

Revision lecture # 1

PART 1

①. Elementary row operations (ero's)

- interchanging rows
- multiplying a row by a non-zero constant
- adding a multiple of one row to another

②. The Gauß-Jordan method (finding the inverse of a matrix, if it has one)

$$A \in M_{n \times n}(\mathbb{R}) \rightsquigarrow \underbrace{[A : I_n]}_{\text{augmented matrix (AM)}}$$

perform ero's on AM to bring it to its reduced echelon form

$$[A : I_n] \sim \dots \sim [I_n : A^{-1}]$$

③. Gaussian elimination (process for solving systems of linear equations using ero's as in finding the reduced echelon form but without so much work)

pivots

→ produces pivots along the main diagonal

→ STOP when the matrix becomes upper triangular

→ SOLVE the system by backward substitution

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \cancel{a_{21}} & a_{22} & a_{23} \\ \cancel{a_{31}} & a_{32} & a_{33} \end{bmatrix} \rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \circ & b_{22} & b_{23} \\ \circ & \cancel{b_{32}} & b_{33} \end{bmatrix} \rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \circ & b_{22} & b_{23} \\ \circ & \circ & c_{33} \end{bmatrix}$$

clear

Upper triangular

OBS $A\underline{x} = \underline{b}$ → then use $[A : \underline{b}]$ in the above sequence of steps

④. Gaussian elimination with partial pivoting

choose pivots as large (in modulus) as possible

⑤. LU decomposition

$$A \in M_{n \times n}(\mathbb{R})$$

$$A = LU$$

lower triang.

upper triangular

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \otimes & 1 & 0 \\ \otimes & \otimes & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} \otimes & \otimes & \otimes \\ 0 & \otimes & \otimes \\ 0 & 0 & \otimes \end{bmatrix}$$

$\otimes \rightarrow$ entries that need to be found

$\otimes \rightarrow \dots$

⑥ LU-pivoting (LU decomposition with partial pivoting)

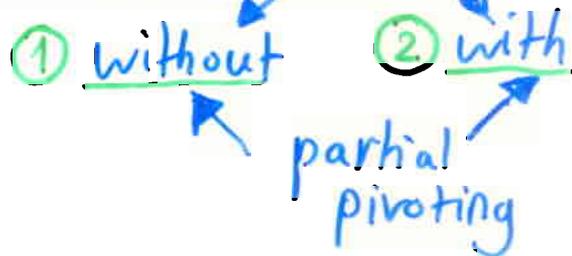
involves interchanging some of the rows

$$PA = LU$$

P = permutation matrix

(obtained from I_n by using the results in the "permutation counter column")

⑦ Solving linear systems using the LU decomposition



① $A\underline{x} = \underline{b}$ where $A = LU$

② $A\underline{x} = \underline{b}$ where $PA = LU$

$$A\underline{x} = \underline{b} \Rightarrow (LU)\underline{x} = \underline{b} \Rightarrow L(\underbrace{U\underline{x}}_{\underline{y}}) = \underline{b} \Rightarrow$$

STEP 1 solve $L\underline{y} = \underline{b}$ \rightarrow find \underline{y} (forward)

STEP 2 solve $U\underline{x} = \underline{y}$ \rightarrow find \underline{x} (backward)

for ②:

$$A\underline{x} = \underline{b} \mid P \Rightarrow PA\underline{x} = P\underline{b} \Rightarrow (LU)\underline{x} = P\underline{b} \Rightarrow L(\underbrace{U\underline{x}}_{\underline{y}}) = P\underline{b}$$

STEP 1 solve $L\underline{y} = P\underline{b}$ \rightarrow find \underline{y} , etc. . . .

STEP 2 solve $U\underline{x} = \underline{y}$ \rightarrow find \underline{x} , etc. . . .

PART 4

① Determinants

$$[A \in M_{n \times n}(\mathbb{R})]$$

• for i, j ($1 \leq i, j \leq n$) \rightsquigarrow minor M_{ij} of the entry a_{ij}

matrix obtained
by deleting $\begin{cases} \text{row } i \\ \text{column } j \end{cases}$

$$A = \begin{bmatrix} 2 & 6 & 0 \\ 3 & 9 & 3 \\ 5 & 1 & 4 \end{bmatrix}$$

$$M_{23} = \begin{bmatrix} 2 & 6 \\ 5 & 1 \end{bmatrix}, \text{ etc. . . .}$$

cofactor A_{ij} of the entry a_{ij}

$$A_{ij} = (-1)^{i+j} \det M_{ij}$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 6 \\ 5 & 1 \end{vmatrix} = -(2-30) = 28, \text{ etc, } \dots$$

Properties (of determinants)

• Cofactor expansion $\begin{cases} \text{by rows} & |A| = \sum_{j=1}^n a_{ij} A_{ij} \\ \text{by columns} & |A| = \sum_{i=1}^n a_{ij} A_{ij} \end{cases}$

• $\det I_n = 1$

• $\det(AB) = (\det A)(\det B)$ $A, B \in M_{n \times n}(\mathbb{R})$

• A invertible $\Leftrightarrow \det A \neq 0$ and $\det(A^{-1}) = \frac{1}{\det A}$

• $\det(A^T) = \det A$

• if $A = \begin{bmatrix} R+S \\ R_2 \\ \vdots \\ R_n \end{bmatrix}$ R_j ($j=2,3,\dots,n$) row vectors
 $R_1, S =$ row vectors

then $\det A = \det \begin{pmatrix} \begin{bmatrix} R \\ R_2 \\ \vdots \\ R_n \end{bmatrix} \\ \begin{bmatrix} S \\ R_2 \\ \vdots \\ R_n \end{bmatrix} \end{pmatrix} + \det \begin{pmatrix} \begin{bmatrix} S \\ R_2 \\ \vdots \\ R_n \end{bmatrix} \\ \begin{bmatrix} R \\ R_2 \\ \vdots \\ R_n \end{bmatrix} \end{pmatrix}$

• determinants & ero's ($A \in M_{n \times n}$)

Ⓘ if any pair of rows in A is interchanged \rightsquigarrow

$(R_i \leftrightarrow R_j)$

the new det changes sign

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 5 & 3 & 2 \\ 6 & 9 & 7 \end{bmatrix} \rightsquigarrow \begin{vmatrix} 5 & 3 & 2 \\ 1 & 3 & 4 \\ 6 & 9 & 7 \end{vmatrix} = - \begin{vmatrix} 1 & 3 & 4 \\ 5 & 3 & 2 \\ 6 & 9 & 7 \end{vmatrix}$$

$$\begin{vmatrix} 6 & 9 & 7 \\ 5 & 3 & 2 \\ 1 & 3 & 4 \end{vmatrix} = - \begin{vmatrix} 1 & 3 & 4 \\ 5 & 3 & 2 \\ 6 & 9 & 7 \end{vmatrix}$$

Ⓐ if $R_i \rightarrow \lambda R_i \rightsquigarrow$ new det. gets multiplied by λ

$$\begin{vmatrix} 1 & 3 & 4 \\ 10 & 6 & 4 \\ 6 & 9 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 4 \\ 2 \cdot 5 & 2 \cdot 3 & 2 \cdot 2 \\ 6 & 9 & 7 \end{vmatrix} = 2 \begin{vmatrix} 1 & 3 & 4 \\ 5 & 3 & 2 \\ 6 & 9 & 7 \end{vmatrix}$$

Ⓑ if $R_i \rightarrow R_i + \lambda R_j \rightsquigarrow$ new det. remains the same

$$\begin{vmatrix} 1 & 3 & 4 \\ 5 & 3 & 2 \\ 6 & 9 & 7 \end{vmatrix} = \begin{vmatrix} 11 & 9 & 8 \\ 5 & 3 & 2 \\ 6 & 9 & 7 \end{vmatrix} \quad \text{etc.} \dots$$

($R_1 \rightarrow R_1 + 2R_2$)

② Similar matrices, diagonalisation, etc

• $A, B \in M_{n \times n}(\mathbb{C})$ are said to be similar if

(\exists) $S \in M_{n \times n}(\mathbb{C})$ s.t. $A = SBS^{-1}$
invertible

- A is diagonalisable if $(\exists) S \in M_{n \times n}(\mathbb{C})$ s.t. invertible

$$A = SDS^{-1}$$

where $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{bmatrix}$ ($\lambda_j =$ eigenvalues of A)

OBS $S = [C_1 : C_2 : \dots : C_n]$

\uparrow \uparrow \uparrow
 eigenv. eigenv. eigenv.
 λ_1 λ_2 λ_n

- $A^k = SD^k S^{-1}$ $D^k = \begin{bmatrix} \lambda_1^k & & 0 \\ & \lambda_2^k & \\ 0 & & \ddots \\ & & & \lambda_n^k \end{bmatrix}$ ($\forall k \geq 1$)

Cramer's Rule

$$A^{-1} = \frac{1}{\det A} \text{Adj}(A)$$

adjoint of A = transpose of the matrix of cofactors of A

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{\text{original matrix}} \rightsquigarrow \underbrace{\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}}_{\text{matrix of cofactors}} \rightsquigarrow \underbrace{\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}}_{\text{Adj}(A)}$$

③ Eigenvectors & eigenvalues

$$A \in M_{n \times n}(\mathbb{C})$$

(λ, \underline{x}) is an eigenpair if $A \underline{x} = \lambda \underline{x}$, $\underline{x} \neq \underline{0}$

\downarrow \downarrow
eigenvector eigenvalue

to find λ : \rightsquigarrow use the characteristic equation of A

$$\det(A - \lambda I_n) = 0$$

\downarrow has n solutions

to find \underline{x} : \rightsquigarrow solve $(A - \lambda I_n) \underline{x} = \underline{0}$

[for each eigenvalue]

(0)

Example for G-J method

$$\left(\begin{array}{ccc|ccc} 2 & 4 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 3 & 5 & 7 & 0 & 0 & 1 \end{array} \right) \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3/2 & 1/2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 3 & 5 & 7 & 0 & 0 & 1 \end{array} \right]_{R_1 \rightarrow \frac{1}{2}R_1}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3/2 & 1/2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 5/2 & -3/2 & 0 & 1 \end{array} \right]_{R_3 \rightarrow R_3 - 3R_1}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 7/2 & 1/2 & -2 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 3/2 & -3/2 & 1 & 1 \end{array} \right]_{\substack{R_1 \rightarrow R_1 - 2R_2 \\ R_3 \rightarrow R_3 + R_2}}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 7/2 & 1/2 & -2 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2/3 & 2/3 \end{array} \right]_{R_3 \rightarrow R_3 \times \frac{2}{3}}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -13/3 & -7/3 \\ 0 & 1 & 0 & -1 & 5/3 & 2/3 \\ 0 & 0 & 1 & -1 & 2/3 & 2/3 \end{array} \right]_{\substack{R_1 \rightarrow R_1 - (7/2)R_3 \\ R_2 \rightarrow R_2 + R_3}}$$



 A^{-1}

(DE 2004) GE without PP \rightarrow find LU decomp for

1

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 3 & 2 \\ 2 & 6 & 3 \end{bmatrix}$$

Use this to solve

$$\begin{cases} x + 2y + z = 8 \\ -x + 3y + 2z = 21 \\ 2x + 6y + 3z = 28 \end{cases}$$

Solution

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 3 & 2 \\ 2 & 6 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 3 \\ 0 & 2 & 1 \end{bmatrix} \begin{array}{l} R_2 \rightarrow R_1 + R_2 \\ R_3 \rightarrow R_3 - 2R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 3 \\ 0 & 0 & -\frac{1}{5} \end{bmatrix} \begin{array}{l} R_3 \rightarrow R_3 - \frac{2}{5}R_2 \end{array}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 2/5 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 3 \\ 0 & 0 & -1/5 \end{bmatrix}$$

$$A \underline{x} = \underline{b}$$

$$b = \begin{bmatrix} 8 \\ 21 \\ 28 \end{bmatrix}$$

$$\underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$(LU)x = \underline{b} \Rightarrow \underbrace{L(Ux)}_{\underline{y}} = \underline{b}$$

(2)

(1) $L\underline{y} = \underline{b}$

(2) $U\underline{x} = \underline{y}$

$$\text{From (1)} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 2/5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 21 \\ 28 \end{bmatrix} \Rightarrow \begin{cases} y_1 = 8 \\ y_2 = 29 \\ y_3 = 2/5 \end{cases}$$

$$(2) \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 3 \\ 0 & 0 & -1/5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 29 \\ 2/5 \end{bmatrix} \Rightarrow \begin{cases} z = -2 \\ y = 7 \\ x = -4 \end{cases}$$

(DE 2004) GE with P.P. \rightsquigarrow find P and L, U s.t.

$$PA = LU$$

$$A = \begin{bmatrix} 2 & -1 & 4 \\ 1 & 0 & 2 \\ 3 & 1 & -5 \end{bmatrix}$$

Then solve $A\underline{x} = \underline{b}$ $\underline{b} = \begin{bmatrix} 17 \\ 8 \\ -10 \end{bmatrix}$

$$AM = \left[\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ 1 & 0 & 2 & 2 \\ 3 & 1 & -5 & 3 \end{array} \right] \sim$$

$$\sim \left[\begin{array}{ccc|c} 3 & 1 & -5 & 3 \\ 1 & 0 & 2 & 2 \\ 2 & -1 & 4 & 1 \end{array} \right] R_1 \leftrightarrow R_2$$

$$\sim \begin{matrix} \frac{1}{3} \\ \frac{2}{3} \end{matrix} \left[\begin{array}{ccc|c} 3 & 1 & -5 & 3 \\ 0 & -1/3 & 11/3 & 2 \\ 0 & -5/3 & 22/3 & 1 \end{array} \right] \begin{matrix} R_2 \rightarrow R_2 - \frac{1}{3} R_1 \\ R_3 \rightarrow R_3 - \frac{2}{3} R_1 \end{matrix}$$

$$\sim \begin{matrix} \frac{1}{3} \\ \frac{3}{2} \end{matrix} \left[\begin{array}{ccc|c} 3 & 1 & -5 & 3 \\ 0 & -5/3 & 22/3 & 1 \\ 0 & -1/3 & 11/3 & 2 \end{array} \right] R_2 \leftrightarrow R_3$$

$$\sim \begin{matrix} \frac{1}{5} \end{matrix} \left[\begin{array}{ccc|c} 3 & 1 & -5 & 3 \\ 0 & -5/3 & 22/3 & 1 \\ 0 & 0 & 11/5 & 2 \end{array} \right] \equiv U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & 1/5 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$Ax = b$$

$$\underline{b} = \begin{bmatrix} 17 \\ 8 \\ -10 \end{bmatrix}$$

(2)

$$PAx = P\underline{b}$$

$$P\underline{b} = \begin{bmatrix} -10 \\ 17 \\ 8 \end{bmatrix}$$

↓

$$\underbrace{L(Ux)}_{\underline{y}} = P\underline{b}$$

$$(1) \quad L\underline{y} = P\underline{b}$$

$$\Rightarrow \begin{cases} y_1 = -10 \\ y_2 = 71/3 \\ y_3 = 33/5 \end{cases}$$

$$(2) \quad L\underline{x} = \underline{y}$$

$$\Rightarrow \begin{cases} \underline{x} = 3 \\ y = -1 \\ x = 2 \end{cases}$$

DE 2004

$$\begin{vmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 2 \\ 3 & 3 & 4 & 4 \\ 4 & 4 & 5 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & 0 & -2 \\ 0 & -3 & 1 & -2 \\ 0 & -4 & 1 & -3 \end{vmatrix} \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \\ R_4 \rightarrow R_4 - 4R_1 \end{array}$$

$$= \begin{vmatrix} -3 & 0 & -2 \\ -3 & 1 & -2 \\ -4 & 1 & -3 \end{vmatrix} = \begin{vmatrix} -3 & 0 & -2 \\ 1 & 0 & 1 \\ -4 & 1 & -3 \end{vmatrix} = (-1)^{3+2} \begin{vmatrix} -3 & -2 \\ 1 & 1 \end{vmatrix} = +1$$

DE 2003

$$\begin{vmatrix} 2 & 1 & 2 & 2 \\ 5 & -7 & 2 & 1 \\ 1 & 0 & -1 & -3 \\ 4 & 1 & -3 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 2 & 2 \\ 19 & 0 & 16 & 15 \\ 1 & 0 & -1 & -3 \\ 2 & 0 & -5 & -2 \end{vmatrix}$$

$$= (-1)^{2+1} \begin{vmatrix} 19 & 16 & 15 \\ 1 & -1 & -3 \\ 2 & -5 & -2 \end{vmatrix} = - \begin{vmatrix} 19 & 35 & 72 \\ 1 & 0 & 0 \\ 2 & -3 & 4 \end{vmatrix}$$

$$\begin{array}{r} 19 \\ \hline 57 \\ 15 \\ \hline 72 \end{array}$$

$$= -(-1)^{2+1} \begin{vmatrix} 35 & 72 \\ -3 & 4 \end{vmatrix} = 1 \cdot 35 + 3 \cdot 72 = 140 + 216 = 356$$

$$= 356$$