

# Revision lecture #2

PART 3

Linear programming  
& Simplex algorithm

Maximise  
(or minimise)

$P(x_1, x_2, \dots, x_n)$  subject to

$$A\underline{x} \leq \underline{b}$$

← Set of inequalities in  $x_1, \dots, x_n$   
(non-trivial constraints)

$$\underline{x} \geq 0$$

↑ trivial  
constraints

$$[x_j \geq 0 \quad j=1, 2, \dots, n]$$

OBS. If  $n=2 \rightsquigarrow$  we can use the graphical method  
 $n>2 \rightsquigarrow$  requires the simplex algorithm.

Example: Maximise  $P=2x_1+3x_2$  subject to

$$x_1 + x_2 \leq 10$$

$$x_1 \geq 0, \quad x_2 \geq 0$$

$$A = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad \underline{b} = \begin{bmatrix} 10 \end{bmatrix}$$

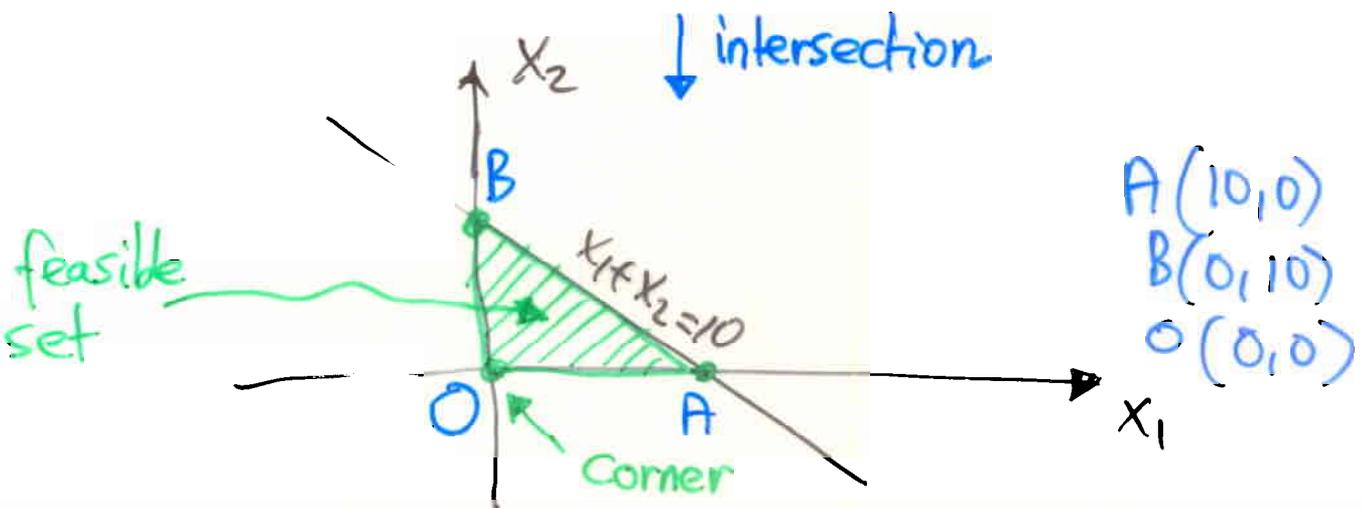
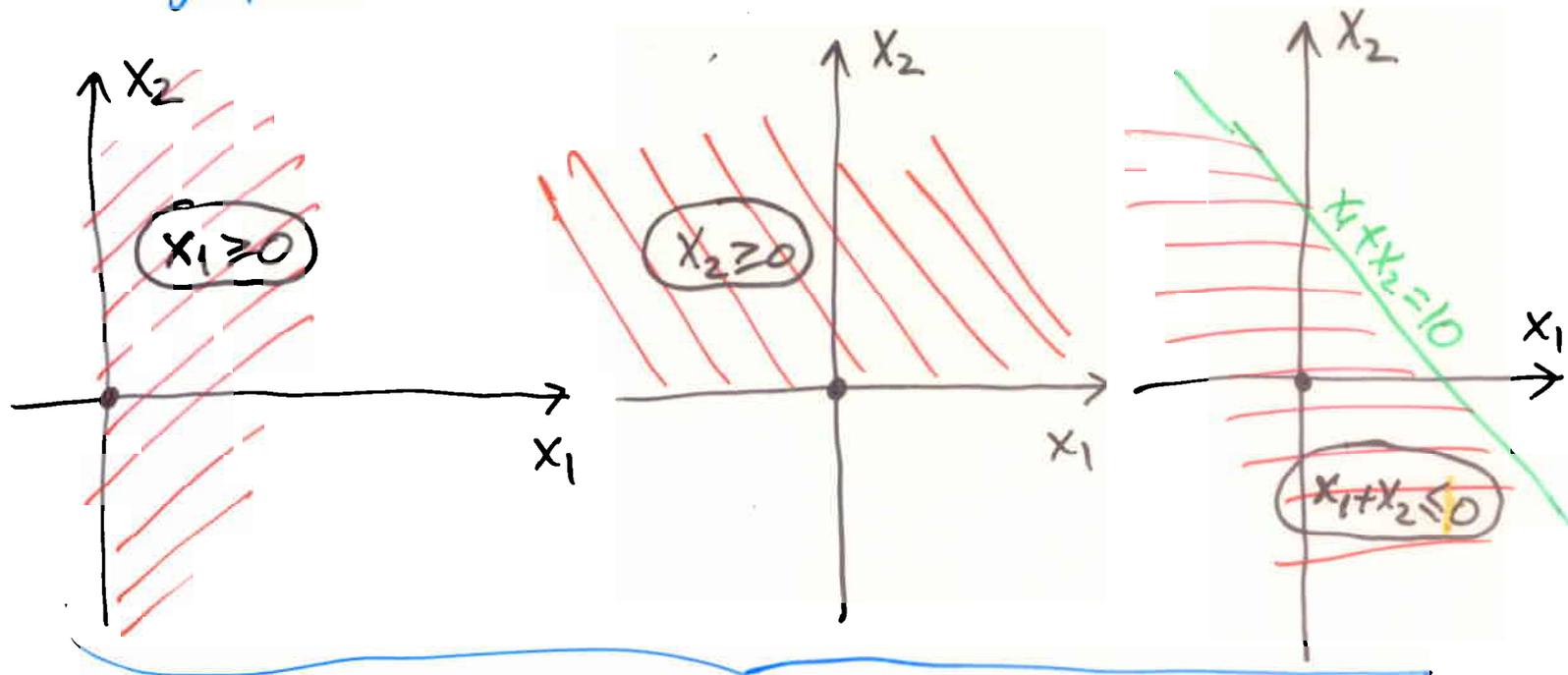
$$x_1 + x_2 \leq 10 \iff A\underline{x} \leq \underline{b} \quad \text{where } \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Another example:  $\left\{ \begin{array}{l} \text{Maximise } P = 2x_1 + 3x_2 \text{ subject to} \\ x_1 + x_2 \leq 10 \quad (1) \\ 5x_1 + 6x_2 \leq 3 \quad (2) \\ x_1 \geq 0, x_2 \geq 0 \end{array} \right.$

$$A = \begin{bmatrix} 1 & 1 \\ 5 & 6 \end{bmatrix} \quad \underline{b} = \begin{bmatrix} 10 \\ 3 \end{bmatrix}$$

Then (1) & (2)  $\Leftrightarrow A\underline{x} \leq \underline{b}$ , etc.

The graphical method for the first example:



$$P(O) = 2 \times 0 + 3 \times 0 = 0$$

$$P(A) = 2 \times 10 + 3 \times 0 = 20$$

$$P(B) = 2 \times 0 + 3 \times 10 = 30$$

$$\left. \begin{array}{l} P(O) = 2 \times 0 + 3 \times 0 = 0 \\ P(A) = 2 \times 10 + 3 \times 0 = 20 \\ P(B) = 2 \times 0 + 3 \times 10 = 30 \end{array} \right\} \Rightarrow \max P = 30 \text{ when } x_1 = 0 \text{ and } x_2 = 10$$

OBS. The more non-trivial constraints  $\rightsquigarrow$  the more difficult the problem becomes (# corners increases)

(DE 2005) Use the graphical method to find the minimum of

$$P = 3u + 2v$$

subject to  $u \geq 0, v \geq 0$ , and

$$\begin{cases} 3u + 5v \geq 15 \\ 2u + v \geq 6 \\ u + v \geq 4 \end{cases}$$

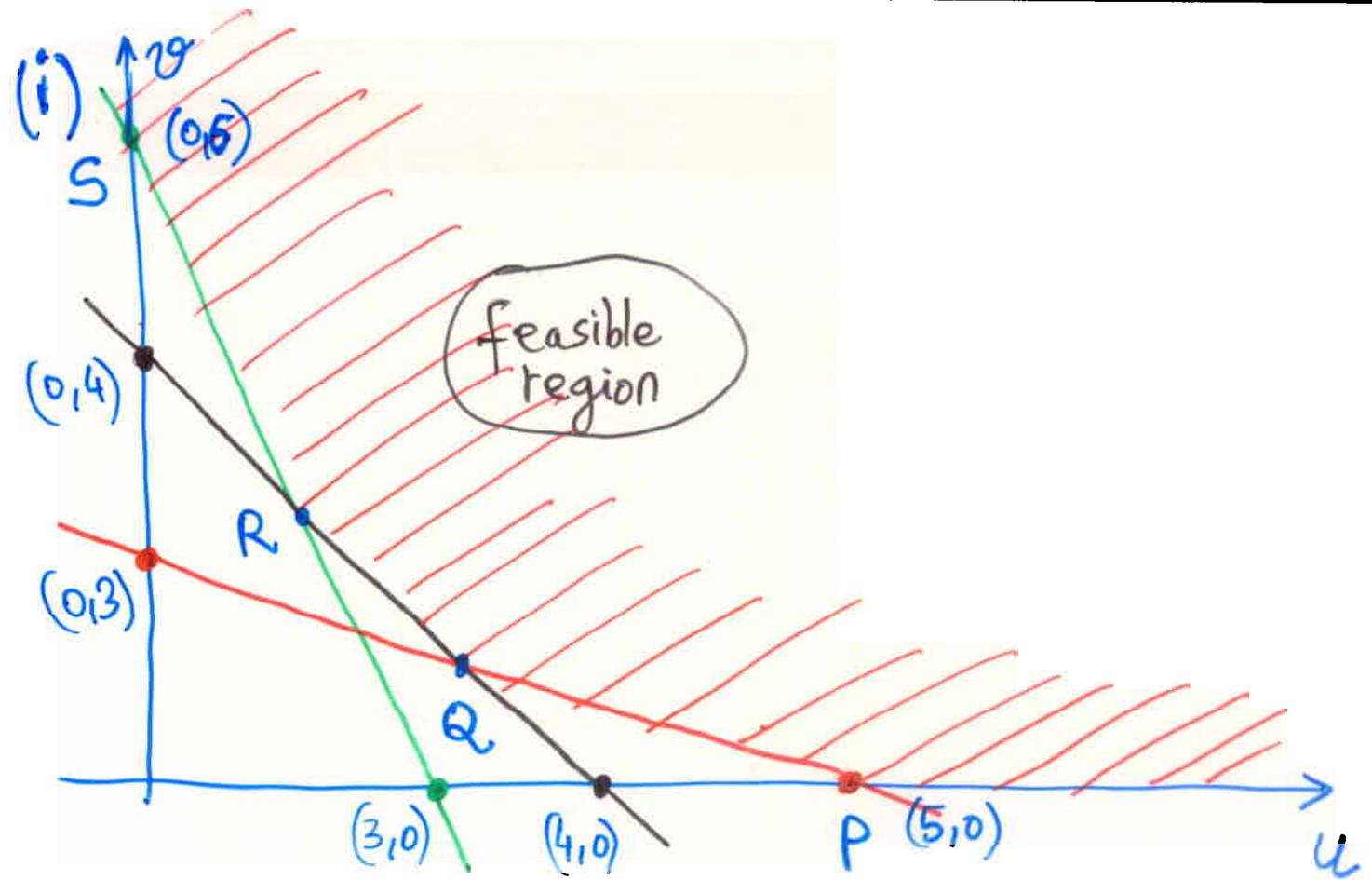
(ii) Use the simplex method to find the maximum of

$$C = 3x_1 + 2x_2$$

subject to  $x_1, x_2 \geq 0$ , and

$$3x_1 + 5x_2 \leq 15$$

$$2x_1 + x_2 \leq 6$$



$$d_1: 3u + 5v - 15 = 0 \quad (\text{red})$$

$$d_2: 2u + v - 6 = 0 \quad (\text{green})$$

$$d_3: u + v - 4 = 0 \quad (\text{black})$$

$$d_1 \cap d_2 = \{Q\}$$

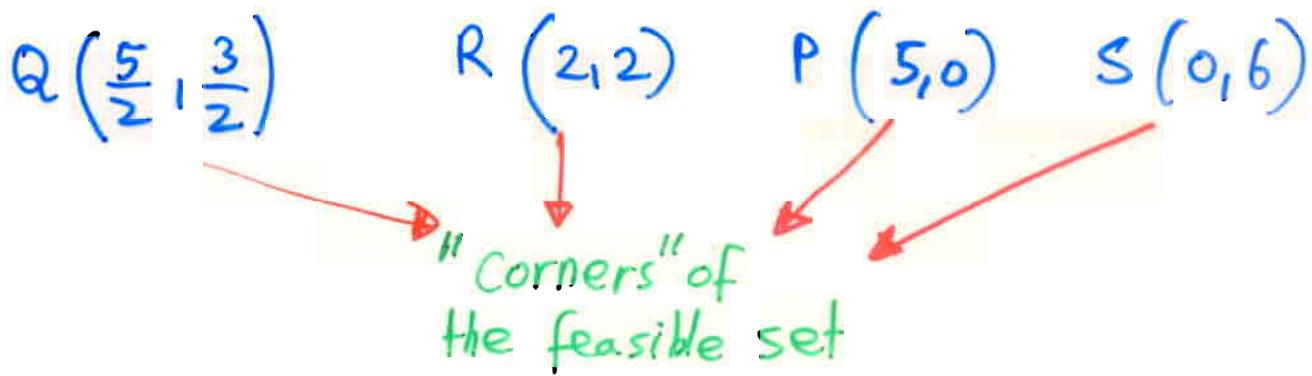
$$\begin{cases} 3u + 5v - 15 = 0 \\ u + v - 4 = 0 \end{cases}$$

$$d_2 \cap d_3 = \{R\}$$

$$\begin{cases} 2u + v = 6 \\ u + v = 4 \end{cases}$$

$$P \rightarrow d_1 \cap (u\text{-axis}) \rightsquigarrow \begin{cases} 3u + 5v - 15 = 0 \\ v = 0 \end{cases}$$

$$S \rightarrow d_2 \cap (v\text{-axis}) \rightsquigarrow \begin{cases} 2u + v - 6 = 0 \\ u = 0 \end{cases}$$



Next  $\rightarrow$  evaluate  $P(Q), P(R), P(S), P(P)$

find which one is the smallest

$$\left. \begin{aligned}
 P(Q) &= 3 \times \left(\frac{5}{2}\right) + 2 \times \left(\frac{3}{2}\right) = \frac{21}{2} \\
 P(R) &= 3 \times 2 + 2 \times 2 = 10 \\
 P(S) &= 2 \times 6 = 12 \\
 P(P) &= 3 \times 5 = 15
 \end{aligned} \right\} \Rightarrow \text{so } P_{\min} = P(R) = 10$$

## (ii) Simplex method

? max  $C = 3x_1 + 2x_2$  subject to

$$3x_1 + 5x_2 \leq 15$$

$$2x_1 + x_2 \leq 6$$

$$x_1 \geq 0, x_2 \geq 0$$

Introduce the slack variables

$$x_3 = 15 - 3x_1 - 5x_2$$

$$x_4 = 6 - 2x_1 - x_2$$

(note that  $x_3, x_4 \geq 0$ )

Idea:  $C = C(x_1, x_2, x_3, x_4)$

re-write  $C$  as  $C = \text{const.} - \sum_{j=1}^4 \lambda_j x_j$   
positive  
(but some of them = 0)

initially  $\rightsquigarrow x_1, x_2 = \text{free variables}$   
 $x_3, x_4 = \text{basic}$

$$\begin{cases} C - 3x_1 - 2x_2 & = 0 & (C) & \text{row ratios} \\ 3x_1 + 5x_2 + x_3 & = 15 & (1) & 15/3 \\ \boxed{2x_1} + x_2 + x_4 & = 6 & (2) & 6/2 \end{cases}$$

$$\begin{cases} 2C & - x_2 + 3x_4 & = 18 & 2(C) + 3(2) \\ 2x_1 + \boxed{7x_2} + 2x_3 - 3x_4 & = 12 & 2(1) - 3(2) & 12/7 \\ 2x_1 + x_2 + x_4 & = 6 & (2) & 6/1 \end{cases}$$

$$\begin{cases} 14C & + 2x_3 + 18x_4 & = 138 & 7(C) + (1) \\ 7x_2 + 2x_3 - 3x_4 & = 12 & & \\ 14x_1 - 2x_3 + 10x_4 & = 30 & 7(2) - (1) & \end{cases}$$

$$14C = 138 - 2x_3 - 18x_4 \Rightarrow C = \frac{138}{14} - \frac{2}{14}x_3 - \frac{18}{14}x_4$$

$$\Rightarrow C_{\max} = \frac{138}{14} = \frac{69}{7} \quad \text{at } x_3 = 0, x_4 = 0$$

need to express these  
in terms of  $x_1$  and  $x_2$

$$x_3 = x_4 = 0 \Rightarrow \begin{cases} x_2 = \frac{12}{7} & \text{use (1')} \\ x_1 = \frac{30}{14} = \frac{15}{7} & \text{(use (2'))} \end{cases}$$

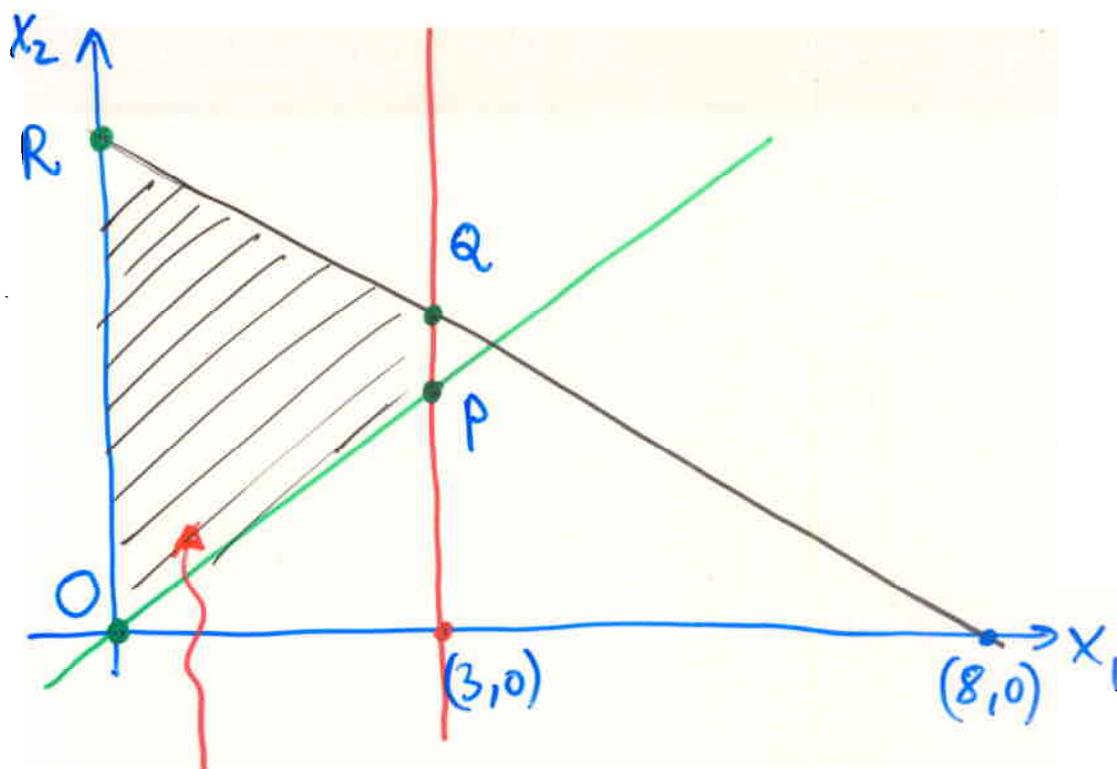
DE 2004 (i) Use the graphical method to find the maximum of  
 $V = 2x_1 + x_2$

subject to  $x_1, x_2 \geq 0$  and

$$\begin{cases} x_1 \leq 3 \\ x_2 \geq x_1 \\ 3x_1 + 4x_2 \leq 24 \end{cases}$$

(ii) Use the Simplex method to solve the same problem.

(i)  $d_1: x_1 - 3 = 0$  (red)  
 $d_2: x_1 - x_2 = 0$  (green) ← find the intersection points  
 $d_3: 3x_1 + 4x_2 - 24 = 0$  (black)



feasible region (four vertices)

$$d_1 \cap d_2 = \{P\}$$

$$\begin{cases} x_1 - 3 = 0 \\ x_1 - x_2 = 0 \end{cases} \rightsquigarrow P(3,3)$$

$$d_1 \cap d_3 = \{Q\}$$

$$\begin{cases} x_1 - 3 = 0 \\ 3x_1 + 4x_2 - 24 = 0 \end{cases} \rightsquigarrow Q\left(3, \frac{15}{4}\right)$$

$$d_3 \cap (x_2\text{-axis}) = \{R\}$$

$$\begin{cases} x_1 = 0 \\ 3x_1 + 4x_2 - 24 = 0 \end{cases} \rightsquigarrow R(0,6)$$

$$\left. \begin{array}{l} V(O) = 0 \\ V(P) = 9 \\ V(Q) = \frac{39}{4} \\ V(R) = 6 \end{array} \right\} \Rightarrow V_{\min} = \frac{39}{4} \text{ at } Q$$

(ii) Introduce slack variables

$$x_3 = 3 - x_1 \geq 0$$

$$x_4 = x_2 - x_1 \geq 0$$

$$x_5 = 24 - 3x_1 - 4x_2 \geq 0$$

Initial tableau

$V$	$-2x_1 - x_2$	$= 0$	$(V)$	
$x_1$	$+x_3$	$= 3$	$(1)$	3
$x_1 - x_2$	$+x_4$	$= 0$	$(2)$	0
$3x_1 + 4x_2$	$+x_5$	$= 24$	$(3)$	6

$V$	$-x_2 + 2x_3$	$= 6$	$(V) + 2(1)$	
$x_1$	$+x_3$	$= 3$		
$-x_2$	$-x_3 + x_4$	$= -3$	$(2) - (1)$	3
$4x_2 - 3x_3$	$+x_5$	$= 15$	$(3) - 3(1)$	$\frac{15}{4}$

$V$	$+3x_3 - x_4$	$= 9$	$(V') - (1)$	
$x_1$	$+x_3$	$= 3$		
$-x_2 - x_3 + x_4$	$+x_5$	$= -3$		3
$-7x_3 + 4x_4$	$+x_5$	$= 3$	$(3') + 4(2')$	$\frac{3}{4}$



$$4V + 5x_3 + x_5 = 39$$

$$\Rightarrow V = \frac{39}{4} - \frac{5}{4}x_3 - \frac{1}{4}x_5 \Rightarrow V_{\max} = \frac{39}{4} \text{ when } x_3 = x_5 = 0$$

(DE 2003) (i) Use the graphical method to find the maximum of

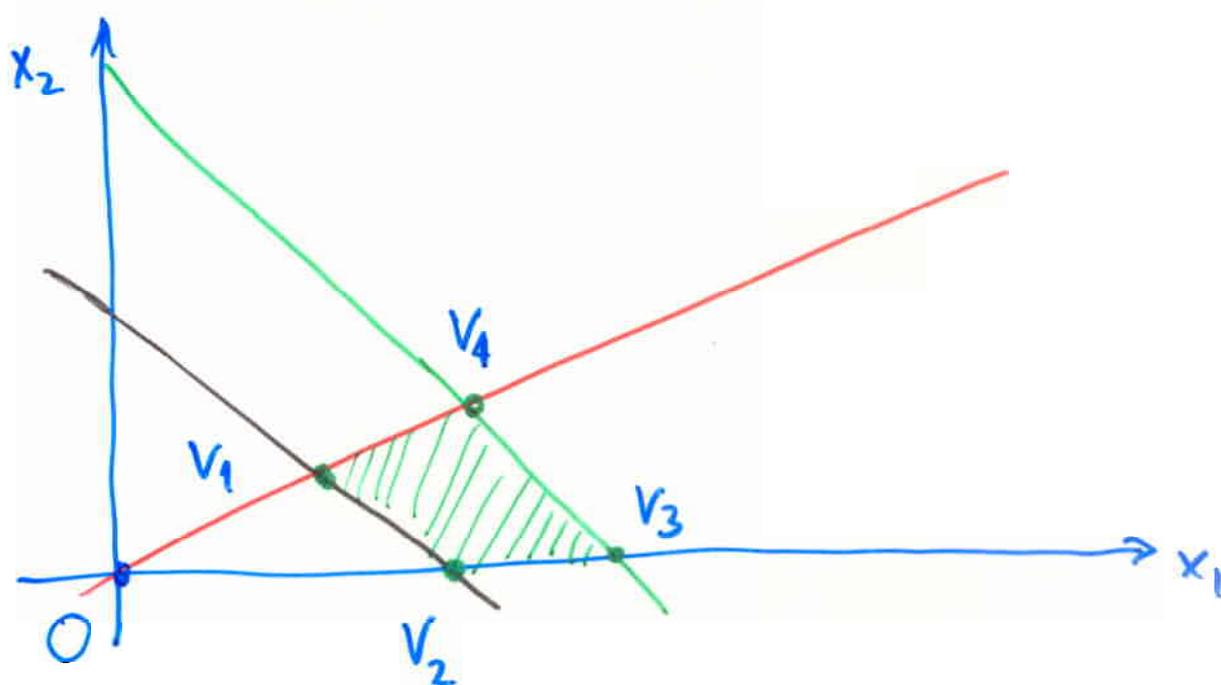
$$P = 7x_1 + 5x_2$$

subject to  $x_1 \geq 0, x_2 \geq 0$  and

$$\begin{cases} 2x_2 \leq x_1 \\ x_1 + x_2 \geq 1 \\ 4x_1 + 3x_2 \leq 6 \end{cases}$$

(ii)  $\rightsquigarrow$  same problem with SA.

$d_1: x_1 - 2x_2 = 0$  (red)  
 $d_2: x_1 + x_2 - 1 = 0$  (black)  
 $d_3: 4x_1 + 3x_2 - 6 = 0$  (green)



$$d_1 \cap d_2 = \{V_1\}$$

$$\begin{cases} x_1 = 2x_2 \\ x_1 + x_2 - 1 = 0 \end{cases} \rightsquigarrow V_1 \left( \frac{2}{3}, \frac{1}{3} \right)$$

$$d_2 \cap (x_1\text{-axis}) = \{V_2\} \quad \begin{cases} x_1 + x_2 = 1 \\ x_2 = 0 \end{cases} \rightsquigarrow V_2 (1, 0)$$

$$d_1 \cap d_3 = \{V_4\} \quad \begin{cases} x_1 - 2x_2 = 0 \\ 4x_1 + 3x_2 - 6 = 0 \end{cases} \rightsquigarrow V_4 \left(\frac{12}{11}, \frac{6}{11}\right)$$

$$d_3 \cap (x_1\text{-axis}) = \{V_3\} \quad \begin{cases} 4x_1 + 3x_2 - 6 = 0 \\ x_2 = 0 \end{cases} \rightsquigarrow V_3 \left(\frac{3}{2}, 0\right)$$

$$P(V_1) = \frac{19}{3}$$

$$P(V_2) = 7$$

$$P(V_3) = \frac{21}{2} = 10\frac{1}{2}$$

$$P(V_4) = \frac{114}{11} = 10\frac{4}{11}$$

$\Rightarrow \max P$  is  $10\frac{1}{2}$  attained at  $V_3$

$$x_1 = \frac{3}{2}, x_2 = 0$$

(ii) slack variables

$$x_3 = x_1 - 2x_2 \geq 0$$

$$x_4 = x_1 + x_2 - 1 \geq 0$$

$$x_5 = 6 - 4x_1 - 3x_2 \geq 0$$

$$P = \text{Const} - \sum_{j=1}^5 \lambda_j x_j$$

free variables:  
 $x_1, x_2$

basic:  
 $x_3, x_4, x_5$

$$\left\{ \begin{array}{l} P - 7x_1 - 5x_2 \\ x_1 - 2x_2 - x_3 \\ \boxed{x_1} + x_2 - x_4 \\ 4x_1 + 3x_2 + x_5 \end{array} \right. = \begin{array}{l} 0 \\ 0 \\ 1 \\ 6 \end{array} \left| \begin{array}{l} (P) \\ (1) \\ (2) \\ (3) \end{array} \right. \quad \begin{array}{l} \\ \\ 1 \\ 3/2 \end{array}$$

↓ "exchange"  $x_1$  and  $x_4$

$$\left\{ \begin{array}{l} P \\ \\ \\ \\ \end{array} \right. \begin{array}{l} + 2x_2 \\ - 3x_2 - x_3 \\ x_1 + x_2 \\ - x_2 \end{array} \begin{array}{l} - 7x_4 \\ + x_4 \\ - x_4 \\ + \boxed{4x_4} + x_5 \end{array} = \begin{array}{l} 7 \\ -1 \\ 1 \\ 2 \end{array} \left| \begin{array}{l} (P) + 7(2) \\ (1) - (2) \\ \\ (3) - 4(2) \end{array} \right. \quad \begin{array}{l} \\ \\ \\ 1/2 \end{array}$$

↓ "exchange"  $x_4$  and  $x_5$

$$\left\{ \begin{array}{l} 4P \\ \vdots \\ \vdots \end{array} \right. \begin{array}{l} + x_2 \\ \\ \\ \end{array} + 7x_5 = 42 \quad \left| \begin{array}{l} 4(P') + 7(3') \\ \\ \end{array} \right.$$

$$P = \frac{42}{4} - \frac{1}{4}x_2 - \frac{7}{4}x_5 \Rightarrow P_{\max} = \frac{21}{2}$$

when  $x_2 = x_5 = 0$

$$(3') \Rightarrow x_4 = \frac{1}{2}$$

$$(2') \Rightarrow x_1 = 1 + \frac{1}{2} = \frac{3}{2} \Rightarrow \boxed{x_1 = \frac{3}{2}} \quad \boxed{x_2 = 0} \quad \text{etc. ....}$$