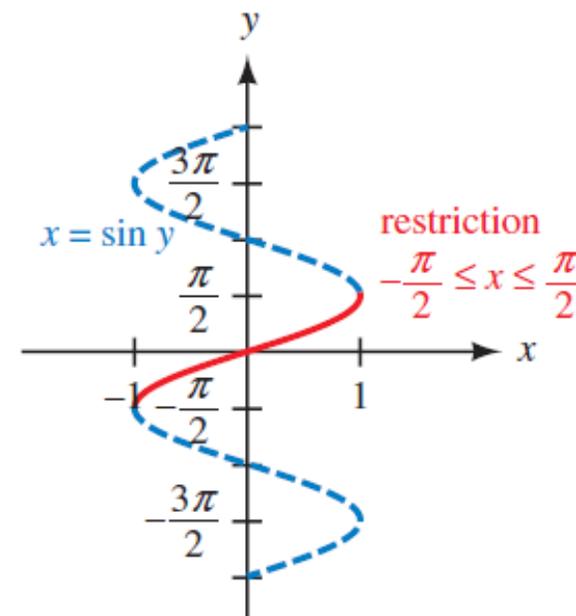
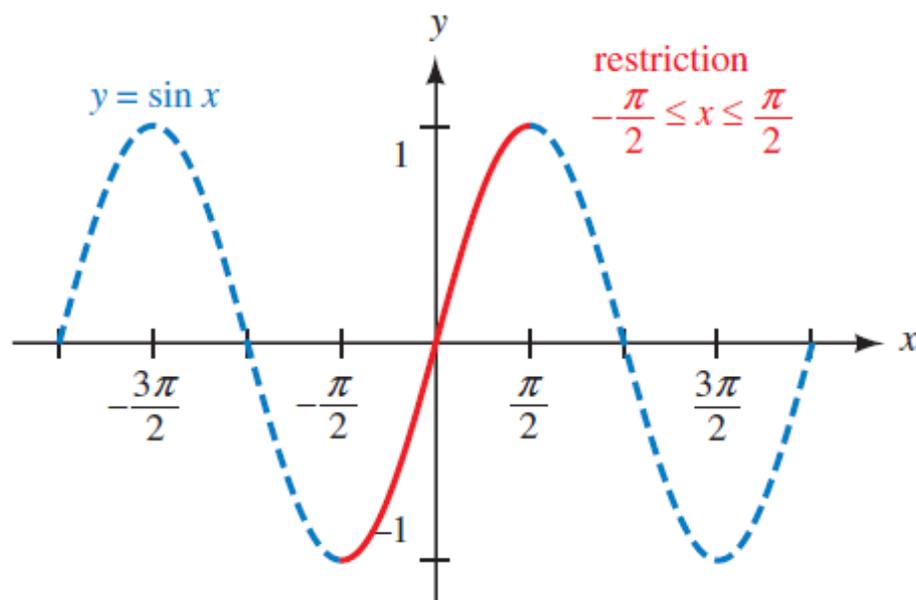
For every value of x in the domain, there are many values of y . The graph of

$$x = \sin(y)$$

fails the vertical line test.



The Inverse Sine Function



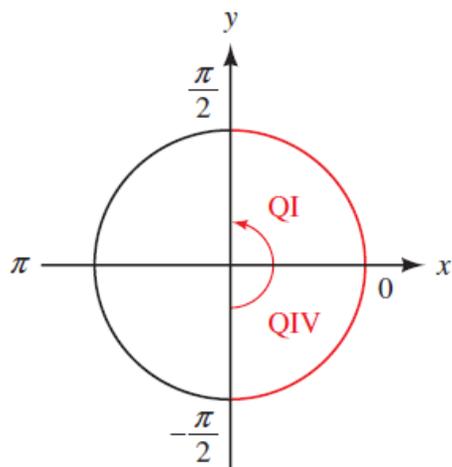
The Inverse Sine Function

NOTATION

The notation used to indicate the inverse sine function is as follows:

Notation	Meaning
$y = \sin^{-1} x$ or $y = \arcsin x$	$x = \sin y$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

In words: y is the angle between $-\pi/2$ and $\pi/2$, inclusive, whose sine is x .



$$\sin^{-1} : [-1, 1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

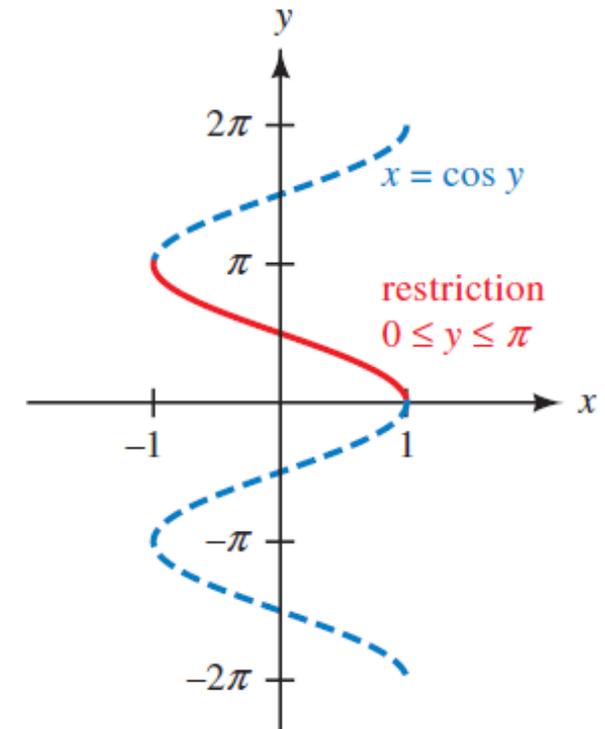
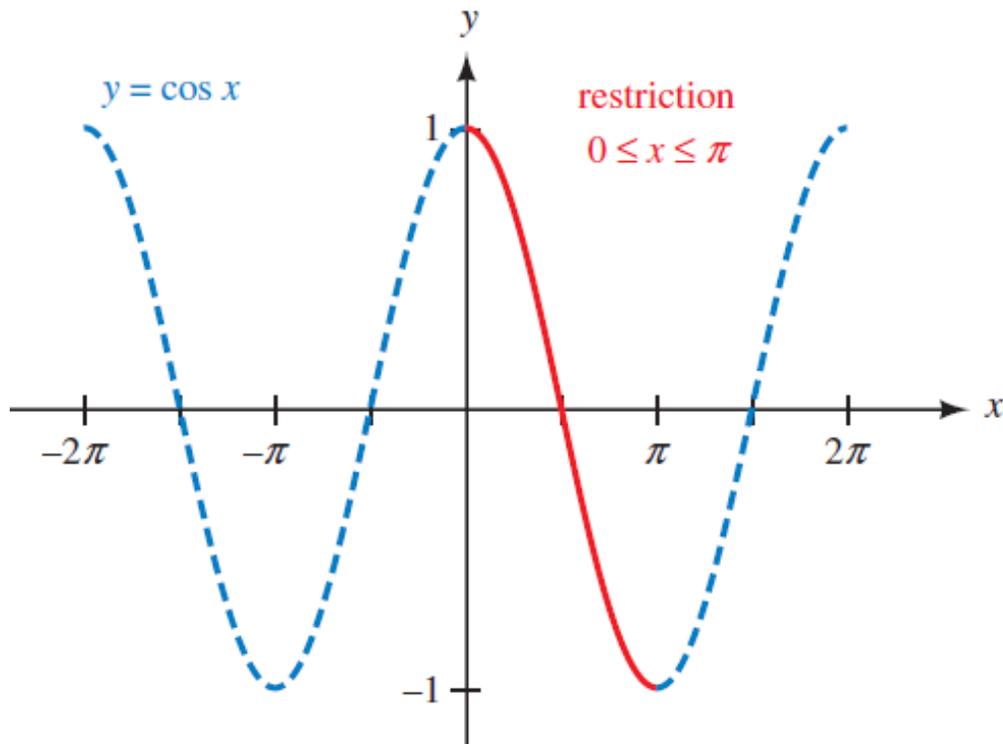
$$\sin(\sin^{-1}(x)) = x \quad \text{for all } -1 \leq x \leq 1$$

$$\sin^{-1}(\sin(y)) = y \quad \text{only for } \frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

The inverse sine function will return an angle between $-\pi/2$ and $\pi/2$, inclusive, corresponding to QIV or QI.



The Inverse Cosine Function (I)



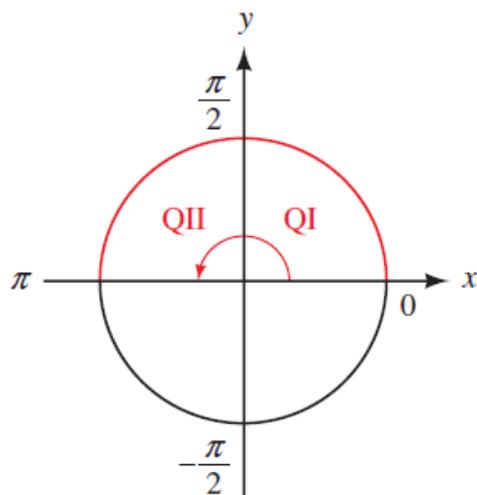
The Inverse Cosine Function (II)

NOTATION

The notation used to indicate the inverse cosine function is as follows:

Notation	Meaning
$y = \cos^{-1} x$ or $y = \arccos x$	$x = \cos y$ and $0 \leq y \leq \pi$

In words: y is the angle between 0 and π , inclusive, whose cosine is x .



$$\cos^{-1} : [-1, 1] \rightarrow [0, \pi]$$

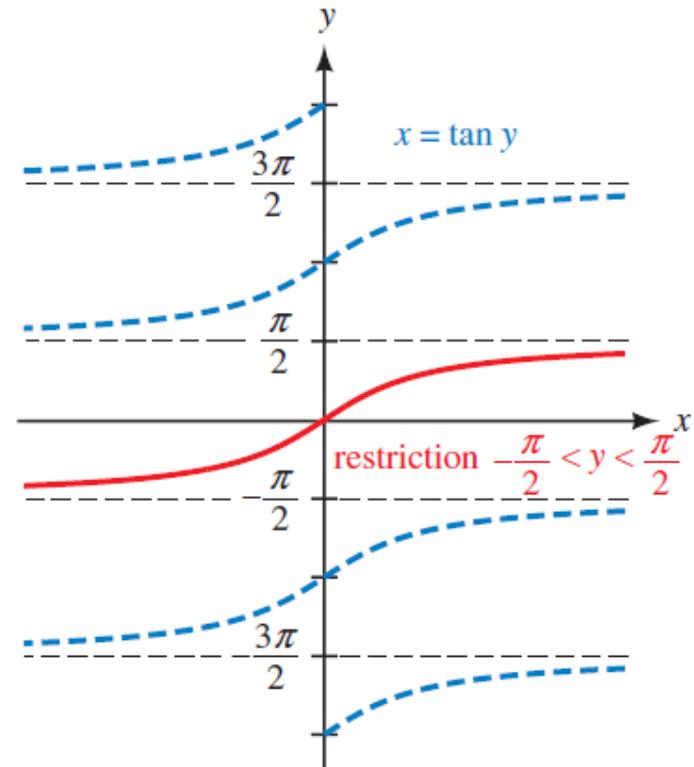
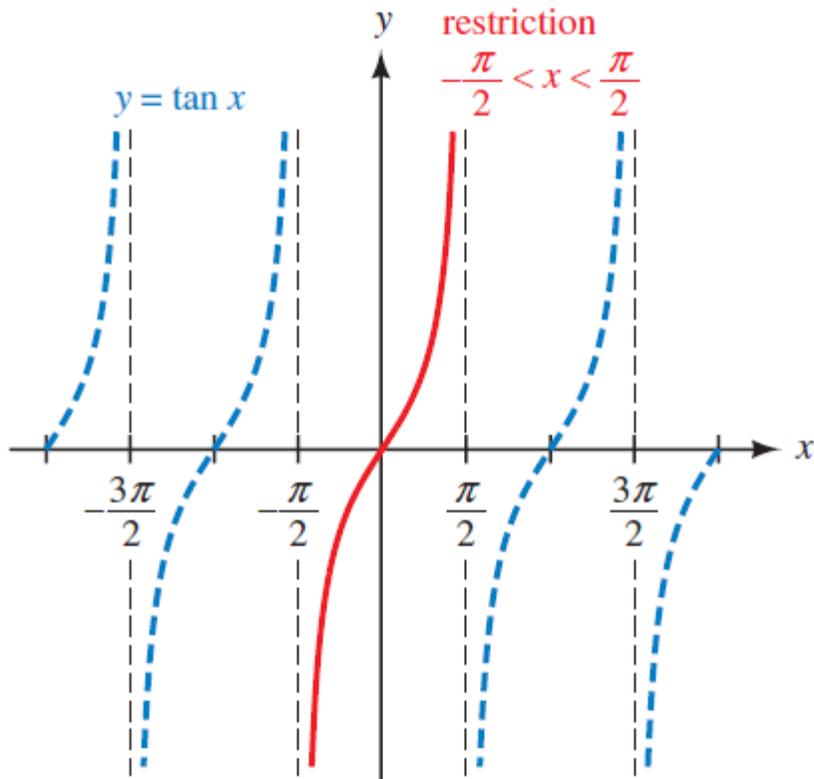
$$\cos(\cos^{-1}(x)) = x \quad \text{for all } -1 \leq x \leq 1$$

$$\cos^{-1}(\cos(y)) = y \quad \text{only for } 0 \leq y \leq \pi$$

The inverse cosine function will return an angle between 0 and π , inclusive, corresponding to QI or QII.



The Inverse Tangent Function



Inverse Tangent Function (II)

NOTATION

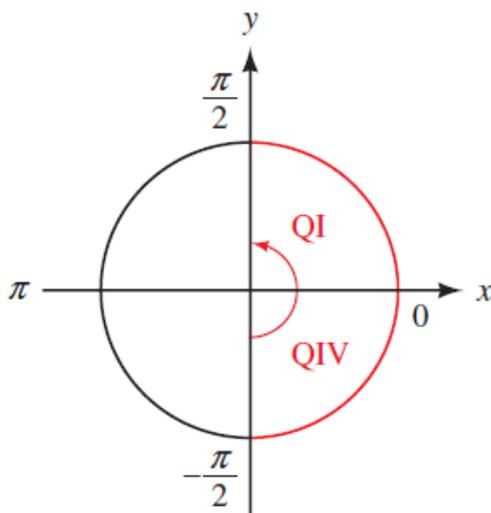
The notation used to indicate the inverse tangent function is as follows:

Notation

Meaning

$$y = \tan^{-1} x \quad \text{or} \quad y = \arctan x \quad x = \tan y \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

In words: y is the angle between $-\pi/2$ and $\pi/2$ whose tangent is x .



$$\tan^{-1} : \mathbb{R} \rightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\tan(\tan^{-1}(x)) = x \quad \text{for all real numbers } x$$

$$\tan^{-1}(\tan(y)) = y \quad \text{only for } \frac{\pi}{2} < y < \frac{\pi}{2}$$

The inverse tangent function will return an angle between $-\pi/2$ and $\pi/2$, corresponding to QIV or QI.

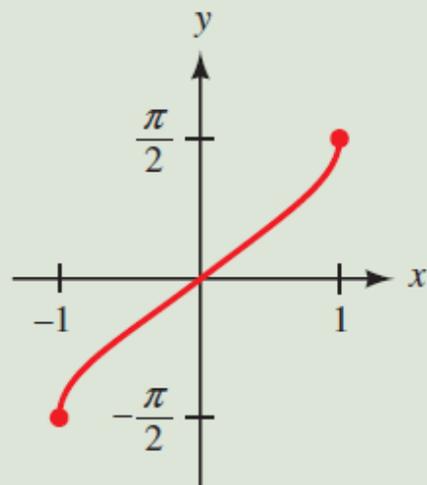


Summary

INVERSE TRIGONOMETRIC FUNCTIONS

Inverse Sine

$$y = \sin^{-1} x = \arcsin x$$

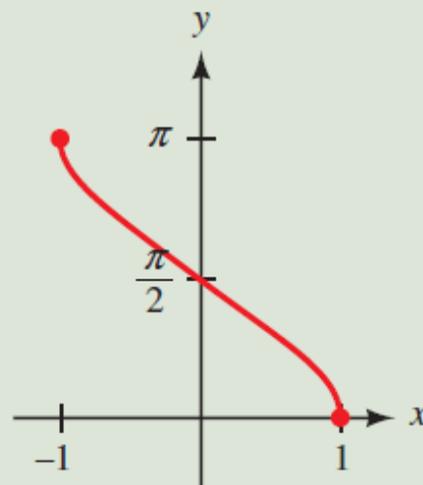


Domain: $-1 \leq x \leq 1$

Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

Inverse Cosine

$$y = \cos^{-1} x = \arccos x$$

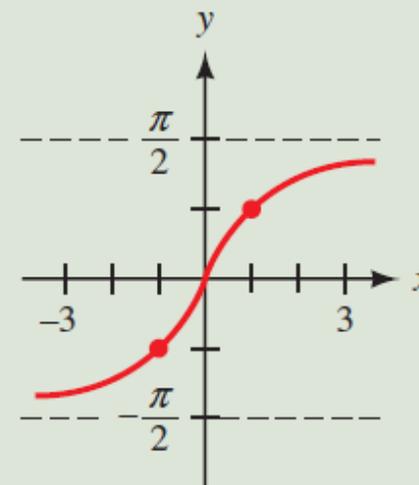


Domain: $-1 \leq x \leq 1$

Range: $0 \leq y \leq \pi$

Inverse Tangent

$$y = \tan^{-1} x = \arctan x$$



Domain: all real numbers

Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$



Examples

I) Calculate (A) $\sin^{-1}\left(\frac{1}{2}\right)$ (B) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ (C) $\tan^{-1}(-1)$

(A) We know (see the notes from last week): $\sin\frac{\pi}{6} = \frac{1}{2}$

Also, $\sin^{-1}(\sin(y)) = y$ for $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$-\frac{\pi}{2} \leq \frac{\pi}{6} \leq \frac{\pi}{2} \rightarrow \sin^{-1}\left(\sin\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6} \Rightarrow \boxed{\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}}$

(B) $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \Rightarrow \cos\left(\pi + \frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2} \Rightarrow \cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

$[\cos(\pi+x) = -\cos x]$

$\cos^{-1}(\cos(y)) = y$ for $0 \leq y \leq \pi$

But $0 \leq \frac{5\pi}{6} \leq \pi$, so $\cos^{-1}\left(\underbrace{\cos\left(\frac{5\pi}{6}\right)}_{-\frac{\sqrt{3}}{2}}\right) = \frac{5\pi}{6} \Rightarrow \boxed{\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}}$

Examples

$$\tan\left(\frac{\pi}{4}\right) = 1 \quad (\text{from last week})$$

$$\Downarrow$$

$$\tan\left(-\frac{\pi}{4}\right) = -1 \quad [\text{since } \tan(-x) = -\tan x \text{ for all } x]$$

Also, $\tan^{-1}(\tan(y)) = y$ for $-\frac{\pi}{2} < y < \frac{\pi}{2}$

$$-\frac{\pi}{2} < -\frac{\pi}{4} < \frac{\pi}{2} \quad \text{Hence, } \tan^{-1}\left(\underbrace{\tan\left(-\frac{\pi}{4}\right)}_{-1}\right) = -\frac{\pi}{4} \Rightarrow \boxed{\tan^{-1}(-1) = -\frac{\pi}{4}}$$

II). Calculate: $\cos^{-1}\left(-\frac{1}{2}\right)$

From last week: $\cos\frac{\pi}{3} = \frac{1}{2}$

$$\cos\left(\pi - \frac{\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2}, \text{ so } \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$$



Examples

$$\cos^{-1}(\cos(y)) = y \quad \text{for } 0 \leq y \leq \pi$$

$0 \leq \frac{2\pi}{3} \leq \pi$ Hence, $\cos^{-1}(\underbrace{\cos\left(\frac{2\pi}{3}\right)}_{-\frac{1}{2}}) = \frac{2\pi}{3} \Rightarrow \boxed{\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}}$

IV) Calculate: $\tan^{-1}(-\sqrt{3})$

$$\tan\left(\frac{\pi}{3}\right) = \sqrt{3} \quad (\text{from last week})$$

$$\Downarrow$$
$$\tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$$

$$\tan^{-1}\left(\overbrace{\tan\left(-\frac{\pi}{3}\right)}^{-\sqrt{3}}\right) = -\frac{\pi}{3}$$
$$\Rightarrow \boxed{\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}}$$

$$\tan^{-1}(\tan(y)) = y \quad \text{for all } -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$-\frac{\pi}{2} < -\frac{\pi}{3} < \frac{\pi}{2}$



Trigonometric equations

A trigonometric equation is simply an equation that contains one or several trigonometric functions.

Examples:

$$2 \sin^2 \theta + 2 \sin \theta - 1 = 0,$$

$$\cos x - 2 \sin x \cos x = 0$$

$$(2 \cos \theta + \sqrt{3})(2 \cos \theta + 1) = 0$$

$$\sqrt{3} \tan \theta - 2 \sin \theta \tan \theta = 0$$

Most trig equations can be reduced to several basic types:

$$\sin \theta = A$$

or

$$\cos \theta = A$$

or

$$\tan \theta = A$$

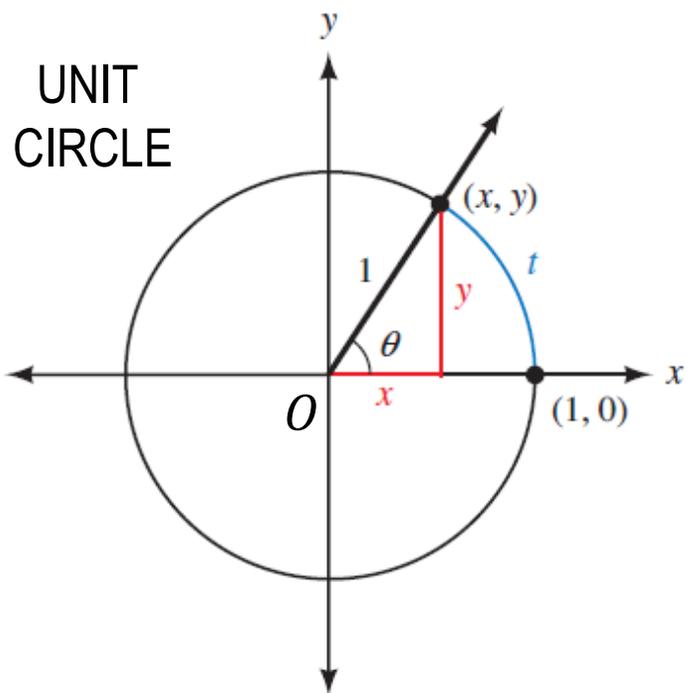
Particular case: $A = 0$

(a). $\sin \theta = 0 \Leftrightarrow \theta = k\pi \quad (k \in \mathbb{Z})$

(b). $\cos \theta = 0 \Leftrightarrow \theta = \frac{\pi}{2} + k\pi \quad (k \in \mathbb{Z})$

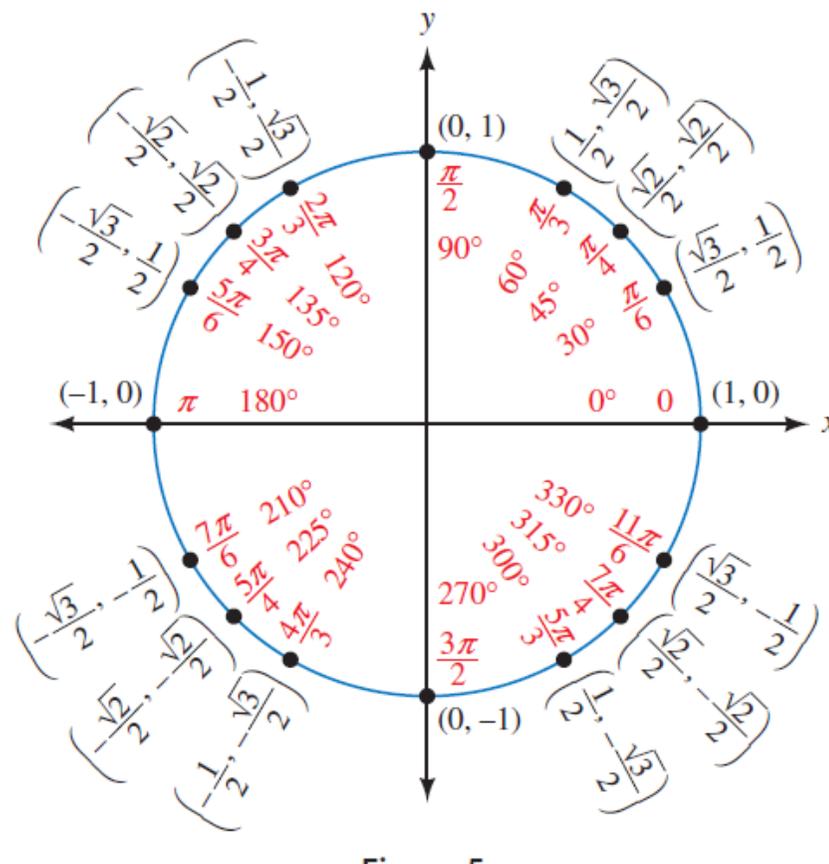
(c). $\tan \theta = 0 \Leftrightarrow \theta = k\pi \quad (k \in \mathbb{Z})$

$\mathbb{Z}: 0, \pm 1, \pm 2, \pm 3, \dots$



$x = \cos \theta$

$y = \sin \theta$





The general case:

$$(a). \sin \theta = \sin \alpha \Leftrightarrow \theta = \alpha + 2k\pi \text{ or } \pi - \alpha + 2k\pi \quad (k \in \mathbb{Z})$$

$$(b). \cos \theta = \cos \alpha \Leftrightarrow \theta = \alpha + 2k\pi \text{ or } -\alpha + 2k\pi \quad (k \in \mathbb{Z})$$

$$(c). \tan \theta = \tan \alpha \Leftrightarrow \theta = \alpha + k\pi \quad (k \in \mathbb{Z})$$

OBS. It is possible to simplify the expression of θ on the first line above:

$$\theta = (-1)^k \alpha + k\pi \quad \text{for } k \in \mathbb{Z}$$

The expression of θ on the second line can be also abbreviated:

$$\theta = \pm \alpha + 2k\pi \quad \text{for } k \in \mathbb{Z}$$



Trig equations: examples

OBS. $\begin{cases} \sin \theta = A \\ \cos \theta = A \end{cases}$ has solutions only if $|A| \leq 1$ OR $-1 \leq A \leq 1$
 $-1 \leq A \leq 1$

$$\sin \theta = \underbrace{A}_{\sin(\sin^{-1}(A))} \Rightarrow \theta = (-1)^k \sin^{-1}(A) + k\pi \quad k = 0, \pm 1, \pm 2, \dots$$

$$\cos \theta = \underbrace{A}_{\cos(\cos^{-1}(A))} \Rightarrow \theta = \pm \cos^{-1}(A) + 2k\pi \quad k = 0, \pm 1, \pm 2, \dots$$

Key Point

To solve trigonometric eqns. you'll have to have an understanding of the inverse trig functions.

Trig equations: examples

$$\text{I). } \sin(2\theta) + \sqrt{2} \cos \theta = 0$$

$$2 \sin \theta \cos \theta$$

$$\cos \theta (2 \sin \theta + \sqrt{2}) = 0$$

$$\cos \theta = 0$$



$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$0 \leq \theta \leq 2\pi$$

want solutions
in this range

$$2 \sin \theta + \sqrt{2} = 0$$
$$\sin \theta = -\frac{\sqrt{2}}{2}$$

$$\theta = \frac{5\pi}{4}, \frac{7\pi}{4}$$

OBS. The original eqn. has 4 solutions in the range $0 \leq \theta \leq 2\pi$



Trig equations: examples

1) $\cos 2\theta + 3\sin\theta = 2$

$1 - 2\sin^2\theta$

$0 \leq \theta < 2\pi$

want solutions
in this range

$2\sin^2\theta - 3\sin\theta + 1 = 0$ ($x = \sin\theta$)

$2x^2 - 3x + 1 = 0$ ← quadratic

$(2x-1)(x-1) = 0 \Rightarrow (2\sin\theta-1)(\sin\theta-1) = 0$

$2\sin\theta - 1 = 0$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$\sin\theta = 1$

$\theta = \frac{\pi}{2}$

if $AB = 0$
then
 $A = 0$ OR $B = 0$

OBS. Original eqn. has
3 solutions in the given range



Trig equations: examples

III), $\sin\left(2\theta - \frac{5\pi}{18}\right) = \underbrace{\frac{\sqrt{3}}{2}}_{\sin \frac{\pi}{3}}$ find all solutions

$$\Rightarrow 2\theta - \frac{5\pi}{18} = (-1)^k \frac{\pi}{3} + k\pi \quad k = 0, \pm 1, \pm 2, \dots$$

$$2\theta = \frac{5\pi}{18} + (-1)^k \frac{\pi}{3} + k\pi$$

$k = \text{even} \rightarrow k = 2m$
 $m = 0, \pm 1, \pm 2, \dots$

$$2\theta = \frac{5\pi}{18} + \frac{\pi}{3} + 2m\pi \Rightarrow 2\theta = \frac{11\pi}{18} + 2m\pi$$

$$\theta = \frac{11\pi}{36} + m\pi \quad m = 0, \pm 1, \pm 2, \dots$$

$k = \text{odd} \rightarrow k = 2m + 1$
 $m = 0, \pm 1, \pm 2, \dots$

$$2\theta = \frac{5\pi}{18} - \frac{\pi}{3} + \pi + 2m\pi \Rightarrow 2\theta = \frac{17\pi}{18} + 2m\pi$$

$$\theta = \frac{17\pi}{36} + m\pi, \quad m = 0, \pm 1, \pm 2, \dots$$



Trig equations: examples

If I want my solutions expressed in degrees, the previous results become:

$$\theta = 55^\circ + 180^\circ m$$

$$m = 0, \pm 1, \pm 2, \dots$$

and

$$\theta = 85^\circ + 180^\circ m$$

$$m = 0, \pm 1, \pm 2, \dots$$

OBS. In this module we work with angles in RADIANS
(not degrees)



Learning Resources

<https://www.mathtutor.ac.uk/trigonometry/trigequations/>

K.A. Stroud and D.J. Booth: Engineering Mathematics, (7th Edition), **pp. 296 -- 303**