

LU factorisation (part II)

- echelon form
- reduced echelon form
- Gaussian elimination
- Gauss-Jordan method

→ obtained by e.r.o.'s

$R_i \leftrightarrow R_j$	(I)
$R_i \rightarrow \lambda R_i$	(II)
$R_i \rightarrow R_i + \lambda R_j$	(III)

$$E_{R_i \leftrightarrow R_j}$$

$$E_{R_i \rightarrow \lambda R_i}$$

$$E_{R_i \rightarrow R_i + \lambda R_j}$$

elementary matrices

Last week: $A \in M_{n \times n}(\mathbb{R})$

$$A = LU$$

lower triangular

upper triangular

$$L = \begin{bmatrix} l_{11} & 0 & 0 & 0 & \dots & 0 \\ l_{21} & l_{22} & 0 & 0 & \dots & 0 \\ l_{31} & l_{32} & l_{33} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{m1} & l_{m2} & l_{m3} & l_{m4} & \dots & l_{mn} \end{bmatrix}$$

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \dots & u_{1n} \\ 0 & u_{22} & u_{23} & \dots & u_{2n} \\ 0 & 0 & u_{33} & \dots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & u_{nn} \end{bmatrix}$$

the elements above the main diagonal are all equal to zero

If A is such that a zero pivot is never encountered when applying Gaussian elimination with Type (III) operations, then A can be factored as the product

$$A = LU$$

• $L =$ lower triangular $U =$ upper triangular

• $l_{ii} = 1$ and $u_{ii} \neq 0$ $(\forall) i = 1, 2, \dots, n$

• the non-zero entries of L :

$l_{ij} =$ multiple of row j that is subtracted from row i in order to clear the (i, j) position during Gaussian elimination

• $U =$ the final result of Gaussian elimination applied to A

OBS Not all non-singular matrices possess an LU factorisation.

e.g. $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix}}_L \underbrace{\begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix}}_U \rightarrow \text{can't find } u_{11} \neq 0$$

zero pivot in the $(1, 1)$ -position

If a row interchange is needed to remove a zero pivot, then no LU factorisation is possible.

However \rightarrow $(\exists) P \in M_{n \times n}(\mathbb{R})$ s.t.

$$\boxed{PA = LU}$$

← permutation matrix

lower triangular upper triangular

$P \rightarrow$ product of permutation matrices \rightarrow still a permutation matrix

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{I_4} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_4}$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_4} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Definition A t -digit, base-10 floating-point number has the form

$$f = \underline{t} . d_1 d_2 \dots d_t \times 10^\epsilon \quad \text{with } d_1 \neq 0$$

$\epsilon = \text{exponent} \rightarrow \text{integers}$
 $0 \leq d_i \leq 9 \rightarrow \text{integers}$
 $t \rightarrow \text{called precision}$

Notation: if $x \in \mathbb{R} \rightsquigarrow \underline{fl(x)}$
floating-point approximation of x

$$fl(x) = \begin{cases} . d_1 d_2 \dots d_t \times 10^\epsilon & \text{if } d_{t+1} < 5 \\ (. d_1 d_2 \dots d_t + 10^{-t}) \times 10^\epsilon & \text{if } d_{t+1} \geq 5 \end{cases}$$

$$fl\left(\frac{3}{80}\right) = fl(.0375) = fl(.375 \times 10^{-1}) = \underline{.38} \times 10^{-1} \\ = \underline{0.038}$$

2-digit base-10 floating point approx. of $3/80$

$$fl\left(\frac{89}{47}\right) = \underline{.189} \times 10$$

3-digit base-10 floating point approx. of $\frac{89}{47}$, etc

$$\begin{cases} -10^{-4}x + y = 1 \\ x + y = 2 \end{cases} \longrightarrow \begin{cases} x = \frac{1}{1.0001} \\ y = \frac{1.0002}{1.0001} \end{cases} \quad (*)$$

Solve using
3-digit arithmetic

①

$$\left(\begin{array}{cc|c} -10^{-4} & 1 & 1 \\ 1 & 1 & 2 \end{array} \right) \sim \left(\begin{array}{cc|c} -10^{-4} & 1 & 1 \\ 0 & 10^4 & 10^4 \end{array} \right)$$

$R_2 \rightarrow R_2 + 10^4 R_1$

$$fe(1+10^4) = fe(.10001 \times 10^5) = .100 \times 10^5 = 10^4$$

$$fe(2+10^4) = fe(.10002 \times 10^5) = .100 \times 10^5 = 10^4$$

②

$$\begin{cases} -10^{-4}x + y = 1 \\ 10^4 y = 10^4 \end{cases} \Rightarrow \begin{cases} y = 1 \\ x = 0 \end{cases}$$

not very close to (*)

③

$$\left(\begin{array}{cc|c} -10^{-4} & 1 & 1 \\ 1 & 1 & 2 \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & 1 & 2 \\ -10^{-4} & 1 & 1 \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array} \right)$$

$R_2 \rightarrow R_2 + 10^{-4} R_1$

$$fe(1+10^{-4}) = fe(.10001 \times 10) = .100 \times 10 = 1$$

$$fe(1+2 \times 10^{-4}) = fe(.10002 \times 10) = .100 \times 10 = 1$$

$$\begin{cases} x + y = 2 \\ y = 1 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 1 \end{cases} \rightarrow \text{both close to the "exact" solution}$$

OBS. If a pivot is very small \rightarrow we may lose significance

\Downarrow AIM

choose pivots as large as possible



partial pivoting

(inspect the rows and maximise the pivots)

Example

$$\begin{cases} x + 2y + 3z = 2 \\ x - 2y + z = 1 \\ -2x + y - z = 0 \end{cases} \leftarrow A\underline{x} = \underline{b}$$

$$[A|\underline{b}] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 1 & -2 & 1 & 1 \\ -2 & 1 & -1 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} -2 & 1 & -1 & 0 \\ 1 & -2 & 1 & 1 \\ 1 & 2 & 3 & 2 \end{array} \right]$$

$| -2 | = 2 > 1$

$R_1 \leftrightarrow R_3$

$$\sim \left[\begin{array}{ccc|c} -2 & 1 & -1 & 0 \\ 0 & -3/2 & 1/2 & 1 \\ 0 & 5/2 & 5/2 & 2 \end{array} \right] \begin{array}{l} R_2 \rightarrow R_2 + \frac{1}{2} R_1 \\ R_3 \rightarrow R_3 + \frac{1}{2} R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} -2 & 1 & -1 & 0 \\ 0 & 5/2 & 5/2 & 2 \\ 0 & -3/2 & 1/2 & 1 \end{array} \right] R_2 \leftrightarrow R_3$$

$$\sim \left[\begin{array}{ccc|c} -2 & 1 & -1 & 0 \\ 0 & 5/2 & 5/2 & 2 \\ 0 & 0 & 2 & 11/5 \end{array} \right] R_3 \rightarrow R_3 + \frac{3}{5} R_1$$

$$\begin{cases} -2x + y + z = 0 \\ \frac{5}{2}y + \frac{5}{2}z = 2 \\ 2z = 11/5 \end{cases} \quad \begin{array}{l} x = -\frac{7}{10} \\ y = -\frac{3}{10} \\ z = \frac{11}{10} \end{array}$$

LU-pivoting

$$(PA = LU)$$

permutation
counter
column

Example

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 0 \\ 4 & 0 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 0 \\ 4 & 0 & 4 \end{bmatrix} \left| \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right.$$

$$\sim \begin{bmatrix} 4 & 0 & 4 & | & 3 \\ 2 & 2 & 0 & | & 2 \\ 1 & 3 & 1 & | & 1 \end{bmatrix} \sim \begin{array}{l} \frac{1}{2} \\ \frac{1}{4} \end{array} \begin{bmatrix} 4 & 0 & 4 & | & 3 \\ 0 & 2 & -2 & | & 2 \\ 0 & 3 & 0 & | & 1 \end{bmatrix}$$

$$\sim \begin{array}{l} \frac{1}{4} \\ \frac{1}{2} \end{array} \begin{bmatrix} 4 & 0 & 4 & | & 3 \\ 0 & 3 & 0 & | & 1 \\ 0 & 2 & -2 & | & 2 \end{bmatrix} \sim \begin{array}{l} \frac{2}{3} \\ \frac{2}{3} \end{array} \begin{bmatrix} 4 & 0 & 4 & | & 3 \\ 0 & 3 & 0 & | & 1 \\ 0 & 0 & -2 & | & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} 4 & 0 & 4 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{2} & \frac{2}{3} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$R_1 \leftrightarrow R_3$

2nd interchange

1st interchange

Example Use LU decomposition with partial pivoting to

Solve

$$\begin{cases} x + y = 2 \\ 2x - z = 1 \\ 2x + 3y - z = 3 \end{cases} \quad A \underline{x} = \underline{b}$$

Step 1

$$PA = LU$$

Step 2

$$A \underline{x} = \underline{b} \Leftrightarrow (PA) \underline{x} = P \underline{b}$$

$$(LU) \underline{x} = P \underline{b} \Rightarrow L(\underbrace{U \underline{x}}_{\underline{y}}) = P \underline{b}$$

Solve $L \underline{y} = P \underline{b}$ (forward substitution)

and then $U \underline{x} = \underline{y}$ (backward substitution)

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 2 & 0 & -1 & 2 \\ 2 & 3 & -1 & 3 \end{array} \right] \sim \left[\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 1 & 1 & 0 & 1 \\ 2 & 3 & -1 & 3 \end{array} \right] \sim$$

$$\begin{array}{l} \sim \\ \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{array} \left[\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 0 & 1 & \frac{1}{2} & 1 \\ 0 & 3 & 0 & 3 \end{array} \right] \sim \frac{1}{2} \left[\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 0 & 3 & 0 & 3 \\ 0 & 1 & \frac{1}{2} & 1 \end{array} \right] \sim$$

$$= \frac{1}{3} \left[\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 1/2 & 1 \end{array} \right]$$

$$P: \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array} \right]$$

$$U: \left[\begin{array}{ccc} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 1/2 \end{array} \right]$$

$$L: \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1/2 & 1/3 & 1 \end{array} \right]$$

$$L\underline{y} = P\underline{b} \Rightarrow \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1/2 & 1/3 & 1 \end{array} \right] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \Rightarrow \begin{cases} y_1 = 1 \\ y_2 = 2 \\ y_3 = \frac{5}{6} \end{cases}$$

$$U\underline{x} = \underline{y} \Rightarrow \left[\begin{array}{ccc} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 1/2 \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5/6 \end{bmatrix} \Rightarrow \begin{cases} x_3 = 5/3 \\ x_2 = 2/3 \\ x_1 = 4/3 \end{cases}$$

Example Use the LU decomposition with partial pivoting to solve

$$\begin{cases} x + 4y + 3z - t = 6 \\ -3x + 2z = -3 \\ 4x + 3y + 2t = 5 \\ 2x - y + 3z - 5t = 6 \end{cases} \rightarrow A\underline{x} = \underline{b}$$

Solution: Step 1 $\rightarrow PA = LU$

Step 2 \rightarrow ~~$Ly = Pb$~~
 $Ly = \underline{p}b$
 $U\underline{x} = \underline{y}$

$$\left[\begin{array}{cccc|c} 1 & 4 & 3 & -1 & 1 \\ -3 & 0 & 2 & 0 & 2 \\ \textcircled{4} & 3 & 0 & 2 & 3 \\ 2 & -1 & 3 & -5 & 4 \end{array} \right] \sim \left[\begin{array}{cccc|c} 4 & 3 & 0 & 2 & 3 \\ -3 & 0 & 2 & 0 & 2 \\ 1 & 4 & 3 & -1 & 1 \\ 2 & -1 & 3 & -5 & 4 \end{array} \right]$$

$$\sim \left[\begin{array}{cccc|c} 4 & 3 & 0 & 2 & 3 \\ -3/4 & 6 & 9/4 & 2 & 3/2 \\ 1/4 & 6 & \textcircled{13/4} & 3 & -3/2 \\ 1/2 & 0 & -5/2 & 3 & -6 \end{array} \right] \sim \left[\begin{array}{cccc|c} 4 & 3 & 0 & 2 & 3 \\ 0 & 13/4 & 3 & -3/2 & 1 \\ 0 & 9/4 & 2 & 3/2 & 2 \\ 0 & -5/2 & 3 & -6 & 4 \end{array} \right]$$

$$\sim \begin{bmatrix} 4 & 3 & 0 & 2 & | & 3 \\ 0 & 13/4 & 3 & -3/2 & | & 1 \\ 9/13 & 0 & -1/13 & 33/13 & | & 2 \\ -10/13 & 0 & 69/13 & -93/13 & | & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1/4 & & & & | & & & & \\ 1/2 & -10/13 & & & | & & & & \\ -3/4 & 9/13 & & & | & & & & \end{bmatrix} \begin{bmatrix} 4 & 3 & 0 & 2 & | & 3 \\ 0 & 13/4 & 3 & -3/2 & | & 1 \\ 0 & 0 & 69/13 & -93/13 & | & 4 \\ 0 & 0 & -1/13 & 33/13 & | & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1/4 & & & & | & & & & \\ 1/2 & -10/13 & & & | & & & & \\ -3/4 & 9/13 & & & | & & & & \end{bmatrix} \begin{bmatrix} 4 & 3 & 0 & 2 & | & 3 \\ 0 & 13/4 & 3 & -3/2 & | & 1 \\ 0 & 0 & 69/13 & -93/13 & | & 4 \\ 0 & 0 & 0 & 56/13 & | & 2 \end{bmatrix} \xrightarrow{-\frac{1}{69}}$$

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 1/2 & -10/13 & 1 & 0 \\ -3/4 & 9/13 & -1/69 & 1 \end{bmatrix}$$