NFM2106/NFE2105
Mathematics

PARTIAL FRACTIONS

Polynomial Functions

A **polynomial** (in x) is a function of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

where $a_0, a_1, a_2, \ldots, a_n$ are real numbers, known as the **coefficients** of the polynomial.

The **degree** (or **order**) of the polynomial $f(x)$ is the largest power of x:

denoted $\deg(f)$

$$\deg(f) = n$$

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E.g., $f(x) = 3x + 1$ $\deg(f) = 1$ $(a_0 = 1, a_1 = 3)$

$f(x) = x^2 + 7x - 5$ $\deg(f) = 2$ $(a_0 = -5, a_1 = 7, a_2 = 1)$

Polynomials (factors & factorisation)

Some polynomials can be written as products of other polynomials – these sub-units are called **factors**.

E.g.

$$2x^3 + 5x^2 + x - 2 = (x + 1)(2x - 1)(x + 2)$$

$$x^3 + 2x^2 + x = x(x + 1)^2$$

$$x^2 - 4 = (x - 2)(x + 2)$$

The RHS of the above identities are known as the **factorisations** of the corresponding polynomials (which appear on the LHS)

Polynomials (quadratics)

$$f(x) = ax^2 + bx + c$$

$$\Delta = b^2 - 4ac$$

the **discriminant**

The two **roots** of this polynomial (i.e., the solutions of $f(x) = 0$) are

$$x_1 = \frac{-b + \sqrt{\Delta}}{2a}$$ and $$x_2 = \frac{-b - \sqrt{\Delta}}{2a}$$

$\Delta > 0$

$\Delta = 0$

$\Delta < 0$

$$f(x) = a(x - x_1)(x - x_2)$$

distinct linear factors

$$f(x) = a(x - x_1)^2$$

repeated linear factor

no *real* factors

$(x_1, x_2$ are complex numbers)

In this case we say that the quadratic is **irreducible**

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Rational Functions

These are **ratios** of polynomial functions: $\dfrac{P(x)}{Q(x)}$

Examples: $\dfrac{3x - 5}{x^2 + 3x + 2}$, $\dfrac{7x + 2}{x^2 + x - 2}$, $\dfrac{x^3 + 4x - 7}{x^5 - 3x^2 + 2x + 1}$, ETC

We are primarily interested in the case: $\boxed{\deg(P) < \deg(Q)}$

If $\deg(P) \geq \deg(Q)$ then we can use **long division** to write

$$\dfrac{P(x)}{Q(x)} = P_1(x) + \dfrac{P_2(x)}{Q(x)}$$

$\deg(P_2) < \deg(Q)$

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Partial Fractions

Rational functions tend to be quite complicated mathematical objects; in particular, the integrals of such functions can rarely be calculated directly. The **method of partial fractions** turns complicated fractions into a sum of simpler ones (which are readily integrated -- more about integration later on).

$$\frac{P(x)}{Q(x)} = \text{sum of (much) \textbf{simpler} rational functions}$$

the complexity of this **sum** depends on the factorisation of the denominator $Q(x)$

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Partial Fractions

Example 1.8: $\dfrac{7x+2}{x^2+x-2}$

$$P(x) = 7x + 2$$

$$Q(x) = x^2 + x - 2$$

Firstly, we **factorise** the bottom of the fraction (denominator): $Q(x) = (x-1)(x+2)$

so that now: $\dfrac{7x+2}{x^2+x-2} = \dfrac{7x+2}{(x-1)(x+2)}$

What we want to do is **split the fraction up into a sum of simpler fractions** (based on the factors calculated above). To this end, we write

$$\frac{7x+2}{x^2+x-2} = \frac{7x+2}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

$A, B \in \mathbb{R}$

to be found

(cont'd)

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Partial Fractions

Add the partial fractions together, and then **cancel** the denominator from both sides.

This will give us the **identity:**

$$7x + 2 = A(x + 2) + B(x - 1) \quad \text{or} \quad 7x + 2 = (A + B)x + (2A - B)$$

the **Substitution** method :

> Substitute a suitable number for x to leave you with just one constant on the RHS

E.g., $x = -2$ and then $x = 1$

(cont'd)

the **Equating Coefficients** method:

> Equate the constant terms and coefficients of x, then solve the equations for A and B simultaneously

E.g.,
$$\begin{cases} 7 = A + B \\ 2 = 2A - B \end{cases}$$

(cont'd)

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$$7x + 2 = A(x + 2) + B(x - 1)$$

$x = -2$:

$$7(-2) + 2 = A(0) + B(-2 - 1)$$

$$-14 + 2 = -3B$$

$$-12 = -3B \implies \boxed{B = 4}$$

$x = 1$:

$$7(1) + 2 = A(1 + 2) + B(0)$$

$$9 = 3A \implies \boxed{A = 3}$$

$$\begin{cases} 7 = A + B & \text{(I)} \\ 2 = 2A - B & \text{(II)} \end{cases}$$

Adding one to the other we get

$$7 + 2 = A + \cancel{B} + 2A - \cancel{B}$$

$$9 = 3A \implies \boxed{A = 3}$$

Plugging A back in (I) or (II) gives $\boxed{B = 4}$

Conclusion:
$$\frac{7x + 2}{x^2 + x - 2} = \frac{3}{x - 1} + \frac{4}{x + 2}$$

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General Rules for Partial Fractions

The decomposition in partial fractions of the rational function $f(x) = \dfrac{P(x)}{Q(x)}$

depends on the factorisation of $Q(x)$

factor in the denominator	example	partial fraction contribution
distinct linear	$(x-a)$	$\dfrac{A}{(x-a)}$
repeated linear	$(x-a)^2$	$\dfrac{A}{(x-a)} + \dfrac{B}{(x-a)^2}$
quadratic	$x^2 + bx + c$ $(\Delta = b^2 - 4c < 0)$	$\dfrac{Ax+B}{x^2+bx+c}$

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General Rules….

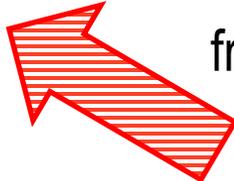Here are some generic examples:

If $Q(x) = x(x-1)(x^2+5)$ then

$$\frac{P(x)}{Q(x)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+5}$$

If $Q(x) = x(x^2+1)^2$ then
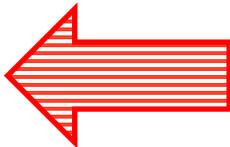
$$\frac{P(x)}{Q(x)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

If $Q(x) = x^3(x^2+x+6)$ then

$$\frac{P(x)}{Q(x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+x+6}$$

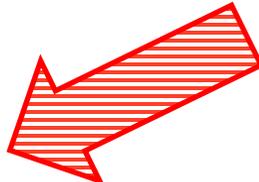The gist of the partial fraction decomposition lies in the determination of the arbitrary constants (*A*,*B*,*C*, etc). This is done by using one of the two methods mentioned before (**'Substitution'** or **'Equating Coefficients'**)

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Further examples

Partial Fractions

A simple example: expand $\dfrac{2x+3}{(x+1)(x+3)}$ in partial fractions.

This means we assume that $\dfrac{2x+3}{(x+1)(x+3)}$ can be written

as $\dfrac{A}{(x+1)} + \dfrac{B}{(x+3)}$ where A and B are numbers (to be found).

$\dfrac{A}{(x+1)}$ and $\dfrac{B}{(x+3)}$ are called partial fractions.

To find A and B put $\dfrac{2x+3}{(x+1)(x+3)} = \dfrac{A}{(x+1)} + \dfrac{B}{(x+3)}$

Multiply throughout by the LCM of all the fractions, i.e. $(x+1)(x+3)$

We get $2x+3 = \dfrac{A(x+1)(x+3)}{(x+1)} + \dfrac{B(x+1)(x+3)}{(x+3)}$

$2x+3 = A(x+3) + B(x+1)$ (1)

There are two methods that we can use to find A and B:

Method 1

Choose values of x to make either the A term or the B term on the right hand side of (1) equal 0.

Put $x = -1$. (1) becomes $2 \times (-1) + 3 = A(2) + B(0)$

then $2A = -2 + 3 = 1$ and so $A = \dfrac{1}{2}$

$$2x+3=A(x+3)+B(x+1) \quad (1)$$

Put $x=-3$. (1) becomes $2\times(-3)+3=A(0)+B(-2)$

so $-2B=-6+3=-3$ and $B=\dfrac{-3}{-2}=\dfrac{3}{2}$

Method 2

Expand the brackets on the right hand side of (1) and collect terms:

$$2x+3=Ax+A3+Bx+B=(A+B)x+(3A+B)$$

Equate x terms $\qquad A+B=2 \quad (2)$

Equate terms not containing x $\quad 3A+B=3 \quad (3)$

Solve the simultaneous equations (2) and (3). Subtract (2) from (3):

we get $2A=1$ and $A=\dfrac{1}{2}$

then from (2) $\dfrac{1}{2}+B=2$ and so $B=2-\dfrac{1}{2}=\dfrac{3}{2}$

From either method $A = \dfrac{1}{2}$ and $B = \dfrac{3}{2}$, i.e.

$$\frac{2x+3}{(x+1)(x+3)} = \frac{\frac{1}{2}}{(x+1)} + \frac{\frac{3}{2}}{(x+3)}$$

or $\quad \dfrac{2x+3}{(x+1)(x+3)} = \dfrac{1}{2(x+1)} + \dfrac{3}{2(x+3)}$

Partial Fractions Example 2

Expand $\dfrac{5x-1}{2x^2+5x-12}$ in partial fractions.

First factorise the bottom line: $2x^2+5x-12=(2x-3)(x+4)$

Then put $\dfrac{5x-1}{2x^2+5x-12}=\dfrac{A}{(2x-3)}+\dfrac{B}{(x+4)}$

Multiply throughout by the LCM, i.e. $(2x-3)(x+4)$

$$5x-1=A(x+4)+B(2x-3) \qquad (4)$$

Use Method 1 -- the quickest in this case.

Put $x=\dfrac{3}{2}$. (4) becomes $5\times\dfrac{3}{2}-1=A(\dfrac{3}{2}+4)+B(0)$

so $A\times\dfrac{11}{2}=\dfrac{13}{2}$ and $A=\dfrac{13}{11}$

$$5x-1=A(x+4)+B(2x-3) \qquad (4)$$

Put $x=-4$. (4) becomes $5\times(-4)-1=A(0)+B(-8-3)$

so $-11B=-21$ and $B=\dfrac{-21}{-11}=\dfrac{21}{11}$

Finally then $\dfrac{5x-1}{2x^2+5x-12}=\dfrac{\frac{13}{11}}{(2x-3)}+\dfrac{\frac{21}{11}}{(x+4)}$

$$=\dfrac{13}{11(2x-3)}+\dfrac{21}{11(x+4)}$$

Partial Fractions Example 3

Expand $\dfrac{2x^2-3x+4}{(x-2)(x+2)(x+3)}$ in partial fractions.

Assume $\dfrac{2x^2-3x+4}{(x-2)(x+2)(x+3)}=\dfrac{A}{(x-2)}+\dfrac{B}{(x+2)}+\dfrac{C}{(x+3)}$

(Three partial fractions needed)

Multiply throughout by the LCM, i.e. $(x-2)(x+2)(x+3)$

We get $2x^2-3x+4=$

$$A(x+2)(x+3)+B(x-2)(x+3)+C(x-2)(x+2) \quad (5)$$

Use Method 1 -- again best in this case.

Put $x=2$ so that $2\times4-3\times2+4=A(4)(5)+B(0)+C(0)$

so $20A=6$ and $A=\dfrac{6}{20}=\dfrac{3}{10}$

$$2x^2 - 3x + 4 =$$
$$A(x+2)(x+3) + B(x-2)(x+3) + C(x-2)(x+2) \quad (5)$$

Put $x = -2$ so that $2 \times 4 - 3 \times (-2) + 4 = A(0) + B(-4)(+1) + C(0)$

so $-4B = 18$ and $B = \dfrac{18}{-4} = -\dfrac{9}{2}$

Put $x = -3$ so that $2 \times 9 - 3 \times (-3) + 4 = A(0) + B(0) + C(-5)(-1)$

so $5C = 31$ and $C = \dfrac{31}{5}$

Finally $\dfrac{2x^2 - 3x + 4}{(x-2)(x+2)(x+3)} = \dfrac{\frac{3}{10}}{(x-2)} - \dfrac{\frac{9}{2}}{(x+2)} + \dfrac{\frac{31}{5}}{(x+3)}$

Or $\dfrac{2x^2 - 3x + 4}{(x-2)(x+2)(x+3)} = \dfrac{3}{10(x-2)} - \dfrac{9}{2(x+2)} + \dfrac{31}{5(x+3)}$

Partial Fractions Example 4

Expand $\dfrac{x^2+3x-1}{(x-2)(x^2+4)}$ in partial fractions.

Now we have a quadratic term, x^2+4, on the bottom line which will not factorise.

We try $\quad \dfrac{x^2+3x-1}{(x-2)(x^2+4)} = \dfrac{A}{(x-2)} + \dfrac{Bx+C}{x^2+4}$

Note the extra term in the second fraction, Bx. Without this the expansion will not work.

Multiply throughout by the LCM, i.e. $(x-2)(x^2+4)$

We get $\quad x^2+3x-1 = A(x^2+4)+(Bx+C)(x-2) \quad$ (6)

We get $\quad x^2 + 3x - 1 = A(x^2 + 4) + (Bx + C)(x - 2) \quad$ (6)

Use Method 1 in (6). The obvious choice is $x = 2$

We get $\quad 4 + 6 - 1 = A(4 + 4) + (Bx + C)(0)$

so $\quad 8A = 9$ and $A = \dfrac{9}{8}$

There are no other obvious choices for Method 1 in (6).
We have to use Method 2, i.e. expand the right hand side:

$$x^2 + 3x - 1 = A(x^2 + 4) + (Bx + C)(x - 2)$$
$$= Ax^2 + 4A + Bx^2 + Cx - 2Bx - 2C$$
$$= (A + B)x^2 + (C - 2B)x + 4A - 2C$$

Equate x^2 terms: $\quad A + B = 1$

Since $A = \dfrac{9}{8}$ we have $B = 1 - \dfrac{9}{8} = -\dfrac{1}{8}$

Equate x terms: $C - 2B = 3$

Since $B = -\dfrac{1}{8}$ we have $C = 3 + 2 \times -\dfrac{1}{8} = \dfrac{11}{4}$

Finally $\dfrac{x^2 + 3x - 1}{(x-2)(x^2+4)} = \dfrac{\dfrac{9}{8}}{(x-2)} + \dfrac{-\dfrac{1}{8}x + \dfrac{11}{4}}{x^2 + 4}$

or $\dfrac{x^2 + 3x - 1}{(x-2)(x^2+4)} = \dfrac{9}{8(x-2)} + \dfrac{-x+22}{8(x^2+4)}$

Partial Fractions Example 5

Expand $\dfrac{2x^2 - x + 2}{(x-2)(x+1)^2}$ in partial fractions.

Now we have a repeated factor, $(x+1)^2$, on the bottom line.

We try $\dfrac{2x^2 - x + 2}{(x-2)(x+1)^2} = \dfrac{A}{(x-2)} + \dfrac{B}{(x+1)} + \dfrac{C}{(x+1)^2}$

Note the extra term, the third fraction, $\dfrac{C}{(x+1)^2}$. Without this term, including the $(x+1)^2$ on the bottom line, the expansion will not work.

Multiply throughout by the LCM of all the fractions,
i.e. $(x-2)(x+1)^2$

Note that as with ordinary fractions, there is no need to use
$(x-2)(x+1)(x+1)^2$

We get $2x^2-x+2=$
$$\frac{A(x-2)(x+1)^2}{(x-2)}+\frac{B(x-2)(x+1)^2}{(x+1)}+\frac{C(x-2)(x+1)^2}{(x+1)^2}$$

Which cancels to give:
$$2x^2-x+2=A(x+1)^2+B(x-2)(x+1)+C(x-2) \quad (7)$$

Use Method 1 in (7). First put $x=2$

We get: $2\times2^2-2+2=A(2+1)^2+B(0)+C(0)$

so $9A=8$ and $A=\dfrac{8}{9}$

$$2x^2 - x + 2 = A(x+1)^2 + B(x-2)(x+1) + C(x-2) \quad (7)$$

The other obvious choice is to put $x = -1$

We get: $2 \times (-1)^2 - (-1) + 2 = A(0) + B(0) + C(-1-2)$

so $-3C = 2 + 1 + 2 = 5$ and $C = \dfrac{5}{-3} = -\dfrac{5}{3}$

There are no other obvious choices for Method 1 in (7).
We have to use Method 2, i.e. expand the right hand side:

$$\begin{aligned}
2x^2 - x + 2 &= A(x+1)^2 + B(x-2)(x+1) + C(x-2) \\
&= A(x^2 + 2x + 1) + B(x^2 - x - 2) + C(x-2) \\
&= Ax^2 + 2Ax + A + Bx^2 - Bx - 2B + Cx - 2C \\
&= (A+B)x^2 + (2A - B + C)x + A - 2B - 2C
\end{aligned}$$

Equate x^2 terms: $A + B = 2$

This enables us to get B the only missing constant.

We have: $\dfrac{8}{9} + B = 2$ so $B = 2 - \dfrac{8}{9} = \dfrac{10}{9}$

So $A = \dfrac{8}{9}$, $B = \dfrac{10}{9}$ and $C = -\dfrac{5}{3}$

Finally $\dfrac{2x^2 - x + 2}{(x-2)(x+1)^2} = \dfrac{\frac{8}{9}}{(x-2)} + \dfrac{\frac{10}{9}}{(x+1)} - \dfrac{\frac{5}{3}}{(x+1)^2}$

$$= \dfrac{8}{9(x-2)} + \dfrac{10}{9(x+1)} - \dfrac{5}{3(x+1)^2}$$

Example 6

Expand $\dfrac{x^4 - x^2 + 1}{x^3(x^2+1)}$ in partial fractions.

Now we have both a repeated factor, x^3, and a quadratic which will not factorise, x^2+1, on the bottom line.

We try $\dfrac{x^4 - x^2 + 1}{x^3(x^2+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{Dx+E}{x^2+1}$

Multiply throughout by the LCM of all the fractions, i.e. $x^3(x^2+1)$. We get:

$$x^4 - x^2 + 1 = Ax^2(x^2+1) + Bx(x^2+1) + C(x^2+1) + (Dx+E)x^3$$

(8)

$$x^4 - x^2 + 1 = A x^2 (x^2 + 1) + B x (x^2 + 1) + C(x^2 + 1) + (D x + E) x^3$$

$$(8)$$

Use Method 1 in (8). Put $x=0$:

$$0 + 1 = A(0) + B(0) + C(0 + 1) + (D(0) + E)(0)$$

so $C=1$. There are no more obvious choices for Method 1

We have to use Method 2, i.e. expand the right hand side of (8):

$$x^4 - x^2 + 1 = A x^4 + Ax^2 + B x^3 + B x + C x^2 + C + D x^4 + Ex^3$$

Equate x^2 terms: $A + C = -1$ so $A = -1 - C = -2$

Equate x^4 terms: $A + D = 1$ so $D = 1 - A = 1 - (-2) = 3$

Equate x terms: $B = 0$

Equate x^3 terms: $B + E = 0$, so $E = 0$

summing up: $A = -2$, $B = 0$, $C = 1$, $D = 0$, $E = 0$ and

$$\frac{x^4 - x^2 + 1}{x^3 (x^2 + 1)} = \frac{-2}{x} + \frac{1}{x^3} + \frac{3 x}{x^2 + 1}$$

Example 7

So far all the partial fractions expansions have been of $\dfrac{p(x)}{q(x)}$ where $p(x)$ is a polynomial of lower order than $q(x)$.

If this is not the case we must do long division to make it the case before it is possible to do partial fractions.

Example: expand $\dfrac{2x^3+6x-2}{(x+1)(x+3)}$ in partial fractions.

First, note that $(x+1)(x+3)=x^2+4x+3$

Then do long division

$$x^2+4x+3\,\big)\,\overline{2x^3+\qquad\ \ 6x-2}$$

$$\underline{2x\ -8}$$

$$2x^3+8x^2+6x$$

$$\underline{}$$

$$-8x^2\qquad\ -2$$

$$\underline{-8x^2-32x-24}$$

$$32x+22$$

This means $\dfrac{2x^3+6x-2}{(x+1)(x+3)}=2x-8+\dfrac{32x+22}{(x+1)(x+3)}$

So now do partial fractions with $\dfrac{32x+22}{(x+1)(x+3)}$

i.e. put $\dfrac{32x+22}{(x+1)(x+3)}=\dfrac{A}{(x+1)}+\dfrac{B}{(x+3)}$

Multiplying by the LCM $(x+1)(x+3)$ gives
$$32x+22=A(x+3)+B(x+1) \quad (9)$$

Use Method 1:

Put $x=-1$. Then from (9): $32(-1)+22=A(-1+3)+B(0)$

so $2A=-10$ and $A=\dfrac{-10}{2}=-5$

Put $x=-3$. Then from (9): $32(-3)+22=A(0)+B(-3+1)$

so $-2B=-74$ and $B=\dfrac{-74}{-2}=\dfrac{74}{2}=37$

Finally $\dfrac{2x^3+6x-2}{(x+1)(x+3)}=2x-8-\dfrac{5}{(x+1)}+\dfrac{37}{(x+3)}$

Example 8

Example: expand $\dfrac{x^5}{(x-1)^2(x^2+x+1)}$ in partial fractions.

Multiply out the brackets $(x-1)^2(x^2+x+1)$

$$=(x-1)(x-1)(x^2+x+1)$$
$$=(x-1)(x^3+x^2+x-x^2-x-1)$$
$$=(x-1)(x^3-1)$$
$$=x^4-x^3-x+1$$

Then do long division

$$x^4-x^3-x+1 \overline{\smash{\big)}\ x^5} \qquad \overset{\displaystyle x+1}{}$$

$$\underline{x^5-x^4 \qquad -x^2+x}$$

$$x^4 \qquad +x^2-x$$

$$\underline{x^4-x^3 \qquad -x+1}$$

$$x^3+x^2 \qquad -1$$

We now have: $\dfrac{x^5}{(x-1)^2(x^2+x+1)}=x+1+\dfrac{x^3+x^2-1}{(x-1)^2(x^2+x+1)}$

Put: $\dfrac{x^3+x^2-1}{(x-1)^2(x^2+x+1)}=\dfrac{A}{(x-1)}+\dfrac{B}{(x-1)^2}+\dfrac{Cx+D}{(x^2+x+1)}$

Multiply throughout by the LCM, i.e. $(x-1)^2(x^2+x+1)$

We get: x^3+x^2-1

$= A(x-1)(x^2+x+1)+B(x^2+x+1)+(Cx+D)(x-1)^2$ (10)

Use Method 1 in (10). Put $x=1$:

So: $1^3+1^2-1=A(0)+B(1^2+1+1)+(Cx+D)(0)$

i.e. $3B=1$ and $B=\dfrac{1}{3}$

We must now use Method 2, i.e. expand the right hand side of (10):

x^3+x^2-1

$= A(x^3+x^2+x-x^2-x-1)+B(x^2+x+1)+(Cx+D)(x^2-2x+1)$

$= A(x^3-1)+B(x^2+x+1)+C(x^3-2x^2+x)+D(x^2-2x+1)$

Equate x^3 terms: $\qquad\qquad A+C=1$

Equate x^2 terms: $\qquad B-2C+D=1$

Equate x terms: $\qquad B+C-2D=0$

Substitute $B=\dfrac{1}{3}$ into the last two equations:

$$-2C+D=1-\frac{1}{3}=\frac{2}{3}$$

$$C-2D=-\frac{1}{3}$$

Multiply the last of these by 2 :

$$2C-4D=-\frac{2}{3}$$

Add to the first:

$$-3D=0 \text{ so } D=0 \text{ and hence } C=-\frac{1}{3}$$

Then: $A+(-\dfrac{1}{3})=1$ so $A=\dfrac{4}{3}$

Summarising: $\quad A = \dfrac{4}{3}, B = \dfrac{1}{3}, C = -\dfrac{1}{3}, D = 0$

Recall: $\dfrac{x^5}{(x-1)^2(x^2+x+1)} = x+1+\dfrac{x^3+x^2-1}{(x-1)^2(x^2+x+1)}$

and: $\dfrac{x^3+x^2-1}{(x-1)^2(x^2+x+1)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x-1)^2} + \dfrac{Cx+D}{(x^2+x+1)}$

which we have shown $= \dfrac{\dfrac{4}{3}}{(x-1)} + \dfrac{\dfrac{1}{3}}{(x-1)^2} - \dfrac{\dfrac{1}{3}x}{(x^2+x+1)}$

Finally $\dfrac{x^5}{(x-1)^2(x^2+x+1)} = x+1+\dfrac{\dfrac{4}{3}}{(x-1)} + \dfrac{\dfrac{1}{3}}{(x-1)^2} - \dfrac{\dfrac{1}{3}x}{(x^2+x+1)}$