Networks

- describe "flows under forces"

    1). fluid flow some irrigation network
    2) flow of electricity (charge) under the force
    (voltage)

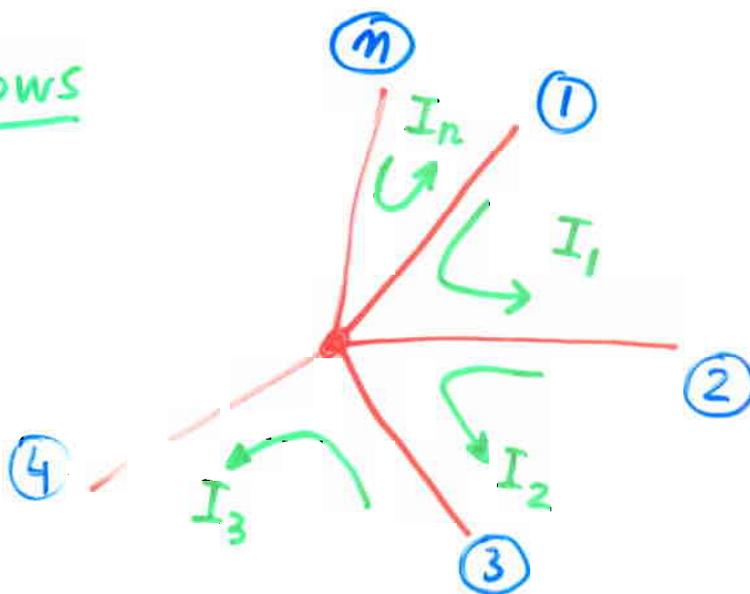- all flows satisfy two properties:

    (i) flow is conserved at a junction
    (ii) the flow is related to the force by some Law
    (e.g. in electricity → Ohm's Law $\boxed{V = RI}$
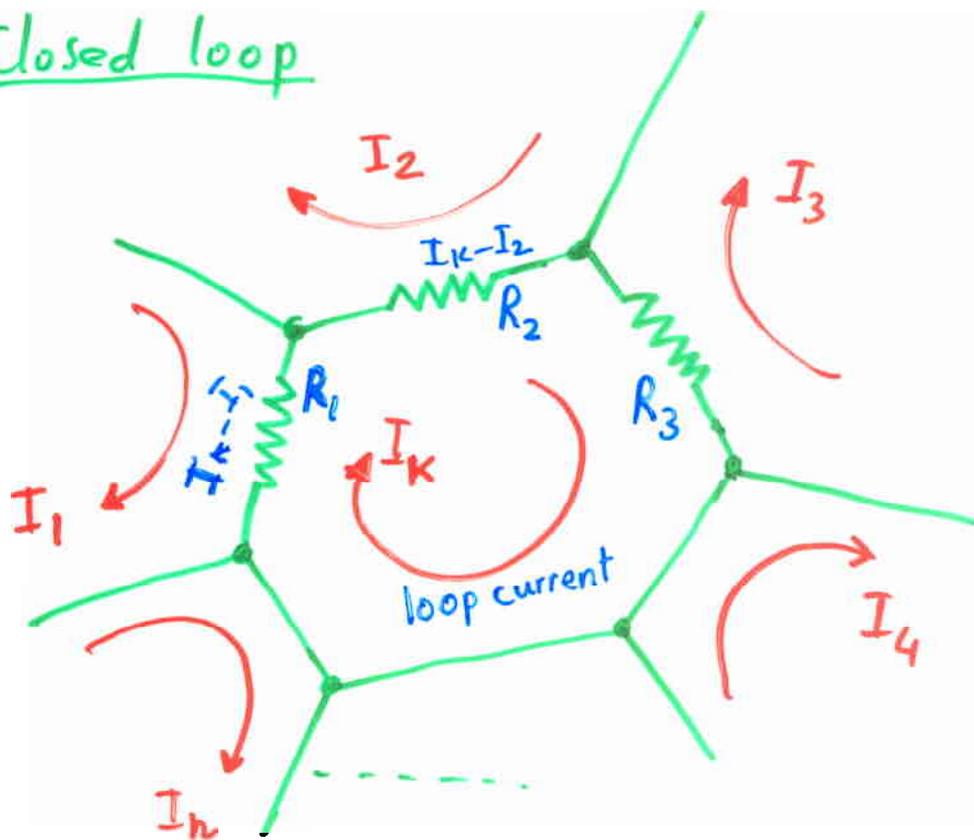
    voltage ↓ Current
    resistance

- loop flows

flow in individual branches
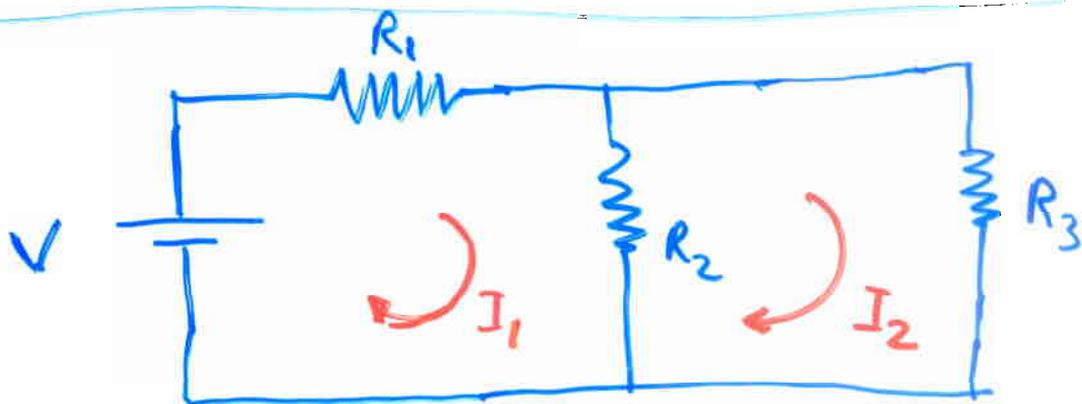
①: $I_1 - I_n$
②: $I_2 - I_1$
③: $I_3 - I_2 ..$ etc

<u>Closed loop</u>

Total ~~flow~~ forces around the closed loop: → net currents in the branches

$$R_1\left(I_k - I_1\right) + R_2\left(I_k - I_2\right) + \cdots + R_n\left(I_k - I_n\right)$$

must be equal to
the force applied around the circuit

<u>Example 1</u>

$\boxed{V = RI}$

<u>loop 1</u>:
$$\begin{cases} R_1 I_1 + R_2\left(I_1 - I_2\right) = V \\ R_3 I_2 + R_2\left(I_2 - I_1\right) = 0 \end{cases}$$

$\left(\begin{array}{l}\text{no source in} \\ \text{this loop}\end{array}\right)$

$$\Rightarrow \begin{bmatrix} R_1+R_2 & -R_2 \\ -R_2 & R_2+R_3 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V \\ 0 \end{bmatrix} \longrightarrow \text{get } I_1, I_2$$

OBS. Note that the loop analysis always leads to a symmetric Coefficient matrix

Wheatstone Bridge (another example)
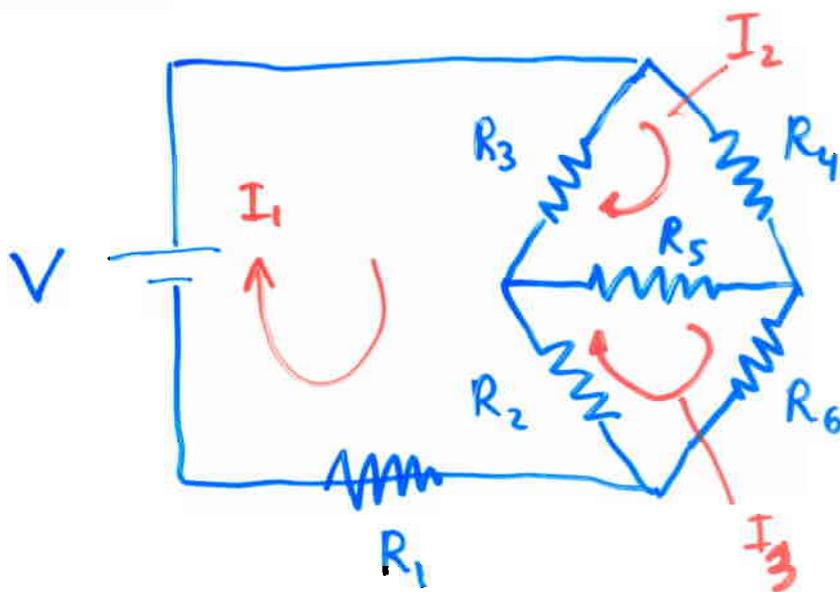
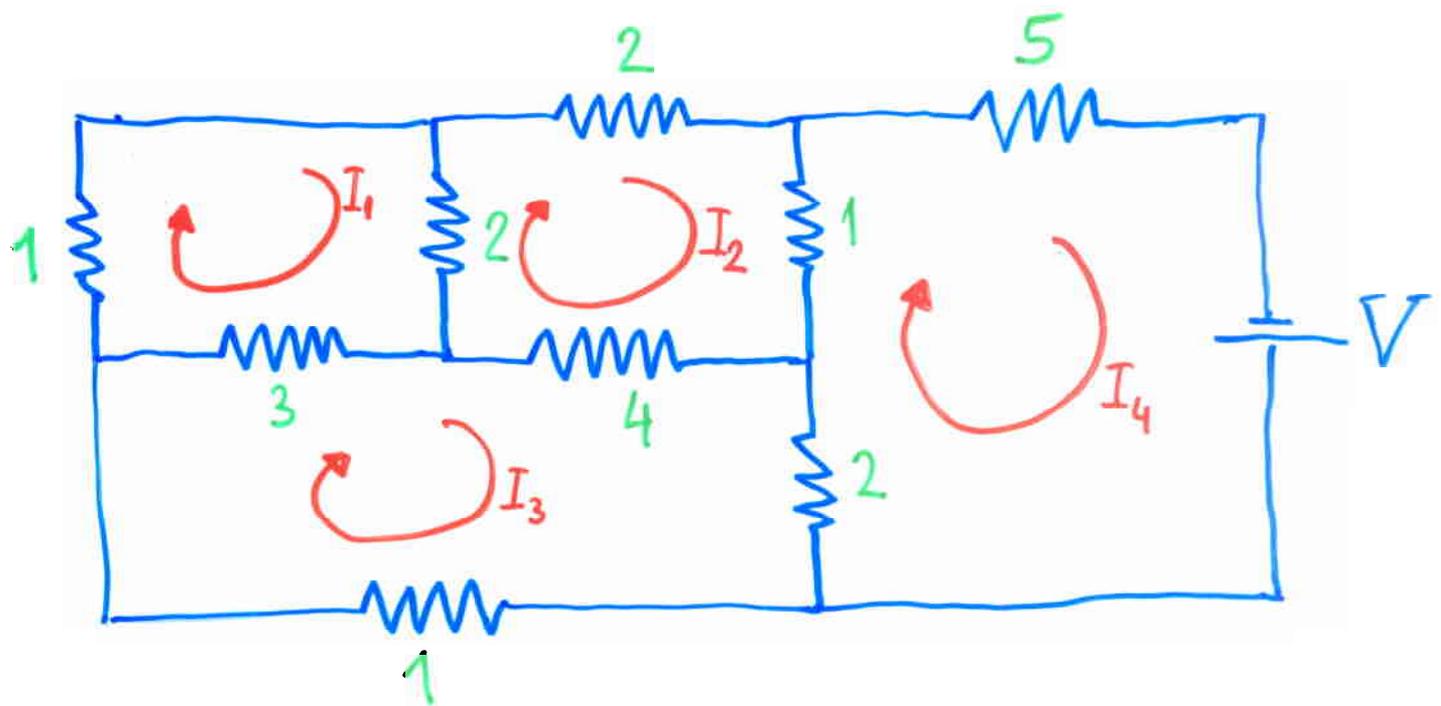loop1: $I_1 R_1 + (I_1 - I_2) R_3 + (I_1 - I_3) R_2 = V$

loop2: $I_2 R_4 + (I_2 - I_3) R_5 + (I_2 - I_1) R_3 = 0$

loop3: $I_3 R_6 + (I_3 - I_1) R_2 + (I_3 - I_2) R_5 = 0$

$$\Rightarrow \begin{bmatrix} R_1+R_2+R_3 & -R_3 & -R_2 \\ -R_3 & R_3+R_4+R_5 & -R_5 \\ -R_2 & -R_5 & R_2+R_5+R_6 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V \\ 0 \\ 0 \end{bmatrix}$$

Example 3 Solve the network so as to find its resistance to the source V

Solution

loop 1: $I_1 + 2(I_1 - I_2) + 3(I_1 - I_3) = 0$

loop 2: $2I_2 + (I_2 - I_4) + 4(I_2 - I_3) + 2(I_2 - I_1) = 0$

loop 3: $I_3 + 3(I_3 - I_1) + 4(I_4 - I_2) + 2(I_3 - I_4) = 0$

loop 4: $5I_4 + 2(I_4 - I_3) + (I_4 - I_2) = V$

$\Downarrow$

$$\begin{bmatrix} 6 & -2 & -3 & 0 \\ -2 & 9 & -4 & -1 \\ -3 & -4 & 10 & -2 \\ 0 & -1 & -2 & 8 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ V \end{bmatrix} \implies$$

Gaussian elimination

$$\begin{bmatrix} 6 & -2 & -3 & 0 & \bigg| & 0 \\ -2 & 9 & -4 & -1 & \bigg| & 0 \\ -3 & -4 & 10 & -2 & \bigg| & 0 \\ 0 & -1 & -2 & 8 & \bigg| & V \end{bmatrix} \sim \begin{bmatrix} 6 & -2 & -3 & 6 & \bigg| & 0 \\ 0 & 25/3 & -5 & -1 & \bigg| & 0 \\ 0 & -5 & 17/2 & -2 & \bigg| & 0 \\ 0 & -1 & -2 & 8 & \bigg| & V \end{bmatrix}$$

$$R_2 \to R_2 + \frac{1}{3} R_1$$
$$R_3 \to R_3 + \frac{1}{2} R_1$$

$$\sim \begin{bmatrix} 6 & -2 & -3 & 0 & \bigg| & 0 \\ 0 & 25/3 & -5 & -1 & \bigg| & 0 \\ 0 & 0 & 11/2 & -13/5 & \bigg| & 0 \\ 0 & 0 & -13/5 & 197/5 & \bigg| & V \end{bmatrix}$$

$$R_3 \to R_3 + \frac{3}{5} R_2$$
$$R_4 \to R_4 + \frac{3}{25} R_2$$

$$\sim \begin{bmatrix} 6 & -2 & -3 & 0 & \bigg| & 0 \\ 0 & 25/3 & -5 & -1 & \bigg| & 0 \\ 0 & 0 & 11/2 & -13/5 & \bigg| & 0 \\ 0 & 0 & 0 & 1829/275 & \bigg| & V \end{bmatrix}$$

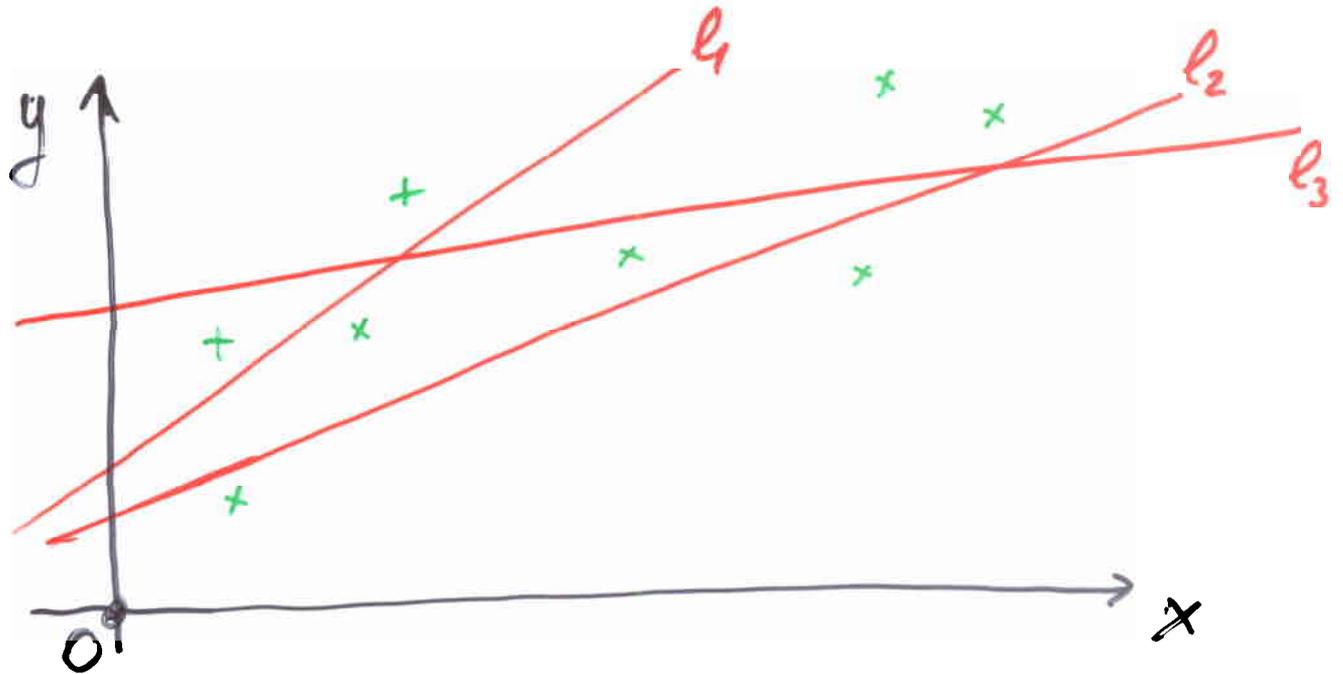$$R_4 \to R_4 + \underbrace{\frac{2}{11} \times \frac{13}{5}}_{\frac{26}{55}} R_3$$

$$\begin{cases} 6I_1 - 2I_2 - 3I_3 = 0 \\[2mm] \dfrac{25}{3} I_2 - 5I_3 - I_4 = 0 \\[2mm] \dfrac{11}{2} I_3 - \dfrac{13}{5} I_4 = 0 \\[2mm] \dfrac{1829}{275} I_4 = V \end{cases}$$

$\longrightarrow$ We only need
$\dfrac{V}{I_4} = \dfrac{\text{resistance}}{\text{to the source } V}$

Least squares fitting

data points: $\{(x_j, y_j) \mid 1 \le j \le n\}$

Main question:

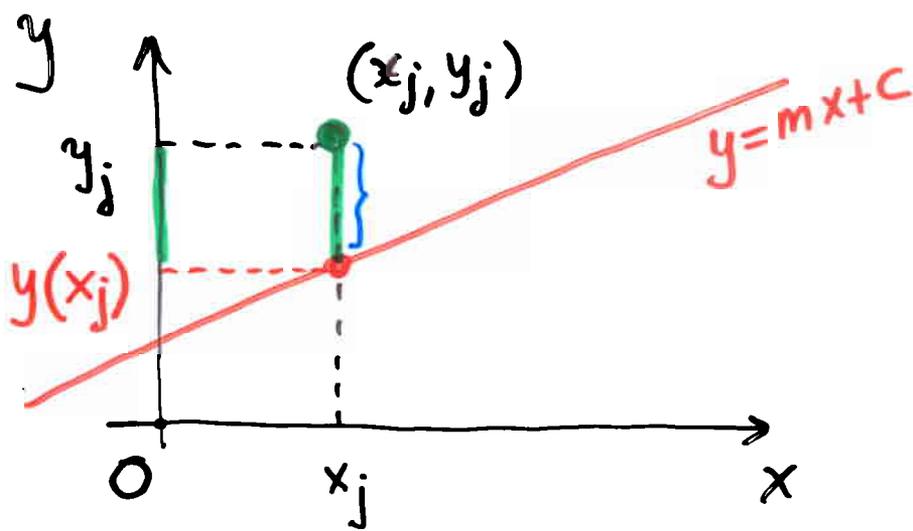> Find the "best" straight line
> $$y = mx + c$$
> which fits them

Aside: every straight line is uniquely determined by 2 numbers

$$y = mx + c$$

$m \rightarrow$ slope

$c \rightarrow$ tells us where the line cuts the y-axis

$$E_j = |y(x_j) - y_j| = |mx_j + c - y_j|$$

error
for (x_j, y_j)

$y = mx + c$

"best fit" $\longrightarrow$ E_j $(1 \leq j \leq n)$ as small as possible

impossible
to achieve

$$\sum_{j=1}^{n} E_j \quad \text{as small as possible}$$

still very hard

minimise $\displaystyle\sum_{j=1}^{n} E_j^2$

$$E = \sum_{j=1}^{n} E_j^2 = \sum_{j=1}^{n} \left(mx_j + c - y_j \right)^2$$

depends on two independent parameters: m & c

want to minimise $E(m,c)$

a function of 2 variables

$$\begin{cases} \dfrac{\partial E}{\partial m}(m,c) = 0 \\[2mm] \dfrac{\partial E}{\partial c}(m,c) = 0 \end{cases}$$

$\longleftarrow$ find m & c that satisfy these two eqns.

$$\frac{\partial E}{\partial m} = \sum_{j=1}^{n} \frac{\partial}{\partial m} \left\{ \left(mx_j + c - y_j \right)^2 \right\}$$

$$= \sum_{j=1}^{m} 2\left(mx_j + c - y_j \right) \underbrace{\frac{\partial}{\partial m} \left\{ \left(mx_j + c - y_j \right) \right\}}_{x_j}$$

$$= \sum_{j=1}^{m} 2\left(mx_j + c - y_j \right) x_j$$

$$\frac{\partial E}{\partial c} = \sum_{j=1}^{m} \frac{\partial}{\partial c} \left\{ \left(mx_j + c - y_j \right)^2 \right\}$$

$$= \sum_{j=1}^{m} 2\left(mx_j + c - y_j \right) \underbrace{\frac{\partial}{\partial c} \left\{ \left(mx_j + c - y_j \right) \right\}}_{1}$$

$$= \sum_{j=1}^{n} 2\left(mx_j + c - y_j \right)$$

$$\begin{cases} \sum_{j=1}^{n} 2x_j\left(\underline{m}x_j + \underline{c} - y_j\right) = 0 \\ \sum_{j=1}^{m} 2\left(\underline{m}x_j + \underline{c} - y_j\right) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} m\left(\overbrace{\sum_{j=1}^{n} x_j^2}^{\alpha}\right) + c\left(\underbrace{\sum_{j=1}^{n} x_j}\right) = \overbrace{\sum_{j=1}^{n} x_j y_j}^{\gamma} \\ m\left(\underbrace{\sum_{j=1}^{n} x_j}_{\beta}\right) + cn = \underbrace{\sum_{j=1}^{n} y_j}_{\delta} \end{cases}$$

$\alpha, \beta, \gamma, \delta \rightarrow$ known quantities defined as above

$$\begin{cases} \alpha m + \beta c \qquad \gamma \\ \beta m + nc \qquad \delta \end{cases} \longrightarrow \text{find } m \text{ and } c$$