Gaussian elimination

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \\ \quad \vdots \qquad \vdots \qquad \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m \end{cases}$$

eliminate by e.r.o.'s

eliminate by e.r.o.'s   · · ·

[step 1]   [step 2]   · · ·

$$A \underline{x} = \underline{b} \quad \rightsquigarrow \quad U\underline{x} = \underline{c}$$

modified RHS of the original system

full matrix

upper triangular matrix

Same vector of unknowns

$$\left[A = (a_{ij}) \right]$$

$$\begin{bmatrix} x & x & x & x & x \\ 0 & x & x & x & x \\ 0 & 0 & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & 0 & x \end{bmatrix} \leftarrow 5\times5 \text{ example}$$

lower triangular

$$A = LU \longrightarrow ? \text{ quick way to find } L$$

Clearing columns

$A \in M_{n \times m}(\mathbb{R})$ $\qquad\qquad a_{11} \neq 0$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & & a_{mm} \end{bmatrix}$$

must be "cleared"

notation:

$R_2 \rightarrow R_2 - \dfrac{a_{21}}{a_{11}} R_1$

$R_3 \rightarrow R_3 - \dfrac{a_{31}}{a_{11}} R_1$

$\vdots$

$R_n \rightarrow R_n - \dfrac{a_{n1}}{a_{11}} R_1$

$$\begin{cases} \lambda_2 = -\dfrac{a_{21}}{a_4} \\[2mm] \lambda_3 = -\dfrac{a_{31}}{a_{11}} \\[2mm] \vdots \\[2mm] \lambda_n = -\dfrac{a_{n1}}{a_{**}} \end{cases}$$

elementary matrices :

$$\begin{cases} E_{R_2 \rightarrow R_2 + \lambda_2 R_1} \\[2mm] E_{R_3 \rightarrow R_3 + \lambda_3 R_1} \\[2mm] \cdots \cdots \\[2mm] E_{R_n \rightarrow R_n + \lambda_n R_1} \end{cases}$$

$E_{R_2 \rightarrow R_2 + \lambda_2 R_1} A \longrightarrow$ clears the first entry in the second row

$$E_{R_3 \to R_3 + \lambda_3 R_1}\left(E_{R_2 \to R_2 + \lambda_2 R_1} A\right) \longrightarrow \text{clears the first entries in the } 2^{nd} \text{ and the } 3^{rd} \text{ rows}$$

$$\vdots$$

$$\underbrace{E_{R_n \to R_n + \lambda_n R_1} \cancel{E_{R_n}} \cdots E_{R_3 \to R_3 + \lambda_3 R_1} E_{R_2 \to R_2 + \lambda_2 R_1} A}_{}$$

$$\underbrace{\text{clears all the entries in the first row which are below } a_{11}}$$

$$Q_1 = E_{R_n \to R_n + \lambda_n R_1} \cdots \cdots E_{R_2 \to R_2 + \lambda_2 R_1}$$

$$\# \quad = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \lambda_2 & 1 & 0 & \cdots & 0 \\ \lambda_3 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ \lambda_n & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$Q_1 A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ 0 & a_{22}^{(1)} & \cdots & a_{2m}^{(1)} \\ 0 & \vdots & & \\ \vdots & & & \\ 0 & a_{m2}^{(1)} & \cdots & a_{nm}^{(1)} \end{bmatrix}$$

has been "cleared" $A^{(1)}$

$$Q_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & \text{----} & 0 \\ x & 1 & 0 & \text{---} & 0 \\ x & 0 & 1 & \text{----} & 0 \\ \vdots & \vdots & \vdots & & \\ x & 0 & 0 & \text{------} & 1 \end{bmatrix}$$

$$Q_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \text{-----} & 0 \\ 0 & 1 & 0 & 0 & \text{------} & 0 \\ 0 & \otimes & 1 & 0 & \text{-----} & 0 \\ \vdots & \vdots & \vdots & \vdots & & \\ 0 & \otimes & 0 & 0 & \text{-----} & 1 \end{bmatrix}$$

$$Q_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \text{------} & 0 \\ 0 & 1 & 0 & 0 & \text{-------} & 0 \\ 0 & 0 & 1 & 0 & \text{------} & 0 \\ 0 & 0 & \boxtimes & 1 & \text{------} & 0 \\ \vdots & \vdots & \vdots & \vdots & & \\ 0 & 0 & \boxtimes & 0 & \text{-------} & 1 \end{bmatrix}$$

and so on ....

Note that :

$$Q_1^{-1} Q_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \text{------} & 0 \\ x & 1 & 0 & 0 & \text{------} & 0 \\ x & \otimes & 1 & 0 & \text{------} & 0 \\ x & \otimes & 0 & 1 & \text{-----} & 0 \\ \vdots & \vdots & \vdots & \vdots & & \\ x & \otimes & 0 & 0 & \text{------} & 1 \end{bmatrix}$$

etc

$$\dot{Q}_1 A = A^{(1)} \implies A = Q_1^{-1} A^{(1)}$$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -\lambda_2 & 1 & 0 & \cdots & 0 \\ -\lambda_3 & 0 & 1 & & \\ \vdots & \vdots & & \ddots & \\ -\lambda_n & 0 & 0 & \cdots & 1 \end{bmatrix}$$

<u>Next</u> ➔ "clear" the elements in the 2nd column, below the 2nd pivot

$$Q_2 Q_1 A = A^{(2)} \implies A = Q_1^{-1} Q_2^{-1} A^{(2)}$$

$$A^{(2)} = \begin{bmatrix} a_{11} & a_{12} & \cdots & & a_{1m} \\ 0 & a_{22}^{(1)} & \cdots & & a_{2m}^{(1)} \\ 0 & 0 & a_{33}^{(2)} & \cdots & a_{3m}^{(2)} \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & a_{n3}^{(2)} & \cdots & a_{nm}^{(2)} \end{bmatrix}$$

The process continues until all entries below the pivots have been "cleared".

$$\cdots \cdot \cdot \cdot \cdot \cdot \cdot \cdot$$

$$A = \underbrace{Q_1^{-1} Q_2^{-1} \cdots Q_k^{-1}}_{?} A^{(k)}$$

➔ upper triangular

Example (quick way to find L)

$$A = \begin{pmatrix} 2 & 2 & 2 \\ 4 & 7 & 7 \\ 6 & 18 & 22 \end{pmatrix} \quad \longrightarrow \quad ? \ L \ \& \ U \ s.t.$$

$$A = LU$$

$$\boxed{\begin{array}{l} R_i \to R_i + \lambda R_j \\ \text{record} (-\lambda) \end{array}}$$

Solution:

$$\begin{pmatrix} 2 & 2 & 2 \\ \textcircled{4} & 7 & 7 \\ \textcircled{6} & 18 & 22 \end{pmatrix} \underset{3}{\overset{2}{\sim}} \begin{pmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & \textcircled{12} & 16 \end{pmatrix} \begin{array}{l} R_2 \to R_2 - \underline{2}R_1 \\ R_3 \to R_3 - \underline{3}R_1 \end{array} \quad \sim$$

$$\sim \quad \underset{4}{} \begin{pmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{pmatrix} \ R_3 \to R_3 - \underline{4}R_2$$

$$\underbrace{\hspace{3cm}}_{U}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \square & 1 & 0 \\ \square & \square & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} \quad \bigotimes$$

$$\underset{\text{produces}}{\underline{Q_1 A}} \longrightarrow \underset{\text{produces}}{\underline{Q_2 (Q_1 A)}} = \underset{U}{\underline{(Q_2 Q_1) A}}$$

produces
the zeros in
the 1st column

produces
the zero
in the 2nd column

U
upper
triangular

$$A = Q_1^{-1} Q_2^{-1} U$$

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

$$Q_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

$$Q_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix} \qquad Q_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix} \qquad \Bigg\} \Rightarrow$$

$$\Rightarrow Q_1^{-1} Q_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix}$$

$$\underbrace{\phantom{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix}}}_{L}$$

$$A = LU \qquad \text{where} \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix}$$

$$\left[\text{Compare with } \otimes \text{ on the previous page}\right]$$

Example (continued) $\longrightarrow$ Solving $A\underline{x} = \underline{b}$ using the LU factorisation

$$\begin{cases} 2x_1 + 2x_2 + 2x_3 = 12 \\ 4x_1 + 7x_2 + 7x_3 = 24 \\ 6x_1 + 18x_2 + 22x_3 = 12 \end{cases}$$

$$\boxed{A\underline{x} = \underline{b}}$$

$$\underline{b} = \begin{pmatrix} 12 \\ 24 \\ 12 \end{pmatrix}$$

Gaussian elimination:

$$U\underline{x} = \hat{\underline{b}}$$

$\underbrace{\phantom{U\underline{x} = \hat{\underline{b}}}}_{\text{backward substitution}}$

$$\hat{\underline{b}} = \begin{pmatrix} 6 \\ 24 \\ 70 \end{pmatrix}$$

LU decomposition

$$(LU)\underline{x} = \underline{b} \iff L\underbrace{(U\underline{x})}_{\underline{y}} = \underline{b}$$

$\boxed{\text{step 1}}$: Solve $L\underline{y} = \underline{b}$ (**forward** substitution)

$\boxed{\text{Step 2}}$: Solve $U\underline{x} = \underline{y}$ (**backward** substitution)

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 12 \\ 24 \\ 12 \end{pmatrix} \implies$$

$$\underbrace{\phantom{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix}}}_{L} \underbrace{\phantom{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}}}_{\underline{y}} \underbrace{\phantom{\begin{pmatrix} 12 \\ 24 \\ 12 \end{pmatrix}}}_{\underline{b}}$$

$$y_1 = 12$$
$$y_2 = 24 - 2y_1 = 0$$
$$y_3 = 12 - 3y_1 - 4y_2 = -24$$

$$\begin{pmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \\ -24 \end{pmatrix} \implies$$

$$\underbrace{\phantom{\begin{pmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{pmatrix}}}_{U} \underbrace{\phantom{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}}_{\underline{x}} \underbrace{\phantom{\begin{pmatrix} 12 \\ 0 \\ -24 \end{pmatrix}}}_{\underline{y}}$$

$$x_3 = -24/4 = -6$$
$$x_2 = (0 - 3x_3)/3 = 6$$
$$x_1 = (12 - 2x_2 - 2x_3)/2 = 6$$

OBS. $\boxed{A \underline{x} = \underline{b}}$ ①

- $[A \mid \underline{b}] \longrightarrow$ echelon form

$$\approx \frac{n^3}{3} \text{ operations}$$

- "clearing" a column in a $n \times n$ matrix $\longrightarrow \approx n^2$ operations.

- LU factorisation for solving ① is relatively cheap when that system must be solved for different $\underline{b}$'s

$$A\underline{x} = \underline{b}_1, \quad A\underline{x} = \underline{b}_2, \quad A\underline{x} = \underline{b}_3, \cdots$$

same coefficient matrix