



University of
HUDDERSFIELD

Introduction to Matrices & Determinants

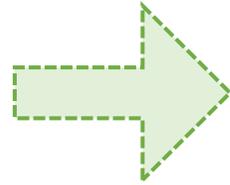
Ciprian D. Coman

OUTLINE

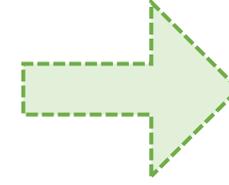
- ❖ Motivation, historical facts, etc
- ❖ Definition of the concept of **matrix**
- ❖ Operations with matrices
(addition, subtraction, scalar multiplication, **matrix multiplication**)
- ❖ **Determinants**

MOTIVATION

$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y = -4 \\ 2x - 5y + 5z = 17 \end{cases}$$



$$AX = b$$



$$X = A^{-1}b$$

A, X, b
new
mathematical
objects

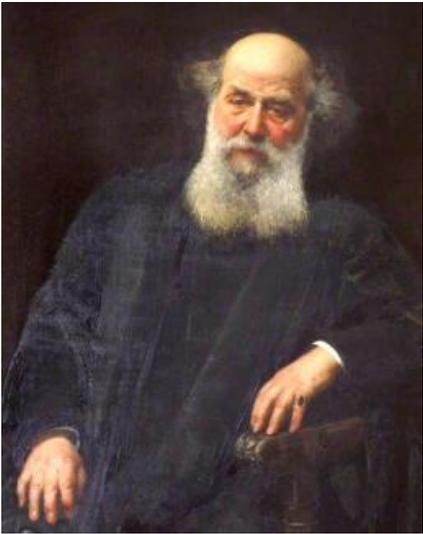
ASIDE:

$$5x = 2 \quad \Rightarrow \quad x = \frac{2}{5} = \frac{1}{5} \times 2 = 5^{-1} \times 2 = (5^{-1})(2)$$

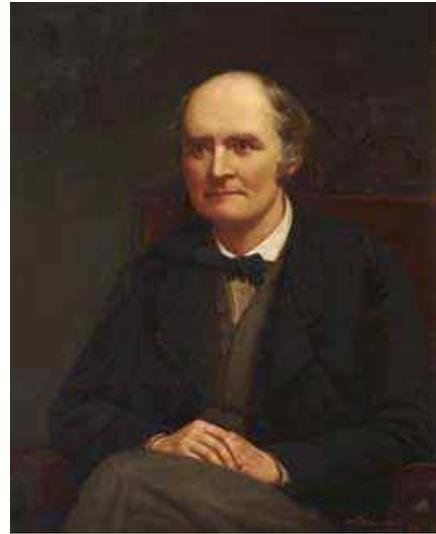
We need to introduce new mathematical objects → **MATRICES**

Historical information

- ❑ First: the Chinese (Han Dynasty, **200-100 B.C.**)
- ❑ “Official” birth date: around **1850**



J.J. Sylvester (1814-1897)



A. Cayley (1821-1895)

There are three types of corn, of which three bundles of the first, two of the second, and one of the third make 39 measures. Two of the first, three of the second and one of the third make 34 measures. And one of the first, two of the second and three of the third make 26 measures. How many measures of corn are contained of one bundle of each type?



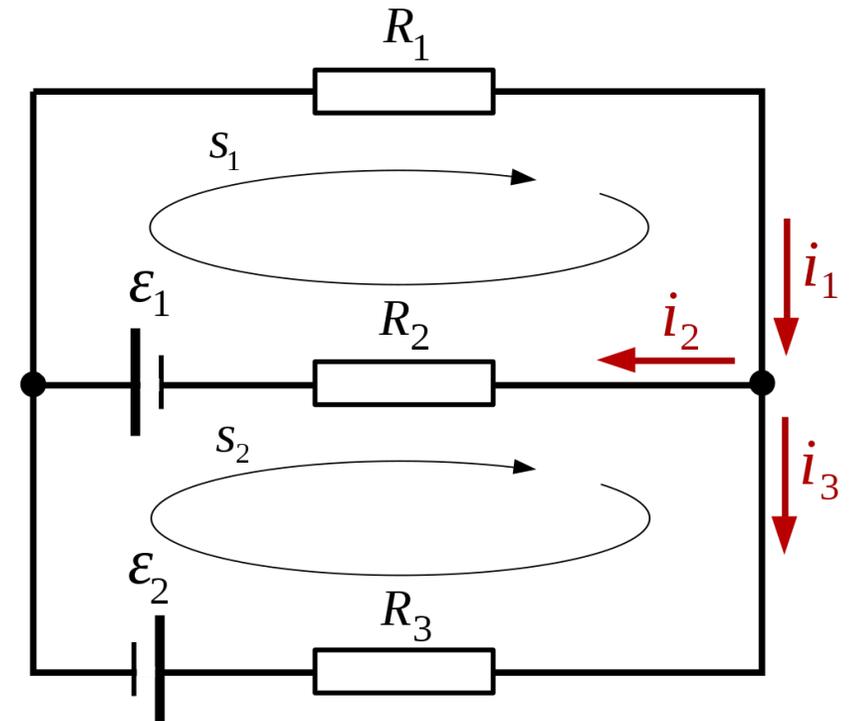
Applications (electrical engineering)

$$R_1 = 100\Omega, \quad R_2 = 200\Omega, \quad R_3 = 300\Omega,$$
$$\mathcal{E}_1 = 3V, \quad \mathcal{E}_2 = 4V$$

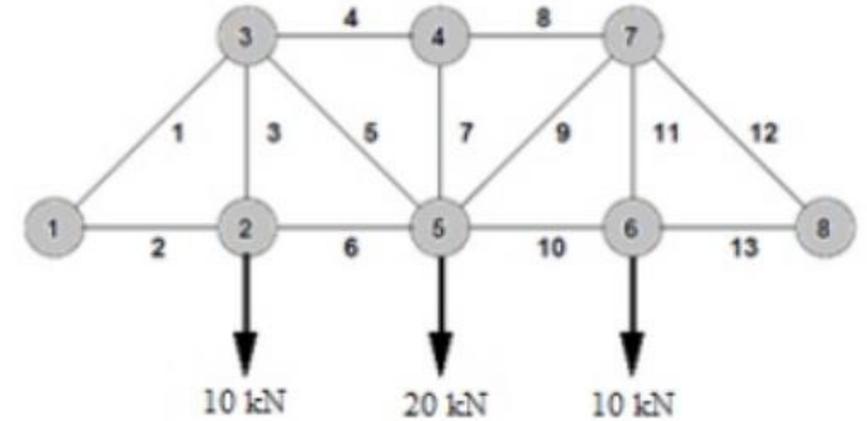
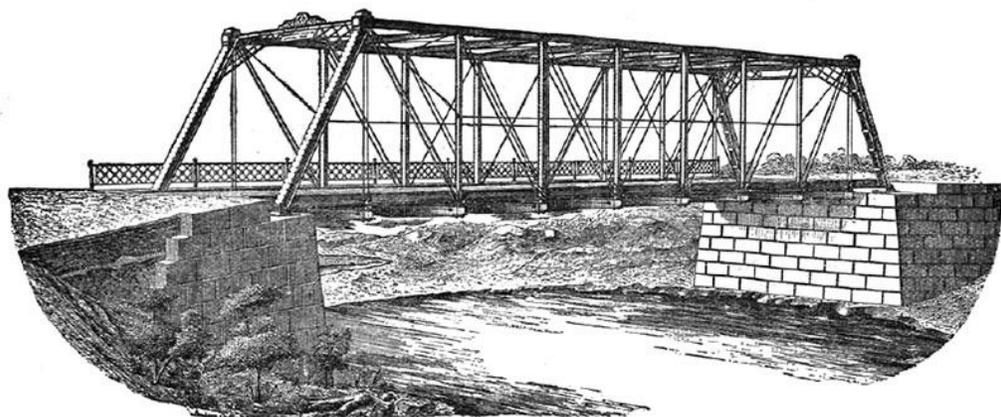
Kirchhoff's first + second laws
Ohm's law

$$\begin{cases} i_1 + (-i_2) + (-i_3) & = 0 \\ R_1 i_1 + R_2 i_2 + 0i_3 & = \mathcal{E}_1 \\ 0i_1 + R_2 i_2 - R_3 i_3 & = \mathcal{E}_1 + \mathcal{E}_2 \end{cases}$$

(3 simultaneous equations in 3 unknowns)



Applications (mechanical engineering)

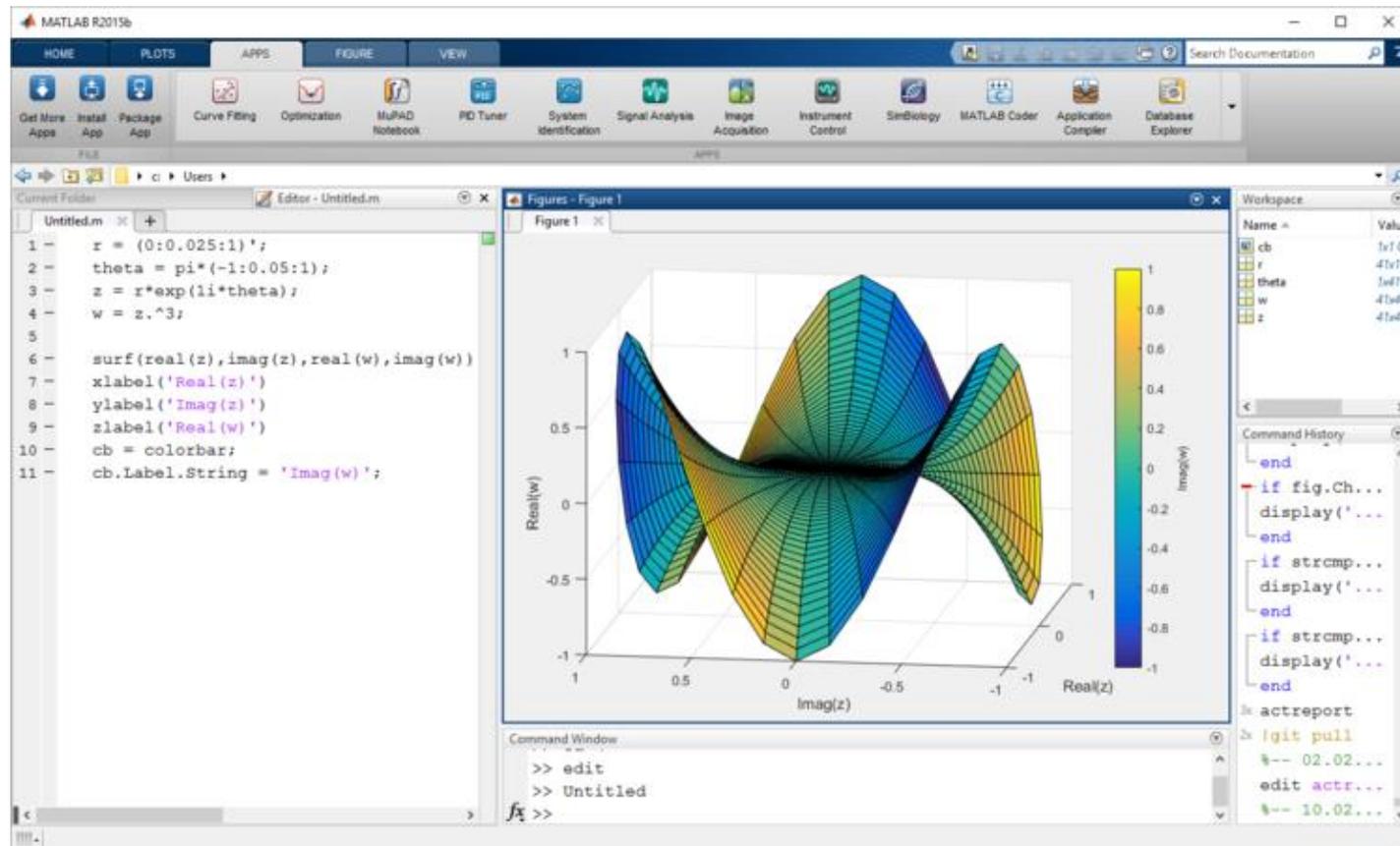


Joint 2:	$-f_2 + f_6 = 0$	$f_3 - 10 = 0$
Joint 3:	$-\alpha f_1 + f_4 + \alpha f_5 = 0$	$-\alpha f_1 - f_3 - \alpha f_5 = 0$
Joint 4:	$-f_4 + f_8 = 0$	$f_7 = 0$
Joint 5:	$-\alpha f_5 - f_6 + \alpha f_9 + f_{10} = 0$	$\alpha f_5 + f_7 + \alpha f_9 - 15 = 0$
Joint 6:	$-f_{10} + f_{13} = 0$	$f_{11} - 10 = 0$
Joint 7:	$-f_8 - \alpha f_9 + \alpha f_{12} = 0$	$-\alpha f_9 - f_{11} - \alpha f_{12} = 0$
Joint 8:	$-f_{13} - \alpha f_{12} = 0$	



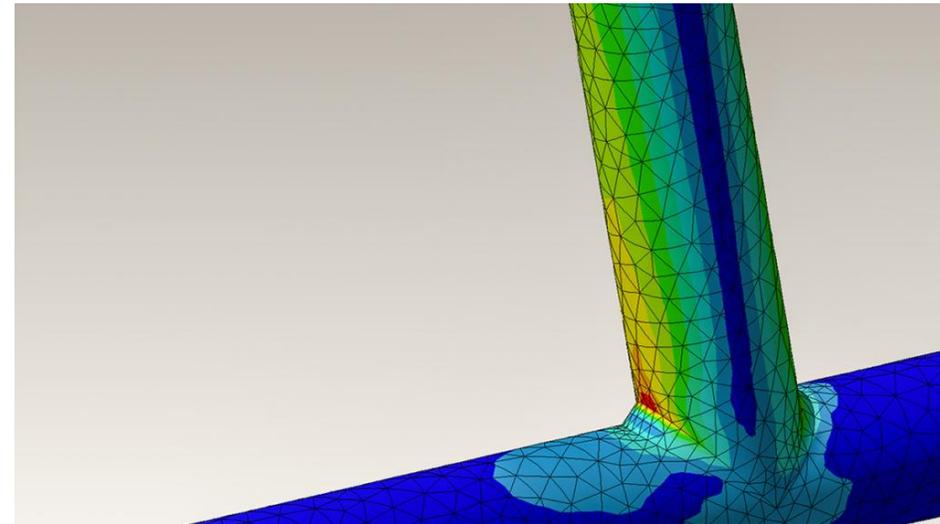
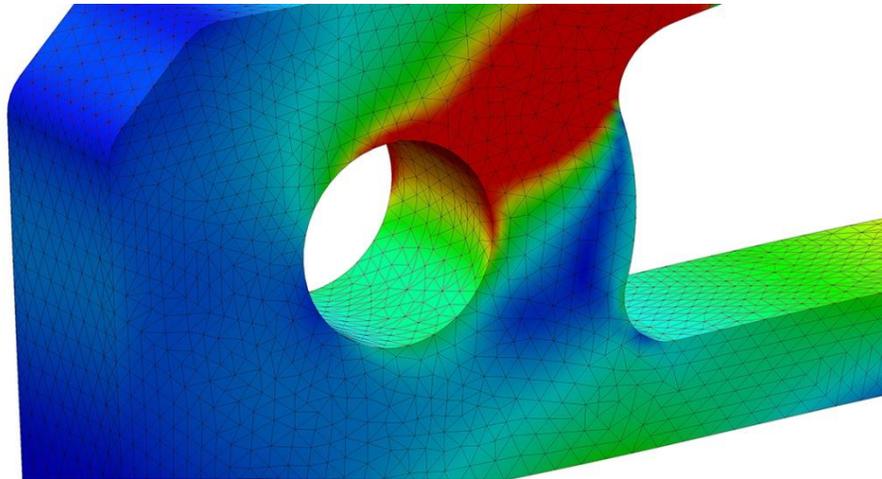
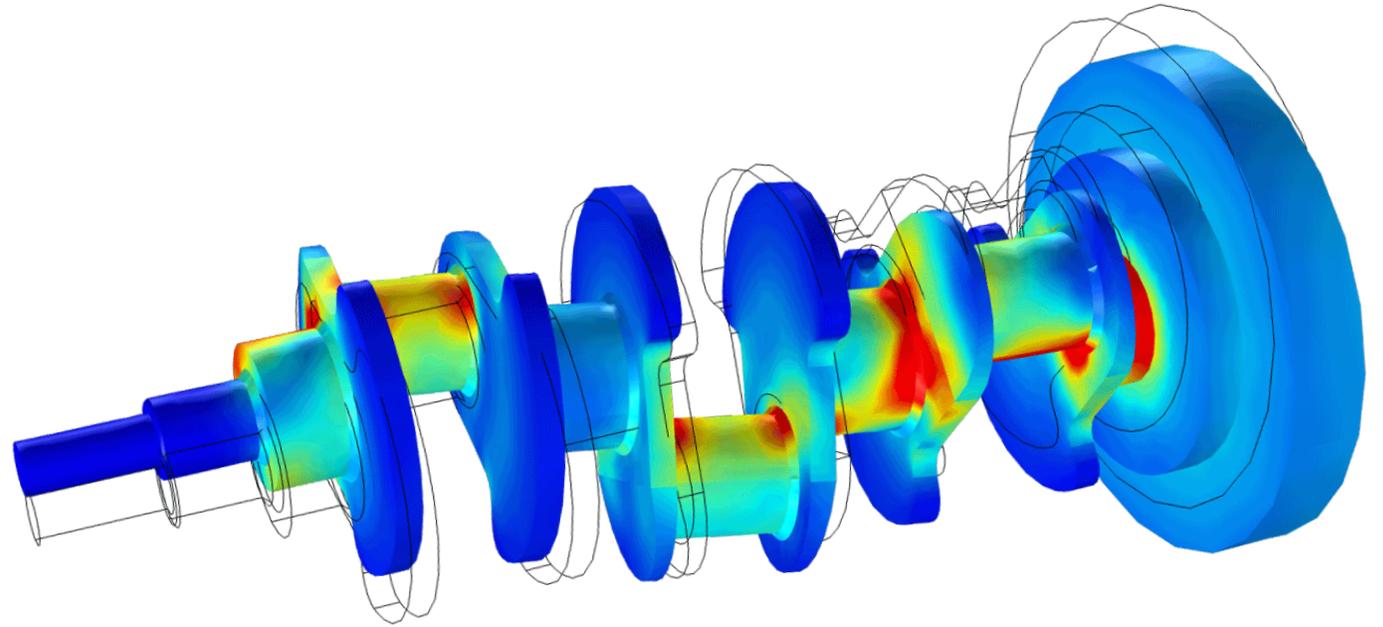
MATrix LABoratory

(computer package/programming language)



Finite Element Analysis (FEA)

- ❑ modern computational method
- ❑ boils down to solving a **large linear system** of simultaneous equations



Matrix → **RECTANGULAR array** of numbers

$$\begin{bmatrix} 1 & 4 \\ 3 & 7 \end{bmatrix}$$

2×2

$$\begin{bmatrix} 0 & 23 \\ 4 & 7 \\ 11 & -1 \end{bmatrix}$$

3×2

$$\begin{bmatrix} 6 & 2 & -4 \\ 9 & 2 & 13 \\ -14 & 5 & 1 \end{bmatrix}$$

3×3

$$\begin{bmatrix} 2 \\ 6 \\ 12 \end{bmatrix}$$

3×1

$$[0 \quad 0 \quad 8]$$

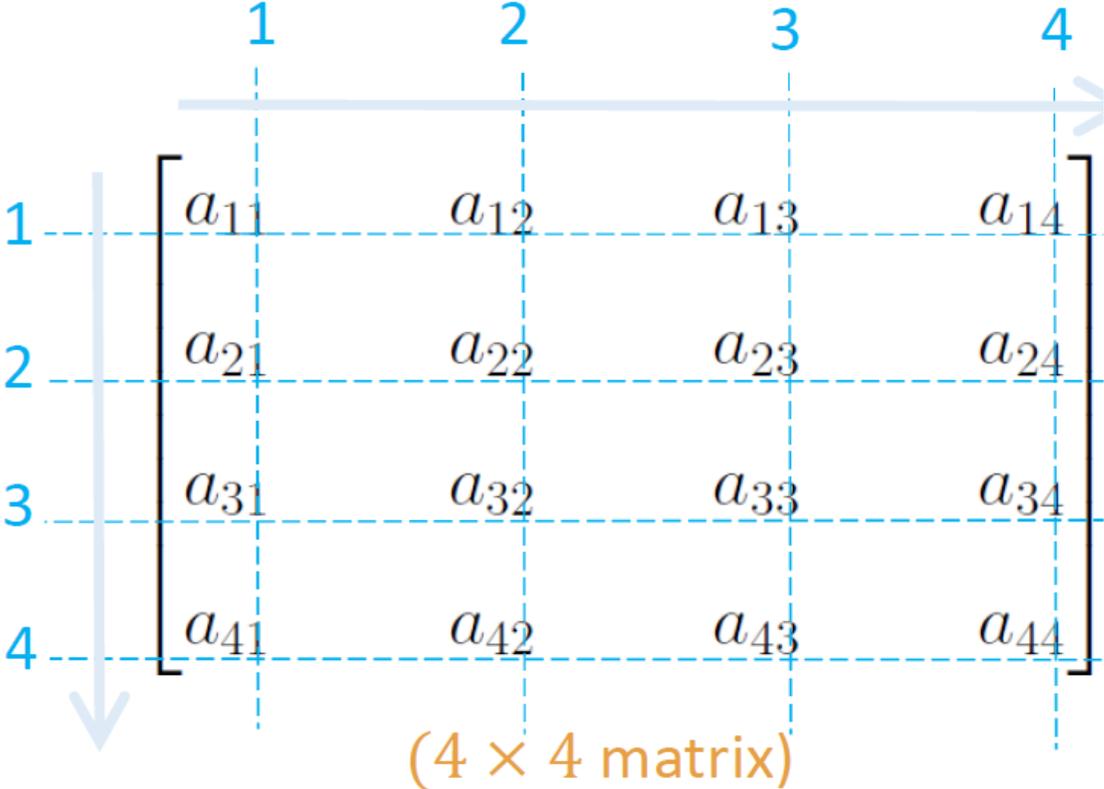
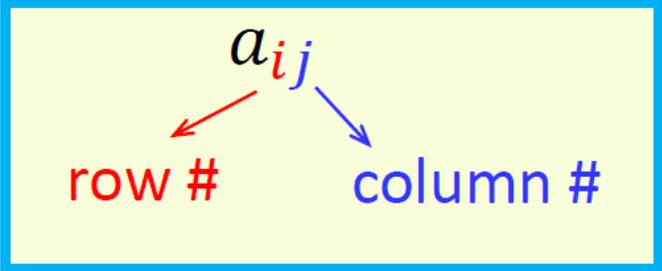
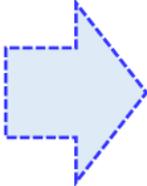
1×3

$$\begin{bmatrix} 9 & 0.23 \\ 19.5 & 0 \\ 99 & -2.67 \end{bmatrix}$$

3×2

NOTATION:

general notation



MATRIX ADDITION:

You can **add** two matrices (of the same size) by adding their corresponding entries.

$$(a) \underbrace{\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}}_{2 \times 2} + \underbrace{\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}}_{2 \times 2} = \begin{bmatrix} -1 + 1 & 2 + 3 \\ 0 - 1 & 1 + 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}}_{2 \times 2}$$

$$(b) \underbrace{\begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix}}_{2 \times 3} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{2 \times 3} = \underbrace{\begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix}}_{2 \times 3}$$

Scalar Multiplication and Matrix Subtraction

For the matrices

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

find (a) $3A$, (b) $-B$, and (c) $3A - B$.

$$(a) \quad 3A = 3 \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(2) & 3(4) \\ 3(-3) & 3(0) & 3(-1) \\ 3(2) & 3(1) & 3(2) \end{bmatrix} = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix}$$

$$(b) \quad -B = (-1) \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ -1 & 4 & -3 \\ 1 & -3 & -2 \end{bmatrix}$$

$$(c) \quad 3A - B = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 12 \\ -10 & 4 & -6 \\ 7 & 0 & 4 \end{bmatrix}$$

Solving a Matrix Equation

Solve for X in the equation $3X + A = B$, where

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix}.$$

SOLUTION

$$3X = B - A \quad \rightarrow \quad X = \frac{1}{3}(B - A)$$

$$X = \frac{1}{3} \left(\begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \right) = \frac{1}{3} \begin{bmatrix} -4 & 6 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} & 2 \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix}.$$

YOUR TURN!

The matrix X that solves the equation $-X + 3A = 3X - B$, where

(6 minutes)

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 12 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 6 & -1 \\ 4 & 0 \end{bmatrix},$$

is:

a). $\begin{bmatrix} 0 & 14 \\ -1 & 6 \end{bmatrix}$

b). $\begin{bmatrix} 3 & 9 \\ 1 & 27 \end{bmatrix}$

c). $\begin{bmatrix} 3 & 2 \\ 1 & 9 \end{bmatrix}$

d). $\begin{bmatrix} -1 & 9 \\ 7 & 28 \end{bmatrix}$



Code: **COMAN862**

Solution

The matrix X that solves the equation $-X + 3A = 3X - B$, where

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 12 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 6 & -1 \\ 4 & 0 \end{bmatrix},$$

is given by:

a). $\begin{bmatrix} 0 & 14 \\ -1 & 6 \end{bmatrix}$

b). $\begin{bmatrix} 3 & 9 \\ 1 & 27 \end{bmatrix}$

c). $\begin{bmatrix} 3 & 2 \\ 1 & 9 \end{bmatrix}$

d). $\begin{bmatrix} -1 & 9 \\ 7 & 28 \end{bmatrix}$

$$-X + 3A = 3X - B \implies 3A + B = X + 3X \implies 3A + B = 4X \implies X = \frac{1}{4}(3A + B)$$

$$X = \frac{1}{4} \left(3 \begin{bmatrix} 2 & 3 \\ 0 & 12 \end{bmatrix} + \begin{bmatrix} 6 & -1 \\ 4 & 0 \end{bmatrix} \right) = \frac{1}{4} \left(\begin{bmatrix} 6 & 9 \\ 0 & 36 \end{bmatrix} + \begin{bmatrix} 6 & -1 \\ 4 & 0 \end{bmatrix} \right) = \frac{1}{4} \begin{bmatrix} 12 & 8 \\ 4 & 36 \end{bmatrix} = \dots$$

THE TRANSPOSE OF A MATRIX:

The **transpose** of a matrix is formed by writing its rows as columns:

$$R_1 \leftrightarrow C_1, \quad R_2 \leftrightarrow C_2, \quad R_3 \leftrightarrow C_3, \quad \text{etc}$$

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \longrightarrow A^T = ?$$

$$B = \begin{bmatrix} 1 & 8 \\ 0 & 1 \\ 3 & 9 \end{bmatrix} \longrightarrow B^T = ?$$

$$C = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 7 & 0 \\ 0 & 5 & 6 \end{bmatrix} \longrightarrow C^T = ?$$

THE TRANSPOSE OF A MATRIX:

The **transpose** of a matrix is formed by writing its rows as columns:

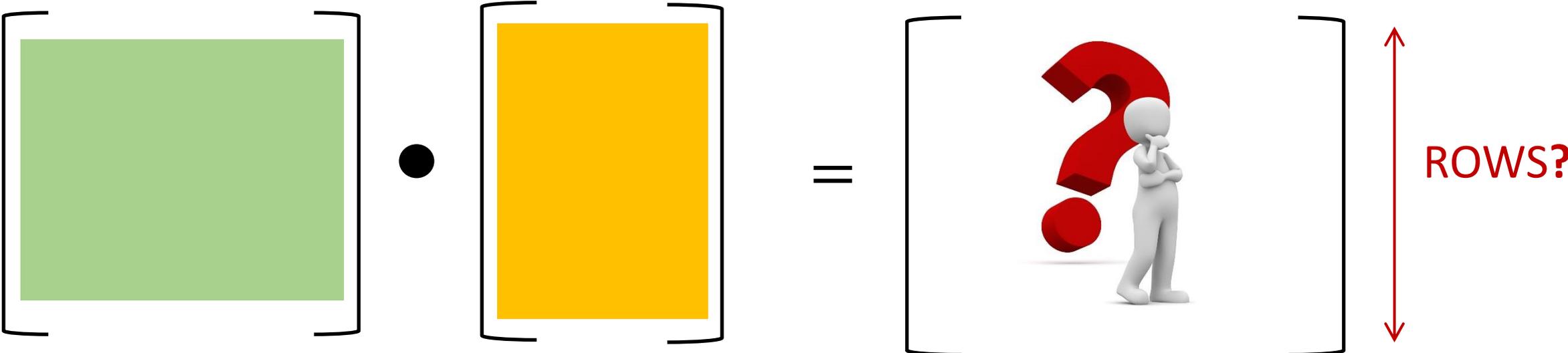
$$R_1 \leftrightarrow C_1, \quad R_2 \leftrightarrow C_2, \quad R_3 \leftrightarrow C_3, \quad \text{etc}$$

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \longrightarrow A^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 8 \\ 0 & 1 \\ 3 & 9 \end{bmatrix} \longrightarrow B^T = \begin{bmatrix} 1 & 0 & 3 \\ 8 & 1 & 9 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 7 & 0 \\ 0 & 5 & 6 \end{bmatrix} \longrightarrow C^T = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 7 & 5 \\ 3 & 0 & 6 \end{bmatrix}$$

MATRIX MULTIPLICATION

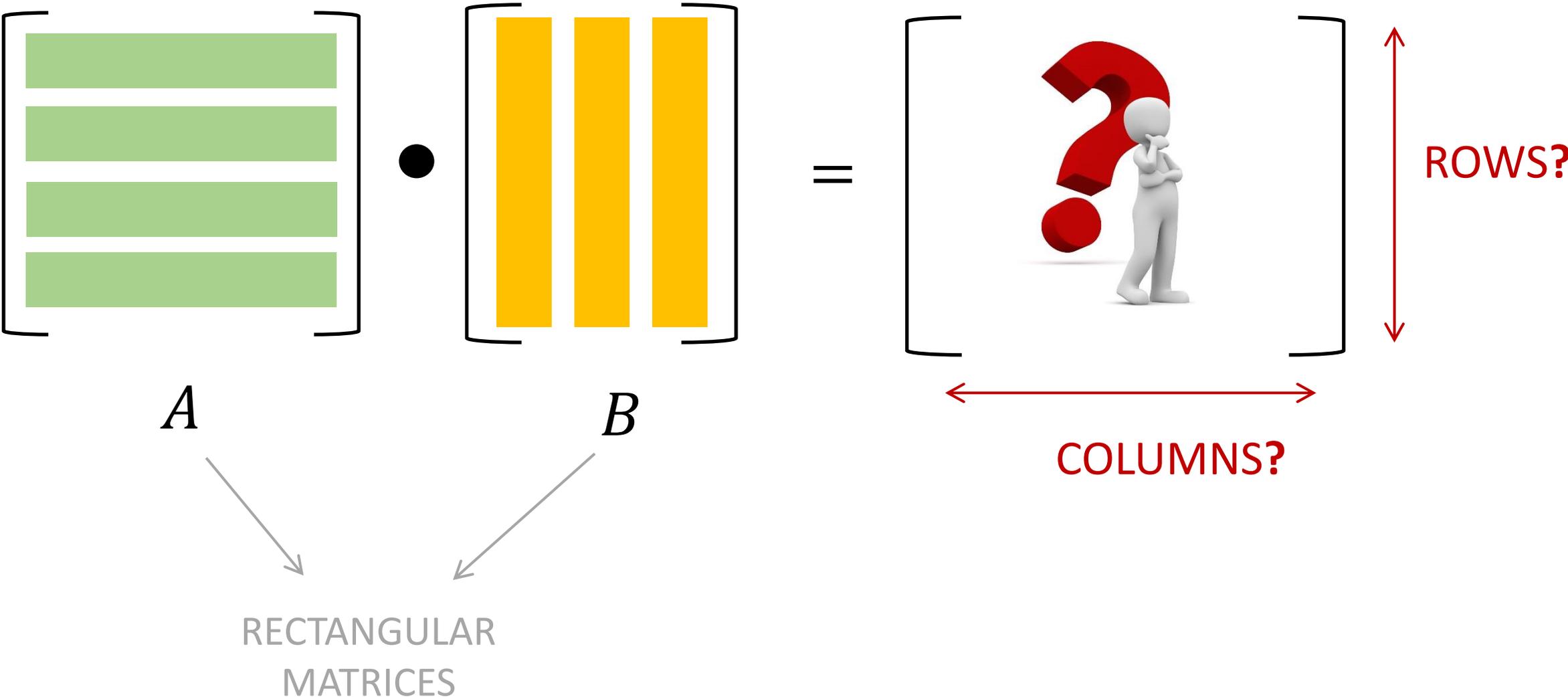


A

B

RECTANGULAR
MATRICES

MATRIX MULTIPLICATION (general idea):



MATRIX MULTIPLICATION (particular case):

$$A = [a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5]$$

ROW (1 × 5)

$$B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

COLUMN (5 × 1)

The product of A and B in this case is **defined** to be:

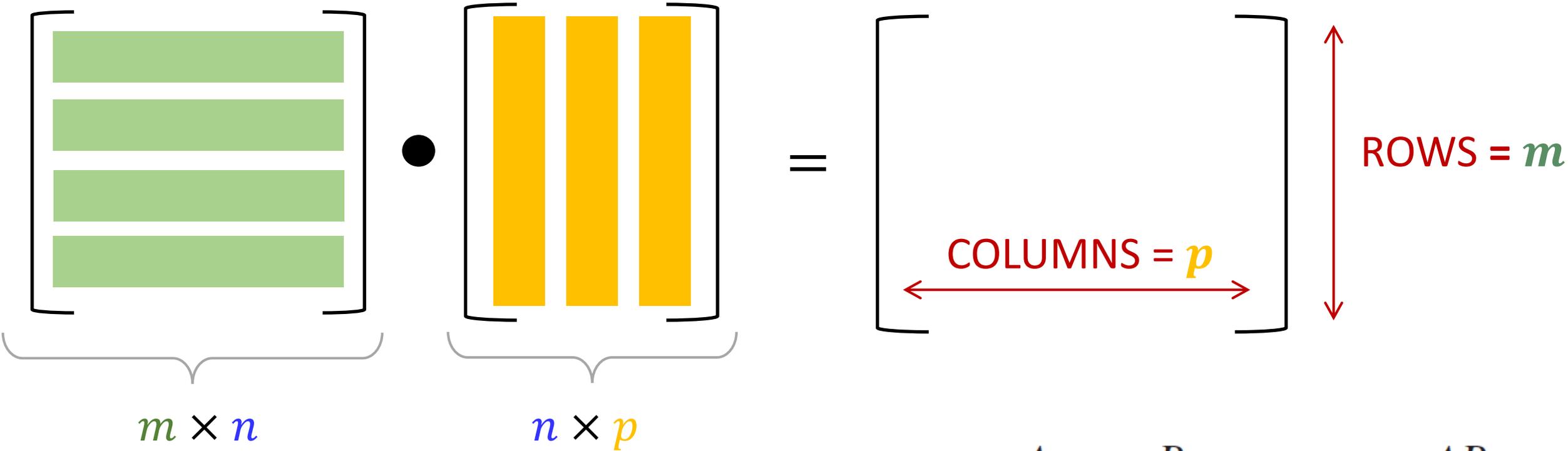
$$AB = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 + a_5b_5$$

this operation works also when the two matrices have an **arbitrary number of elements**

We shall refer to this as the **dot product of a row and a column**.

Note that this operation makes sense only as long as we have the **same number of elements** in both matrices.

MATRIX MULTIPLICATION (further restrictions):



$$\begin{matrix} A & B & = & AB. \\ m \times n & n \times p & & m \times p \\ \uparrow & \uparrow & \uparrow & \uparrow \\ \text{equal} & & & \\ \text{size of } AB & & & \end{matrix}$$

Finding the Product of Two Matrices

Find the product AB , where

$$A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}.$$

3×2 2×2

SOLUTION

First note that the product AB is defined because A has size 3×2 and B has size 2×2 . Moreover, the product AB has size 3×2 and will take the form

$$\begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}.$$

3×2 2×2 3×2

$$\begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 \\ -4 \end{bmatrix} \begin{matrix} 2 \\ 1 \end{matrix} = \begin{bmatrix} -9 & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$$

$c_{11} = (-1)(-3) + (3)(-4) = -9$

$$\begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -9 & 1 \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$$

$c_{12} = (-1)(2) + (3)(1) = 1$

$$\begin{aligned} c_{21} &= (4)(-3) + (-2)(-4) = -4 \\ c_{22} &= (4)(2) + (-2)(1) = 6 \\ c_{31} &= (5)(-3) + (0)(-4) = -15 \\ c_{32} &= (5)(2) + (0)(1) = 10 \end{aligned}$$

The product is

$$AB = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -9 & 1 \\ -4 & 6 \\ -15 & 10 \end{bmatrix}$$

YOUR TURN!

(6 minutes)

Consider the following matrices: $A = \begin{bmatrix} 3 & 1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 33 & 11 \\ 22 & 55 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 9 & 0 \end{bmatrix}$.

Which of the following matrices is NOT defined:

a). $CA - BB$

b). $C^T - AB$

c). $BC - A^T$

d). $CA - AC$

Code: **COMAN862**



Solution

Consider the following matrices:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 33 & 11 \\ 22 & 55 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 9 & 0 \end{bmatrix}$$

3×2 2×2 2×3

Which of the following matrices is NOT defined:

a). $CA - BB$

$2 \times 2 - 2 \times 2$

b). $C^T - AB$

$3 \times 2 - 3 \times 2$

c). $BC - A^T$

$2 \times 3 - 2 \times 3$

d). $CA - AC$

$2 \times 2 - 3 \times 3$

not of the same size!

THE IDENTITY MATRIX:

For numbers:

$$2^{-1} \cdot 2 = 1$$

$$23^{-1} \cdot 23 = 1$$

The **inverse** of a (square) matrix A is denoted by A^{-1} and has the property that

$$AA^{-1} = A^{-1}A = I$$

identity matrix

$$I = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{2 \times 2}$$

$$I = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{3 \times 3}$$

ETC.

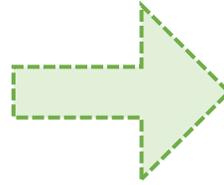
THE IDENTITY MATRIX:

$$(a) \begin{bmatrix} 3 & -2 \\ 4 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & 0 \\ -1 & 1 \end{bmatrix}$$

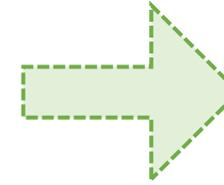
$$(b) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$$

Back to the start...

$$\begin{cases} 1x - 2y + 3z = 9 \\ -1x + 3y + 0z = -4 \\ 2x - 5y + 5z = 17 \end{cases}$$



$$AX = b$$



$$X = A^{-1}b$$

$$\underbrace{\begin{bmatrix} 1 & -2 & 3 \\ -1 & 3 & 0 \\ 2 & -5 & 5 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_X = \underbrace{\begin{bmatrix} 9 \\ -4 \\ 17 \end{bmatrix}}_b$$

3×3 3×1 3×1

Under what conditions
can we find A^{-1} ?

Short answer: $\det(A) \neq 0$

Each **square** matrix A has a **determinant**, a real number denoted either by $\det(A)$ or $|A|$

2 x 2 matrices:

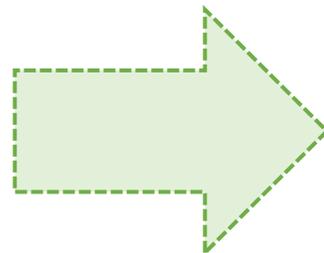
The **determinant** of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is given by

$$\det(A) = |A| = a_{11}a_{22} - a_{21}a_{12}$$

An easy way for remembering:



$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

The diagram shows the determinant calculation with blue arrows indicating the path from a_{11} to a_{22} and from a_{21} to a_{12} .

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Examples: Find the determinant of each matrix:

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 3 \\ 2 & 4 \end{bmatrix}$$

$$\begin{aligned} |A| &= \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} \\ &= 2(2) - 1(-3) \\ &= 4 + 3 = 7 \end{aligned}$$

$$\begin{aligned} |B| &= \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} \\ &= 2(2) - 4(1) \\ &= 4 - 4 = 0 \end{aligned}$$

$$\begin{aligned} |C| &= \begin{vmatrix} 0 & 3 \\ 2 & 4 \end{vmatrix} \\ &= 0(4) - 2(3) \\ &= 0 - 6 = -6 \end{aligned}$$

MINORS & COFACTORS

$$\begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 0 \\ 2 & 5 & 6 \end{bmatrix}$$



$$\begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 0 \\ 2 & 5 & 6 \end{bmatrix}$$



$$\begin{bmatrix} 2 & 1 \\ 4 & 0 \end{bmatrix}$$

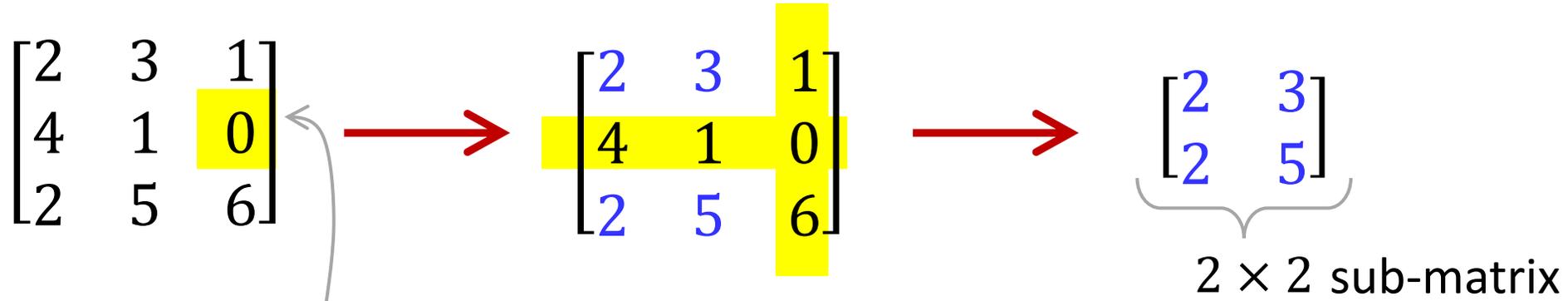
2×2
sub-matrix



the **determinant** of this is
the **MINOR** of the highlighted element (5)
in the original matrix:

$$\begin{vmatrix} 2 & 1 \\ 4 & 0 \end{vmatrix} = (2)(0) - (4)(1) = 0 - 4 = -4$$

MINORS & COFACTORS

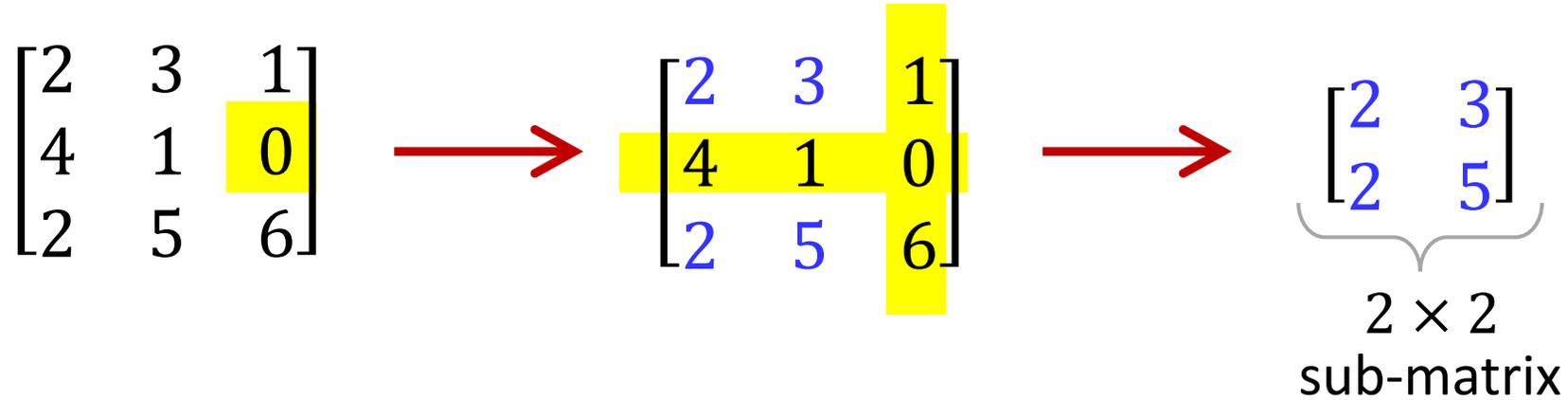


pick this element

remove highlighted
row & column

$$\begin{vmatrix} 2 & 3 \\ 2 & 5 \end{vmatrix} = (2)(5) - (2)(3) = 10 - 6 = 4$$

MINORS & COFACTORS



$$\begin{vmatrix} 2 & 3 \\ 2 & 5 \end{vmatrix} = (2)(5) - (2)(3) = 10 - 6 = 4$$

Observation:

We can repeat this process **for each element** of the original matrix
(9 minors in total)

COFACTOR \longrightarrow \pm MINOR

$$C = (-1)^{ROW \# + COL \#} M$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 0 \\ 2 & 5 & 6 \end{bmatrix}$$

$$C_{32} = (-1)^{3+2} M_{32} = -M_{32} = -(-4) = 4$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 0 \\ 2 & 5 & 6 \end{bmatrix}$$

$$C_{23} = (-1)^{2+3} M_{23} = -M_{23} = -(4) = -4$$

DETERMINANTS OF 3×3 MATRICES:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

C_{11}

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$+ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

C_{12}

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$- \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

C_{13}

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$+ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example:

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

$$\rightarrow M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1 \rightarrow C_{11} = M_{11} = -1$$

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

$$\rightarrow M_{12} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = -5 \rightarrow C_{12} = -M_{12} = 5$$

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

$$\rightarrow M_{13} = \begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = 4 \rightarrow C_{13} = M_{13} = 4$$

Example:

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ = (0)(-1) + (2)(5) + (1)(4) = 14$$

$$\text{or } \det(A) = 14$$

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

$$\rightarrow M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1 \rightarrow C_{11} = M_{11} = -1$$

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

$$\rightarrow M_{12} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = -5 \rightarrow C_{12} = -M_{12} = 5$$

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

$$\rightarrow M_{13} = \begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = 4 \rightarrow C_{13} = M_{13} = 4$$

YOUR TURN!

(10 minutes)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

The determinant of the above matrix is:

a). 0

b). -6

c). 6

d). 12

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REMINDER:

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

C_{11}	C_{12}	C_{13}
$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$	$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$	$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$
$+ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$	$- \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$	$+ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Solution:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\rightarrow M_{11} = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = -3 \rightarrow C_{11} = M_{11} = -3$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\rightarrow M_{12} = \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = -6 \rightarrow C_{12} = -M_{12} = 6$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\rightarrow M_{13} = \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = -3 \rightarrow C_{13} = M_{13} = -3$$

Solution:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ = (1)(-3) + (2)(6) + (3)(-3) = 0$$

$$\text{or } \det(A) = 0$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\rightarrow M_{11} = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = -3 \rightarrow C_{11} = M_{11} = -3$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\rightarrow M_{12} = \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = -6 \rightarrow C_{12} = -M_{12} = 6$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\rightarrow M_{13} = \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = -3 \rightarrow C_{13} = M_{13} = -3$$

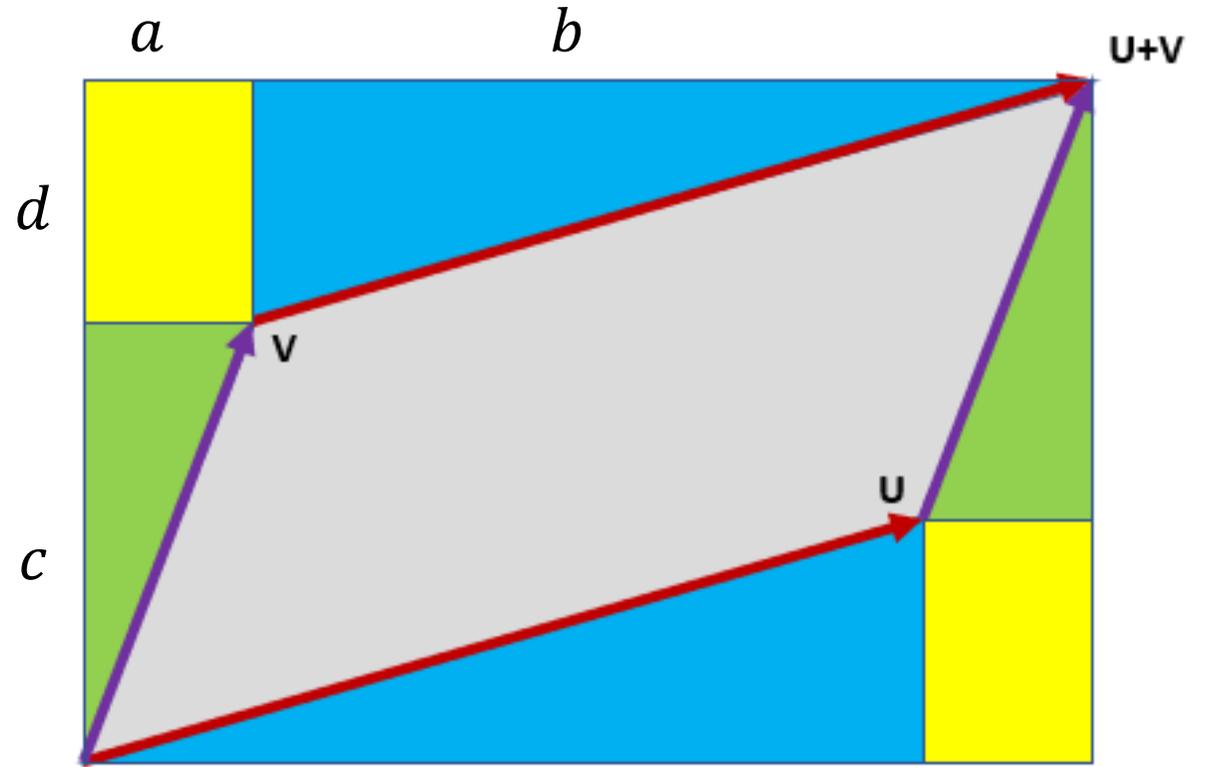
2x2 determinants as AREA:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

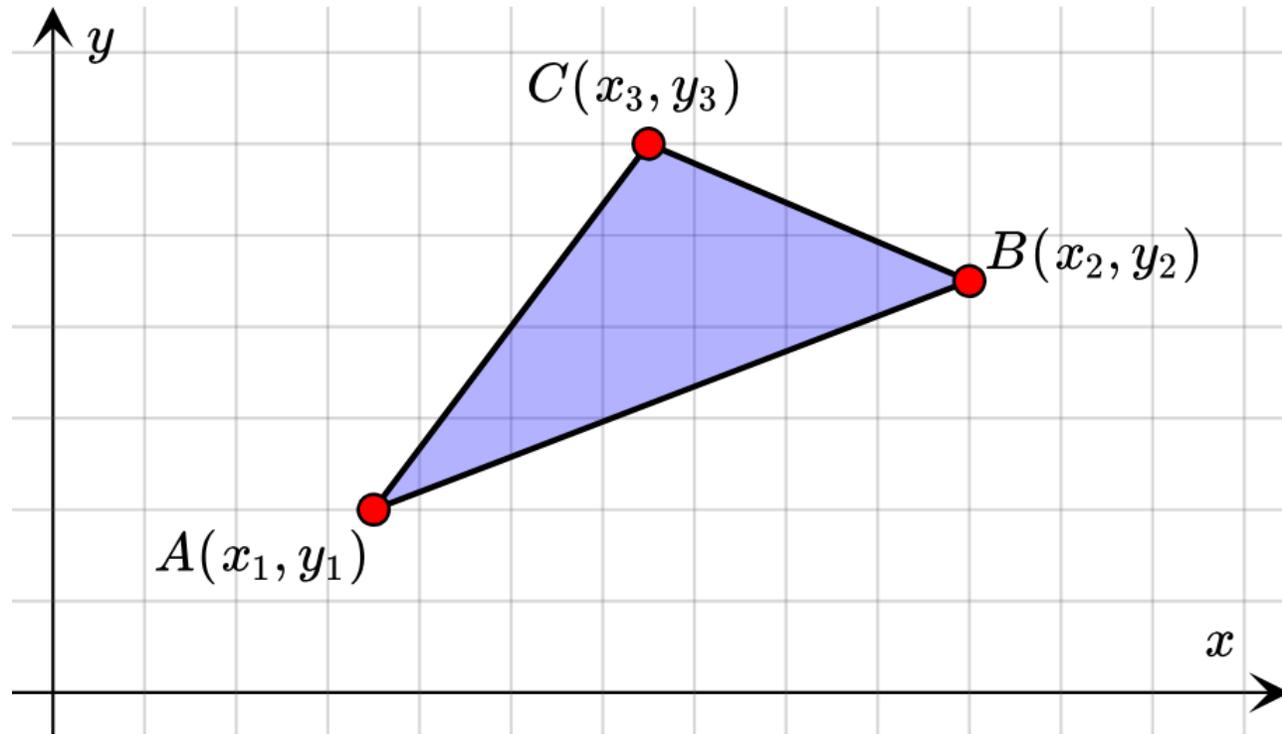
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$$\mathbf{v} = \begin{bmatrix} a \\ c \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} b \\ d \end{bmatrix}$$

$\det(A) = \mathbf{area}$ of parallelogram



Area of a triangle:



$$\text{Area of triangle ABC} = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

YOUR TURN!

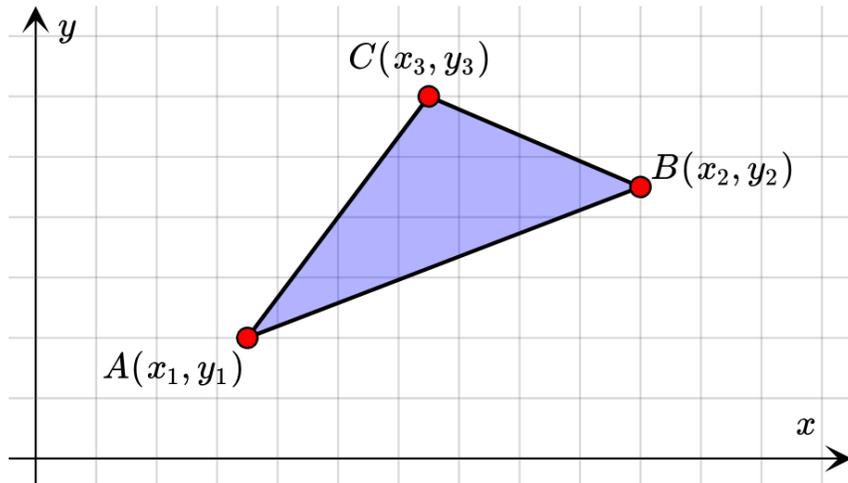
(6 minutes)

Using the above formula, the area of the triangle with vertices

$$A(2, 5), \quad B(12, 5), \quad C(3, 7)$$

is a). 24 b). 10 c). 36 d). 44

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$$\text{Area of triangle ABC} = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

The Inverse of a Matrix

Show that B is the inverse of A , where

$$A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}.$$

Using the definition of an inverse matrix, you can show that B is the inverse of A by showing that $AB = I = BA$, as follows.

$$AB = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 + 2 & 2 - 2 \\ -1 + 1 & 2 - 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 + 2 & 2 - 2 \\ -1 + 1 & 2 - 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

THE ADJOINT OF A MATRIX:

Recall from UNIT 2 that the cofactor C_{ij} of a matrix A is defined as $(-1)^{i+j}$ times the determinant of the matrix obtained by deleting the i th row and the j th column of A . If A is a square matrix, then the **matrix of cofactors** of A has the form

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}.$$

The transpose of this matrix is called the **adjoint** of A and is denoted by $\text{adj}(A)$. That is,

$$\text{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}.$$

Finding the Adjoint of a Square Matrix

Find the adjoint of

$$A = \begin{bmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix}.$$

The cofactor C_{11} is given by

$$\begin{bmatrix} \textcircled{-1} & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix} \rightarrow C_{11} = (-1)^2 \begin{vmatrix} -2 & 1 \\ 0 & -2 \end{vmatrix} = 4.$$

$$\begin{bmatrix} \begin{vmatrix} -2 & 1 \\ 0 & -2 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} & \begin{vmatrix} 0 & -2 \\ 1 & 0 \end{vmatrix} \\ -\begin{vmatrix} 3 & 2 \\ 0 & -2 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ 1 & -2 \end{vmatrix} & -\begin{vmatrix} -1 & 3 \\ 1 & 0 \end{vmatrix} \\ \begin{vmatrix} 3 & 2 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 4 & 1 & 2 \\ 6 & 0 & 3 \\ 7 & 1 & 2 \end{bmatrix} \xrightarrow{\text{TRANSPOSE}} \text{adj}(A) = \begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix}.$$

The adjoint of a matrix A can be used to find the inverse of A .

If A is an $n \times n$ invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A).$$

If A is a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the adjoint of A is simply

$$\text{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Moreover, if A is invertible, then

$$A^{-1} = \frac{1}{|A|} \text{adj}(A) = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

Using the Adjoint of a Matrix to Find Its Inverse

Use the adjoint of

$$A = \begin{bmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix}$$

to find A^{-1} .

- WE HAVE ALREADY CALCULATED THE ADJOINT FOR THIS
- THE DETERMINANTS IS 3

$$A^{-1} = \frac{1}{|A|} \text{adj}(A) = \frac{1}{3} \begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & 2 & \frac{7}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & 1 & \frac{2}{3} \end{bmatrix}.$$