

NFM2106/NFE2105

Mathematics

Discrete Random Variables

Random Variables

experiment

sample space: Ω

events: A, B, \dots

 $P(A) = \frac{n(A)}{n(\Omega)}$

probability of A

Random Variables

In general, a R.V. is a **function** from Ω to the real numbers.

Since the outcomes of the experiment for which Ω is the **sample space** are random, each **number** produced by the function is random as well.

Discrete random variable = a R.V. that can take on only a finite or at most countably infinite number of values.

Random Variables

The rate of unemployment in the UK is approximately 4%

Suppose we randomly choose 2 UK adults.

Let X be the number of adults in our sample that are unemployed.

**possible
outcomes:**

EE

EU

UE

UU

probabilities:

$$0.96 \times 0.96$$

$$0.96 \times 0.04$$

$$0.04 \times 0.96$$

$$0.04 \times 0.04$$

$$0.9216$$

$$0.0384$$

$$0.0384$$

$$0.0016$$

Random Variables

Values of X:	0	1	2
Probabilities:	0.9216	0.0768	0.0016

OBS. 1). What we have above represents the **probability distribution** for the particular random variable X considered on the next slide.

2). We have a listing of all possible values together with their probabilities of occurring.

Notation: X = name of the random variable

x =value of the R.V.

$p(X = x)$ \rightarrow abbreviated $p(x)$

RV (definitions)

A **probability distribution** for a random discrete variable X is a listing of all the possible values of X and their probabilities of occurring.

$$X = \{x_1, x_2, x_3, \dots, x_n\}$$

$$p_i = P(X = x_i)$$

represent the *frequencies* at which the events $\{X = x_i\}$ occur

probability
distribution
function

$$p(x) = \begin{cases} p_i & \text{if } x = x_i \\ 0 & \text{otherwise} \end{cases}$$

PROPERTIES:

$$(I) \quad p_i \geq 0$$

$$(II) \quad \sum_{i=1}^n p_i = 1$$

Examples

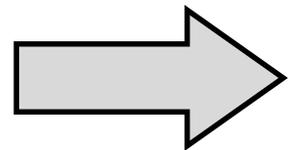
A coin is thrown 3 times, and the sequence of Heads (H) and Tails (T) is observed.

$$\Omega = \{HHH, HHT, HTT, HTH, TTT, TTH, THH, THT\} \equiv \{\omega_1, \omega_2, \dots, \omega_8\}$$

X = total # of H's

$$\begin{array}{cccc} X(\omega_1) = 3 & X(\omega_3) = 1 & X(\omega_5) = 0 & X(\omega_7) = 2 \\ X(\omega_2) = 2 & X(\omega_4) = 2 & X(\omega_6) = 1 & X(\omega_8) = 1 \end{array}$$

Thus, this R.V. can take on only the values: 0, 1, 2, 3



Probability distribution (frequency function)

$$P(X = 0) = \frac{1}{8}$$

$$P(X = 1) = \frac{3}{8}$$

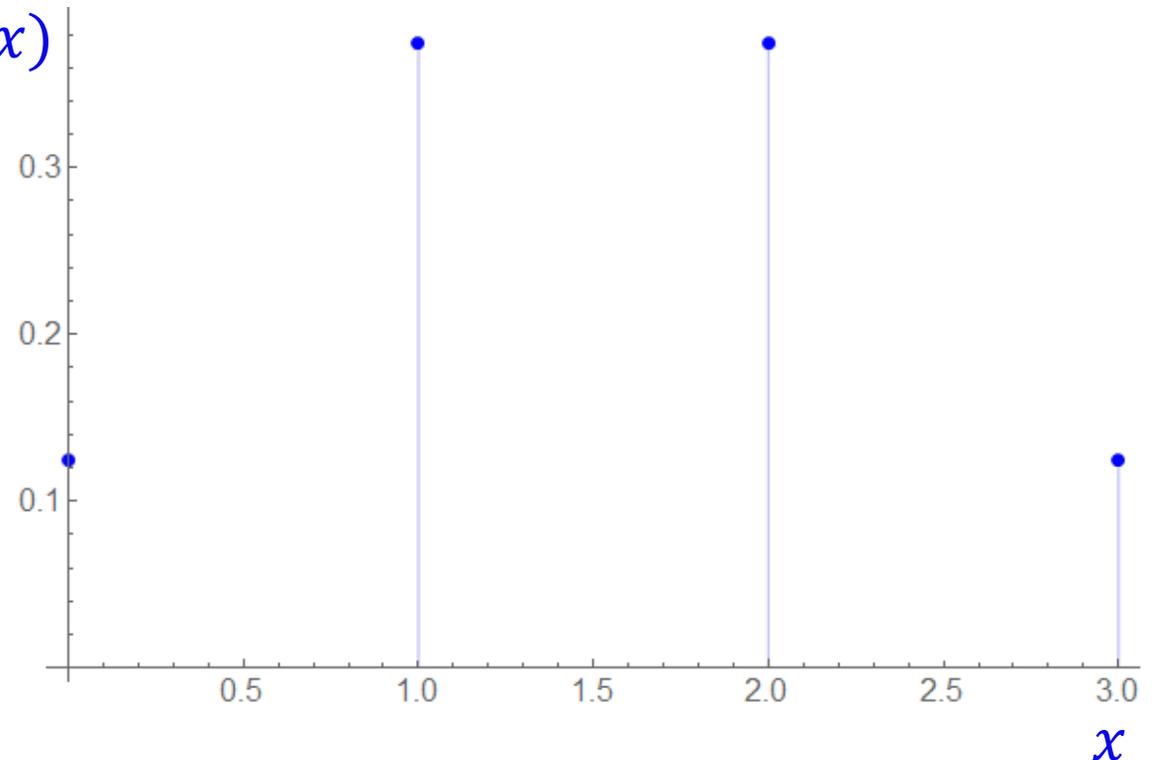
$$P(X = 2) = \frac{3}{8}$$

$$P(X = 3) = \frac{1}{8}$$

the **probability distribution**
for the number of **Heads**
in 3 coin flips



$P(X = x)$



Another discrete probability distribution

Assume X to be the same R.V., but change the experiment to 5 coin flips.

$$P(X = 0) = \frac{1}{32}$$

$$P(X = 1) = \frac{5}{32}$$

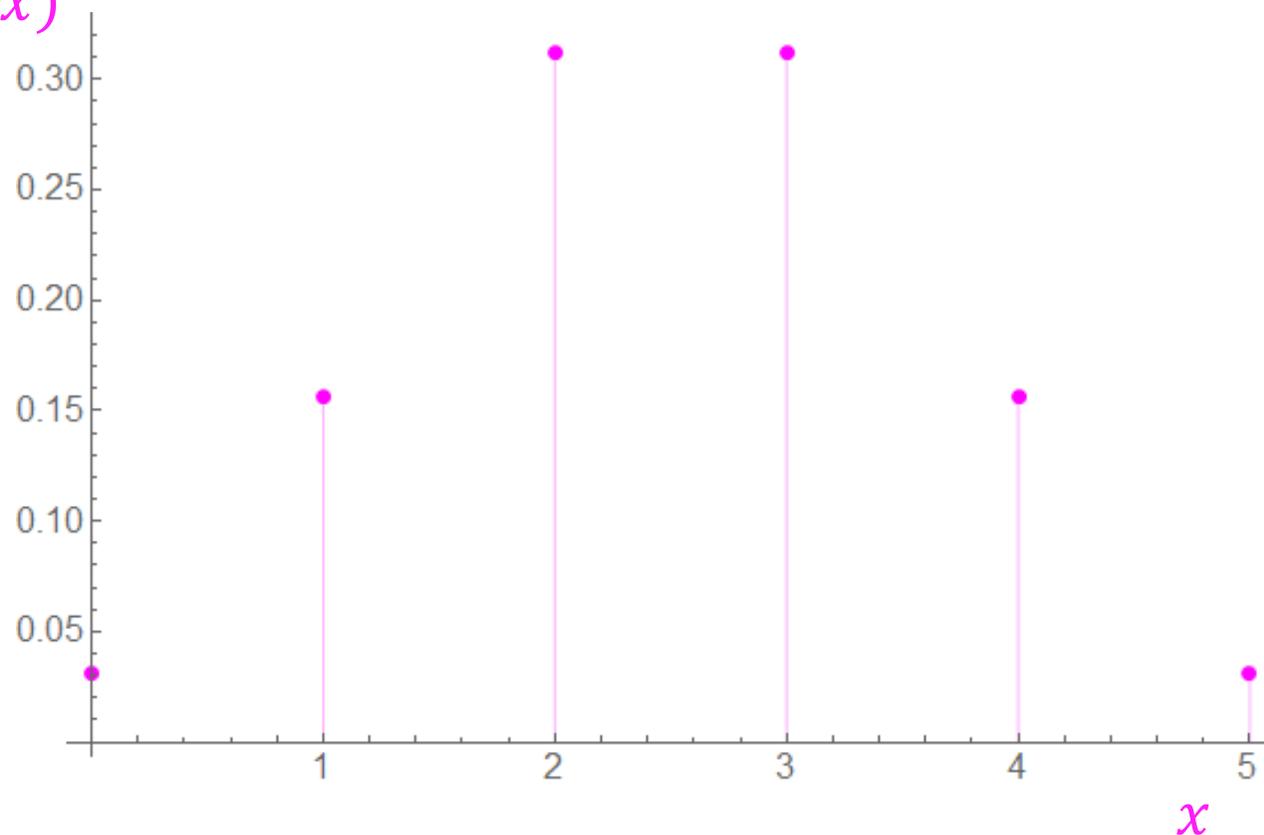
$$P(X = 2) = \frac{10}{32}$$

$$P(X = 3) = \frac{10}{32}$$

$$P(X = 4) = \frac{5}{32}$$

$$P(X = 5) = \frac{1}{32}$$

$$P(X = x)$$



Two dice

The possible outcomes of throwing two dice give the following:

$$\Omega = \{(1, 1)(1, 2)(1, 3)(1, 4)(1, 5)(1, 6) \\ (2, 1)(2, 2)(2, 3)(2, 4)(2, 5)(2, 6) \\ (3, 1)(3, 2)(3, 3)(3, 4)(3, 5)(3, 6) \\ (4, 1)(4, 2)(4, 3)(4, 4)(4, 5)(4, 6) \\ (5, 1)(5, 2)(5, 3)(5, 4)(5, 5)(5, 6) \\ (6, 1)(6, 2)(6, 3)(6, 4)(6, 5)(6, 6)\}$$



36 possible
outcomes

Consider the R.V.:

X = the sum of the two numbers

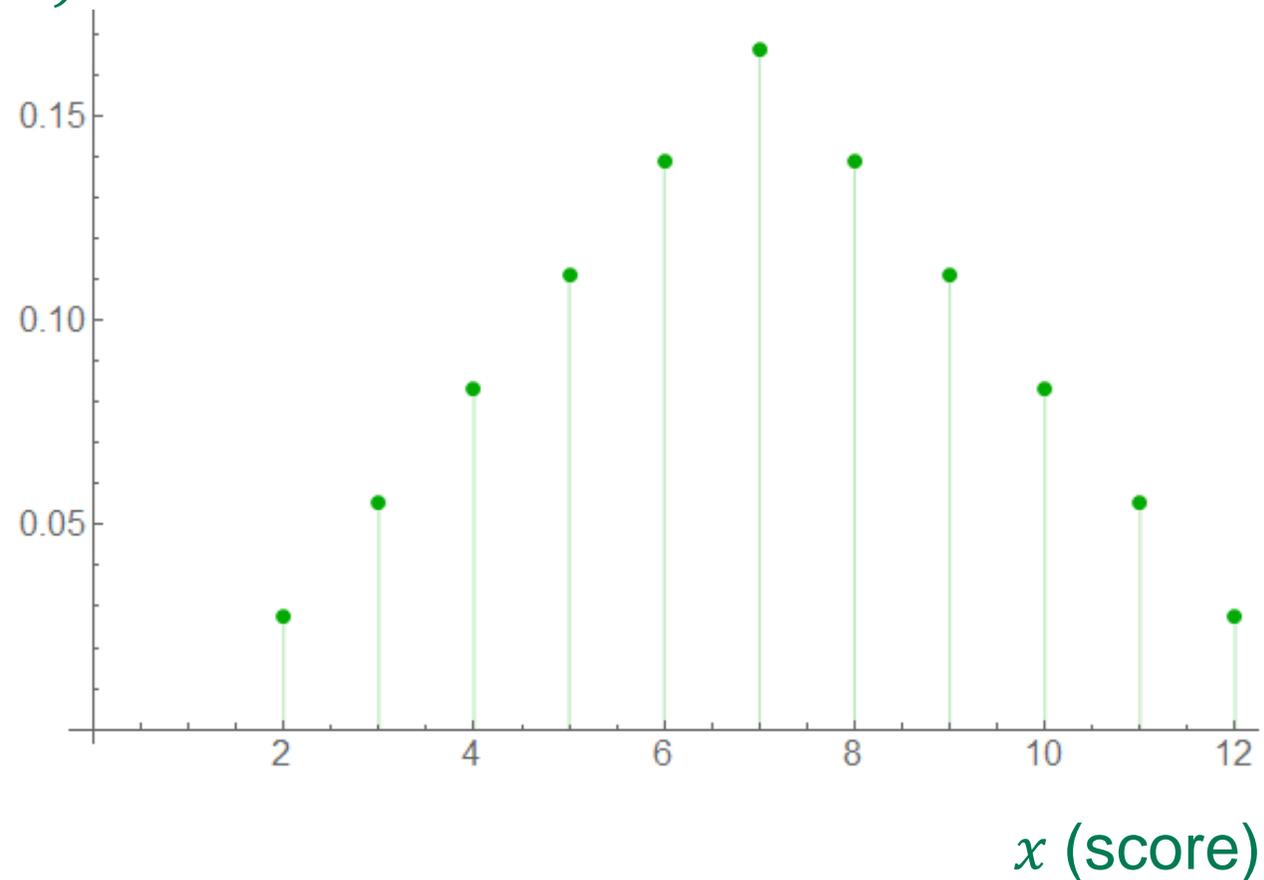
Two dice (probability distribution for X)

 $P(X = x)$

$$P(X = 7) = \frac{6}{36} = \frac{1}{6}$$

$$P(X = 11) = \frac{2}{36} = \frac{1}{18}$$

ETC.



Cumulative Distribution Function (CDF)

$$F(x) = P(X \leq x)$$

Properties:

$$0 \leq F(x) \leq 1 \quad -\infty < x < \infty$$

$$F(-\infty) = 0 \quad F(\infty) = 1$$

$$P(x_1 < X \leq x_2) = F(x_2) - F(x_1)$$

$$P(X > x_1) = 1 - F(x_1)$$

CDF for discrete RVs

$$X = \{x_1, x_2, x_3, \dots, x_n\}$$

$$p_i = P(X = x_i)$$

$$F(x) = \sum_{x_i \leq x} p_i$$

Example

x	1	2	4	6
p	0.2	0.5	0.1	0.2

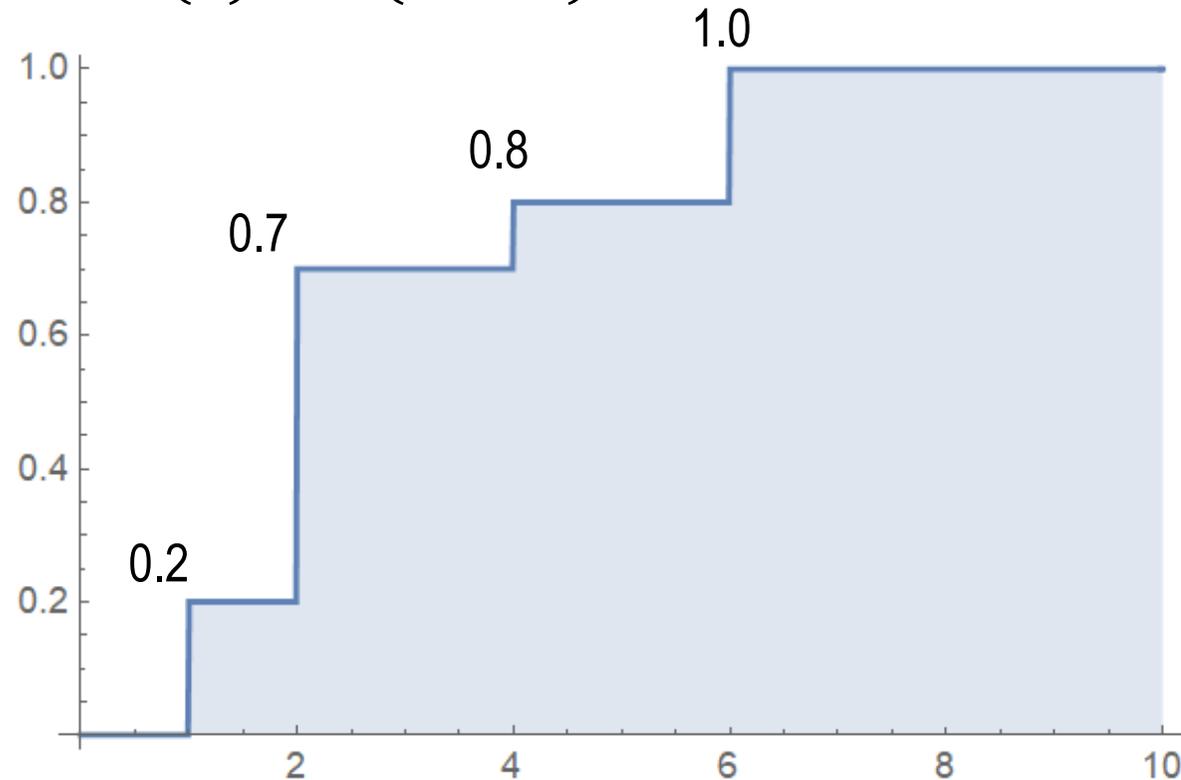
Find

$$P(X \leq 4.5)$$

$$P(X > 4.5)$$

$$P(1.5 < X \leq 4.5)$$

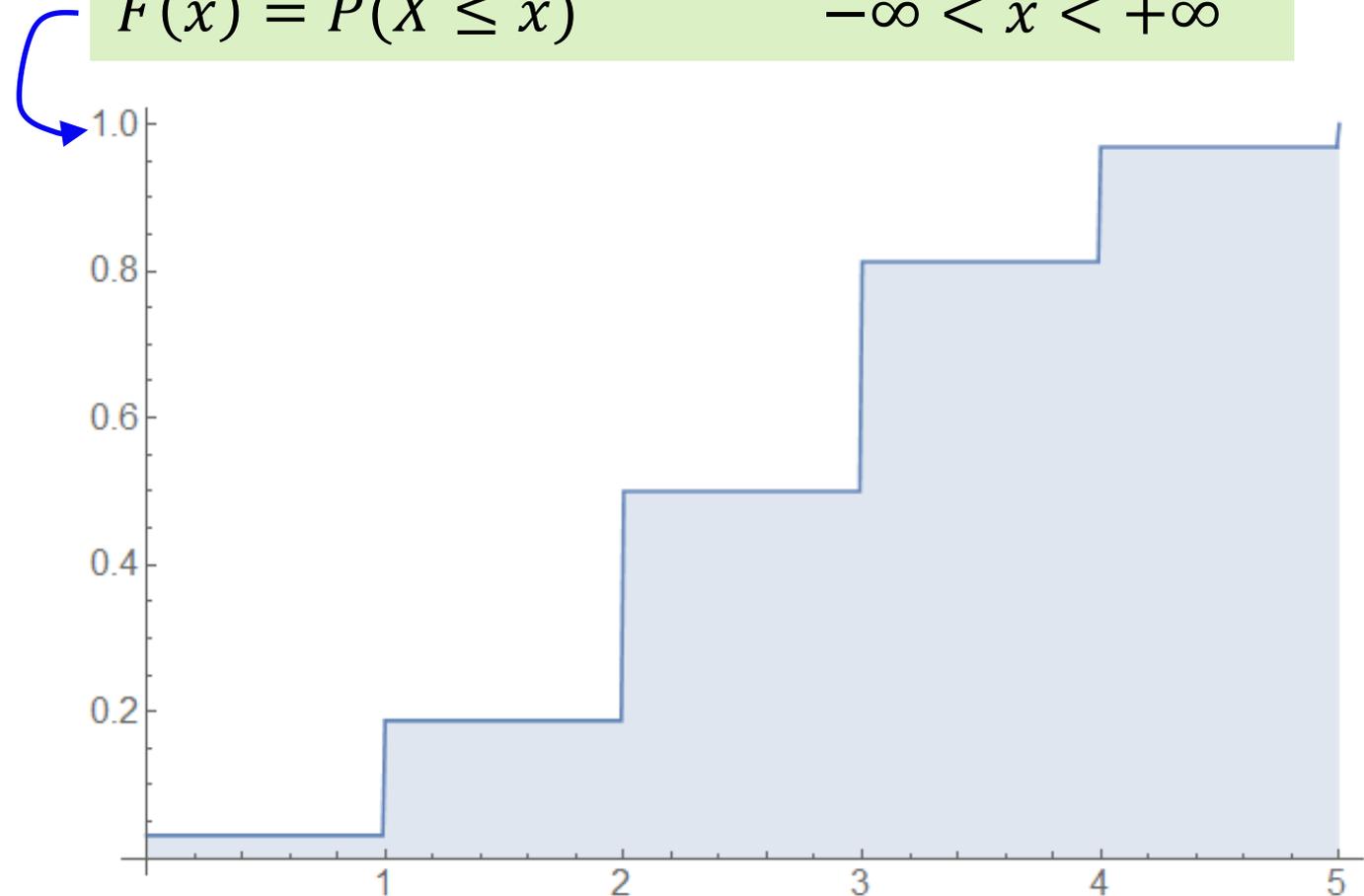
$$F(x) = P(X \leq x)$$



Another example of CDF

For a random variable X , this is defined by

$$F(x) = P(X \leq x) \quad -\infty < x < +\infty$$



5 coin flips
 $X = \#$ of heads (H)

Poisson distribution (discrete)

X = # of events that occur in time T at a rate λ

(i.e. there are, on average, λT events)

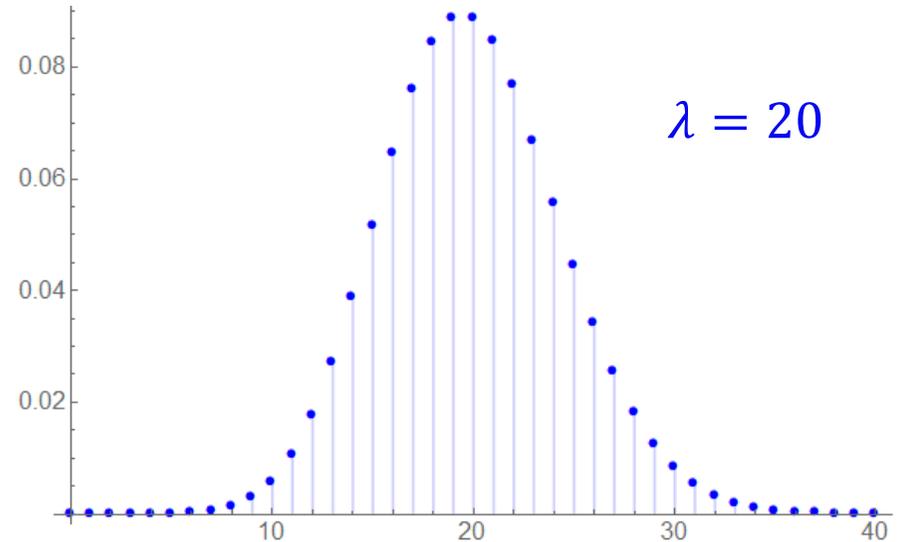
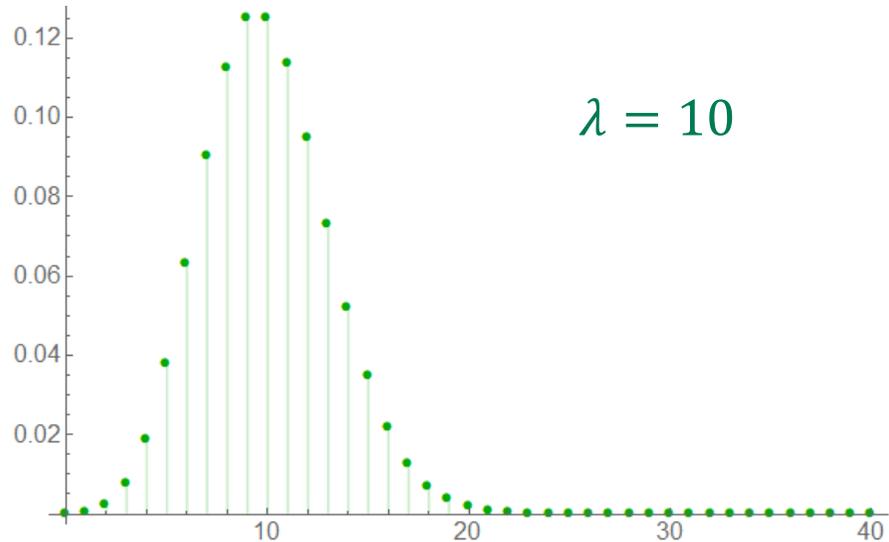
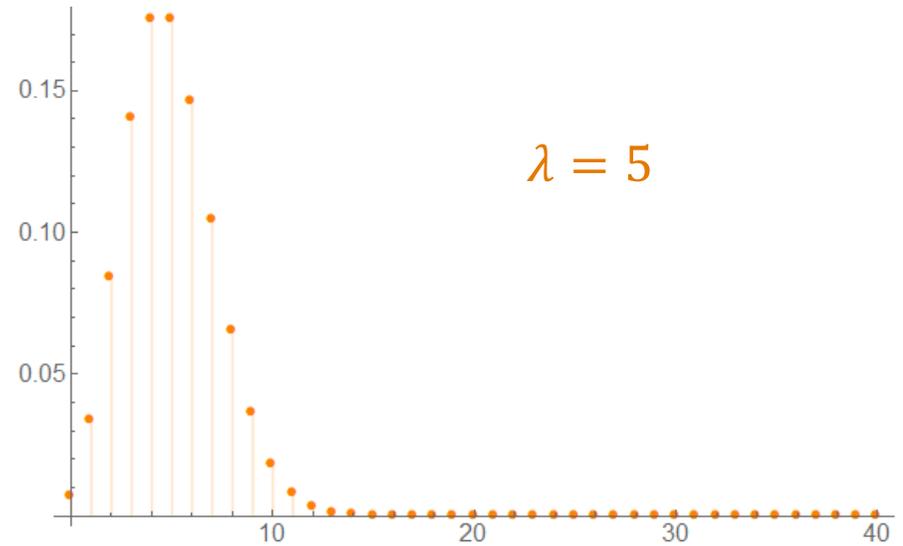
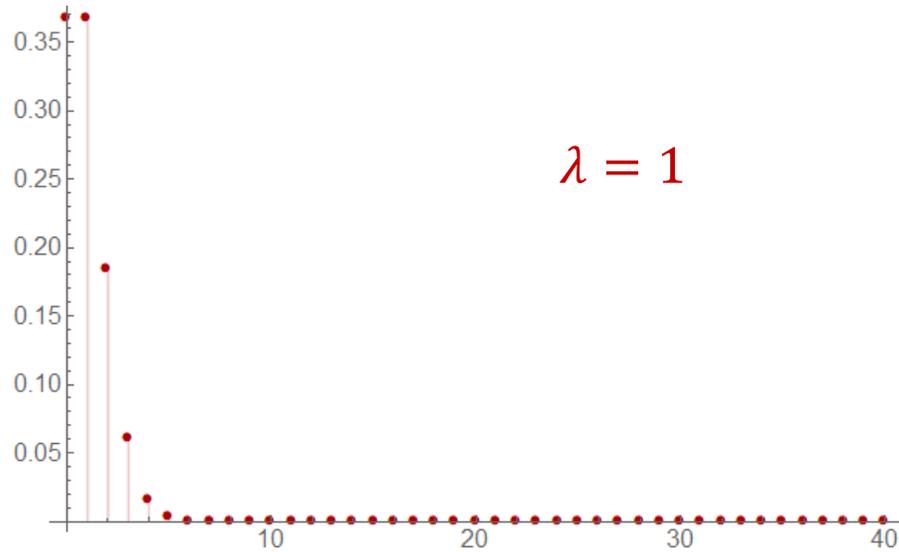
- a). the events are independent of each other
- b). the events cannot occur simultaneously

$$P(X = n) = \frac{\lambda^n}{n!} \exp(-\lambda)$$

OBS.:

- ✗ in this formula n is a positive integer
- ✗ λ is a quantity that must be worked out based on the context of the question we are trying to answer

Poisson distributions



Poisson distribution (example)

3 buses arrive at a stop every 10 minutes. What is the probability for:

- No buses arriving in a ten-minute period?
- Up to two buses arriving within a ten-minute period?
- More than three arriving?

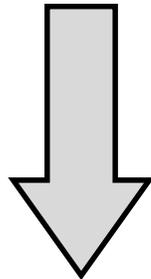
$$P(X = 0) = \frac{\lambda^0}{0!} e^{-\lambda} \text{ where } \lambda = 3/10 \text{ minutes}$$
$$= e^{-3} = 0.05.$$

$$P(\text{up to } 2) = P(0) + P(1) + P(2)$$
$$= e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2 e^{-\lambda}}{2} = e^{-\lambda} \left(1 + \lambda + \frac{\lambda^2}{2} \right) = 0.4232$$

Poisson distribution (example)

$$\begin{aligned} P(\text{more than } 3) &= (P(4) + P(5) + \dots) \\ &= 1 - \underbrace{(P(0) + P(1) + P(2) + P(3))}_{\text{already calculated}} \end{aligned}$$

$$P(3) = \frac{\lambda^3}{3!} e^{-\lambda} = 0.224$$



$$P(\text{more than } 3) = 1 - (0.4232 + 0.224) = 0.3528$$

Poisson distribution (another example)

On average, one shopper enters a given store every 15 seconds. What is the probability that in a given interval of one minute, **zero shoppers** enter the store? **Four shoppers?** **Eight shoppers?**

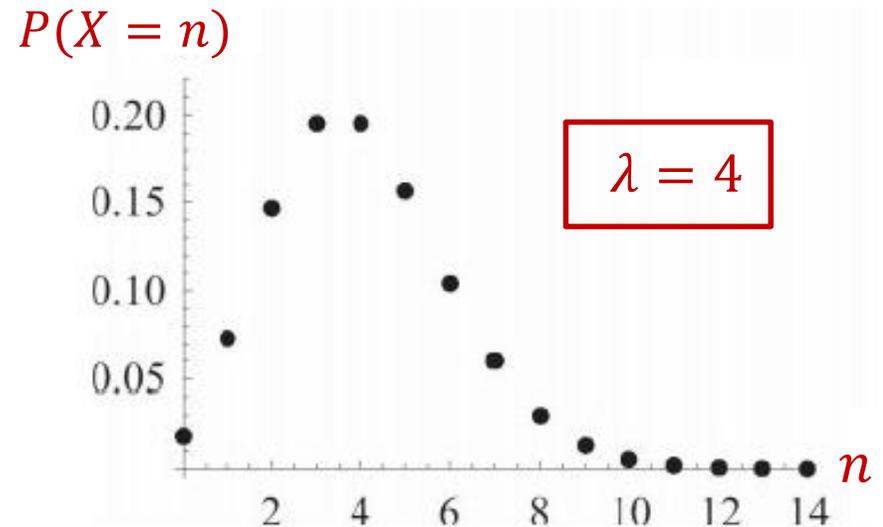
$\lambda = 4$ (shoppers per minute)

$X =$ # shoppers within one minute

$$P(X = 0) = \frac{4^0 e^{-4}}{0!} = e^{-4} \approx 0.018 \approx 2\%$$

$$P(X = 4) = \frac{4^4 e^{-4}}{4!} = \frac{32}{3} \cdot e^{-4} \approx 0.195 \approx 20\%$$

$$P(X = 8) = \frac{4^8 e^{-4}}{8!} = \frac{512}{315} \cdot e^{-4} \approx 0.030 \approx 3\%$$



Expectation value

Let X be a random variable, i.e.

$$X = \{x_1, x_2, x_3, \dots, x_n\}, \quad p_i = P(X = x_i)$$

The **expectation value** (or simply the **mean**) of this R.V. is

$$\mu_X = E(X) = \sum_{i=1}^n p_i x_i$$

For a real function $f: \mathbb{R} \rightarrow \mathbb{R}$ the expectation value of $Y = f(X)$ is given by

$$E(Y) = \sum_{i=1}^n p_i f(x_i)$$

Variance & standard deviation

The **variance** gives the degree of spread of the distribution about the mean

$$\text{Var}(X) = E[(X - \mu)^2] \quad (\mu = E(X))$$

This can be written as

$$\text{Var}(X) = \sum_{i=1}^n p_i (x_i - \mu)^2$$

Standard deviation:

$$\sigma_X = \sqrt{\text{Var}(X)} \implies \sigma_X^2 = \text{Var}(X)$$

The relationship between σ_X and μ_X

$$\begin{aligned}\underline{\text{Var}(X)} &= \sum_{i=1}^n p_i (x_i^2 - 2x_i\mu + \mu^2) \\ &= \sum_{i=1}^n p_i x_i^2 - 2\mu \sum_{i=1}^n p_i x_i + \mu^2 \sum_{i=1}^n p_i \\ &= E(X^2) - 2\mu^2 + \mu^2 = \underline{E(X^2) - \mu^2} \implies \sigma^2 = E(X^2) - \mu^2\end{aligned}$$

This can be written in a symmetric form that is easy to remember:

$$\sigma^2 = E(X^2) - (E(X))^2$$

OBS.: The standard deviation is σ , **not** σ^2 !!

Example: rolling a single die

Here the possible outcomes or events are rolling one of the numbers $\{1, 2, 3, 4, 5, 6\}$, which all occur with probability $1/6$.

$$\mu = \sum_{i=1}^6 p_i x_i = \frac{1}{6} (1 + 2 + 3 + 4 + 5 + 6) = \frac{7}{2} = 3.5$$

$$\begin{aligned} \sigma^2 &= E(X^2) - \mu^2 = \sum_{i=1}^6 p_i x_i^2 - \mu^2 = \frac{1}{6} (1^2 + 2^2 + \dots + 6^2) - \mu^2 \\ &= \frac{91}{6} - \frac{49}{4} = \frac{35}{12} = 2 \frac{11}{12} \implies \sigma = \left(\frac{35}{12}\right)^{1/2} = 1.7078 \dots \end{aligned}$$

The **standard deviation** < **mean**, this means the distribution is “narrow” and is not describing wild fluctuations

Examples

