

Example $A = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix} \xrightarrow{?} \text{diagonalise}$

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 8 & 16 \\ 4 & 1-\lambda & 8 \\ -4 & -4 & -11-\lambda \end{vmatrix} = \begin{vmatrix} 5-\lambda & 8 & 16 \\ 0 & -(\lambda+3) & -(\lambda+3) \\ -4 & -4 & -11-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 5-\lambda & 8 & 8 \\ 0 & -(\lambda+3) & 0 \\ -4 & -4 & -7-\lambda \end{vmatrix} = -(\lambda+3) \begin{vmatrix} 5-\lambda & 8 \\ -4 & -7-\lambda \end{vmatrix}$$

$$= -(\lambda+3) \begin{vmatrix} 1-\lambda & 1-\lambda \\ -4 & -7-\lambda \end{vmatrix} = -(\lambda+3)(1-\lambda)(-7-\lambda+4) \\ = -(\lambda+3)^2(\lambda-1)$$

Eigenvalues $\lambda_1 = -3, \lambda_2 = -3, \lambda_3 = 1$
 repeated eigenv.

To diagonalise $\xrightarrow{}$ $S = [P_1 : P_2 : P_3]$
 eigenvecs. for $\lambda_1 = -3$ ↓ eigenvector for λ_3

[will have to be linearly independent]

$$A + 3I_3 = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 8 & 16 \\ 4 & 4 & 8 \\ -4 & -4 & -8 \end{bmatrix}$$

Eigenvector for $\lambda_1 = \lambda_2 = -3$:

$$\begin{bmatrix} 8 & 8 & 16 \\ 4 & 4 & 8 \\ -4 & -4 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$8x_1 + 8x_2 + 16x_3 = 0 \Rightarrow x_1 + x_2 + 2x_3 = 0$$

$$x_1 = -\alpha - 2\beta \quad (x_2 = \alpha, x_3 = \beta)$$

arbitrary
real numbers

$$\begin{bmatrix} -\alpha - 2\beta \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -\alpha \\ \alpha \\ 0 \end{bmatrix} + \begin{bmatrix} -2\beta \\ 0 \\ \beta \end{bmatrix}$$

$$= \alpha \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{p_1} + \beta \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}_{p_2}$$

$$p_1 \rightarrow \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$p_2 \rightarrow \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{For } \lambda = 1 \rightarrow p_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

(nothing special about this one)

$$S = \begin{bmatrix} \begin{matrix} -1 \\ 1 \\ 0 \end{matrix} & \begin{matrix} -2 \\ 0 \\ 1 \end{matrix} & \begin{matrix} 2 \\ 1 \\ -1 \end{matrix} \end{bmatrix}$$

eigenvectors
for $\lambda_1 = \lambda_2 = -3$

eigenvector
for $\lambda_3 = 1$

$$D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Check $A = SDS^{-1}$, etc....

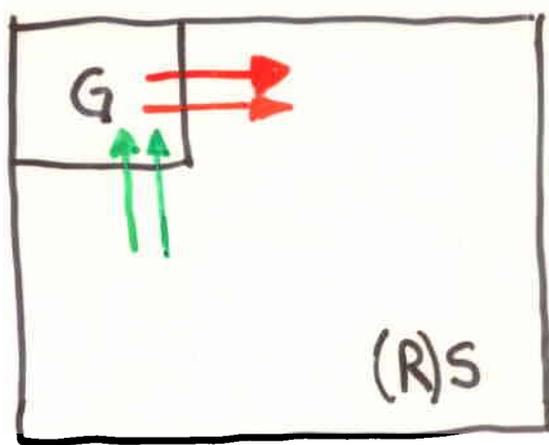
OBS S can also be taken as

$$S = \begin{bmatrix} -2 & -1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}, \text{ etc. .}$$

Markov Processes

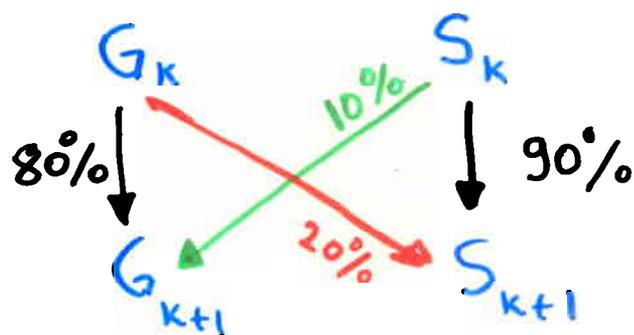
Every year 20% of the population of Glasgow leave to the other parts of Scotland, and in this same period 10% of the people from the rest of Scotland move to Glasgow.

What is the population distribution between Glasgow and the rest of Scotland after n years?



$G \rightarrow RS$ 20%
 $RS \rightarrow G$ 10%

year k :



year $(k+1)$:

G_k = population of Glasgow in year k

S_k = " " " " Scotland " " "

$$G_{k+1} = 80\% G_k + 10\% S_k$$

$$S_{k+1} = 20\% G_k + 90\% S_k$$



$$\underbrace{\begin{bmatrix} G_{k+1} \\ S_{k+1} \end{bmatrix}}_{\underline{u}_{k+1}} = \underbrace{\begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}}_A \underbrace{\begin{bmatrix} G_k \\ S_k \end{bmatrix}}_{\underline{u}_k}$$

⇓

$$\underline{u}_{k+1} = A \underline{u}_k$$

for all $k \geq 0$

recurrence
relation

$$\underline{u}_1 = A \underline{u}_0$$

$$\underline{u}_2 = A \underline{u}_1 = A(A \underline{u}_0) = A^2 \underline{u}_0$$

$$\underline{u}_3 = A \underline{u}_2 = A(A^2 \underline{u}_0) = A^3 \underline{u}_0$$

⋮

$$\underline{u}_n = A \underline{u}_{n-1} = A(A^{n-1} \underline{u}_0) = A^n \underline{u}_0$$

$$\underline{u}_n = A^n \underline{u}_0$$

→ gives the evolution
of the population distribution

Question ⇒ What happens as $n \rightarrow \infty$?

Hint: diagonalise A

$$\det(A - \lambda I) = \begin{vmatrix} 0.8 - \lambda & 0.1 \\ 0.2 & 0.9 - \lambda \end{vmatrix} = \lambda^2 - 1.7\lambda + 0.7$$

$$= \lambda^2 - \lambda - 0.7\lambda + 0.7$$

$$= \lambda(\lambda - 1) - 0.7(\lambda - 1)$$

$$= (\lambda - 0.7)(\lambda - 1)$$

Eigenvalues $\rightarrow \lambda_1 = 1$ and $\lambda_2 = 0.7$

Next \rightarrow find the corresponding eigenvectors

$$\boxed{\lambda_1 = 1} \quad A - \lambda_1 I = A - I = \begin{bmatrix} -0.2 & 0.1 \\ 0.2 & -0.1 \end{bmatrix}$$

$$\sim \begin{bmatrix} -0.2 & 0.1 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \quad -2x_1 + x_2 = 0$$

i.e. $x_2 = 2x_1$

eigenvector $\rightarrow x_1 = \alpha, \quad x_2 = 2\alpha$ $\underline{x} = \begin{bmatrix} \alpha \\ 2\alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\downarrow$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\boxed{\lambda_2 = 0.7} \quad A - \lambda_2 I = A - 0.7I = \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

eigenvector $\rightsquigarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$A = SDS^{-1}$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 0.7 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \rightsquigarrow S^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

Aside

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^n = SD^nS^{-1}$$

$$= \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \underbrace{\begin{bmatrix} 1^n & 0 \\ 0 & 0.7^n \end{bmatrix}}_{D^n} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1 + 2(0.7)^n & 1 - 0.7^n \\ 2(1 - 0.7^n) & 2 + 0.7^n \end{bmatrix}$$

$$\underline{u}_n = A^n \underline{u}_0$$

$$\underline{u}_0 = \begin{bmatrix} G_0 \\ S_0 \end{bmatrix} \quad \underline{u}_n = \begin{bmatrix} G_n \\ S_n \end{bmatrix}$$

$$\Downarrow$$
$$\begin{bmatrix} G_n \\ S_n \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 + 2(0.7)^n & 1 - 0.7^n \\ 2(1 - 0.7^n) & 2 + 0.7^n \end{bmatrix} \begin{bmatrix} G_0 \\ S_0 \end{bmatrix}$$



$$= \frac{1}{3} \begin{bmatrix} [1 + 2(0.7)^n]G_0 + (1 - 0.7^n)S_0 \\ 2(1 - 0.7^n)G_0 + (2 + 0.7^n)S_0 \end{bmatrix}$$

$$= \frac{1}{3} \left\{ G_0 \begin{bmatrix} 1 + 2(0.7)^n \\ 2(1 - 0.7^n) \end{bmatrix} + S_0 \begin{bmatrix} 1 - 0.7^n \\ 2 + 0.7^n \end{bmatrix} \right\}$$

$$\textcircled{*} = \frac{1}{3} \left\{ G_0 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2G_0(0.7)^n \begin{bmatrix} 1 \\ -1 \end{bmatrix} + S_0 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - S_0(0.7)^n \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

$$= \frac{1}{3} (G_0 + S_0) \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{0.7^n}{3} (2G_0 - S_0) \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Aside

$$\begin{bmatrix} a+b \\ c+d \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} + \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} a \\ d \end{bmatrix} + \begin{bmatrix} b \\ c \end{bmatrix} \quad \textcircled{*}$$

$$\lim_{n \rightarrow \infty} 0.7^n = 0 \Rightarrow \lim_{n \rightarrow \infty} u_n = \frac{1}{3} (G_0 + S_0) \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

i.e.
$$\begin{cases} G_n \rightarrow \frac{1}{3} (G_0 + S_0) \\ S_n \rightarrow \frac{2}{3} (G_0 + S_0) \end{cases} \quad \text{as } n \rightarrow \infty$$

OBS. (I) It is easier to calculate A^n as $n \rightarrow \infty$ before we perform the multiplication in \boxtimes

(II) The limits of G_n & S_n as $n \rightarrow \infty$ depend only on G_0 & S_0 . They are in the same proportion as the entries in any eigenvector corresponding to $\lambda_1 = 1$.

A Markov process with n state variables

$$\underline{u}^{(k)} = [x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}] \quad (k \geq 0)$$

is characterised by a recurrence of the form

$$\underline{u}^{(k+1)} = A \underline{u}^{(k)}$$

where $A \in M_{n \times n}(\mathbb{R})$ is the transition matrix, and

$$x_i^{(k)} \geq 0, \quad x_1^{(k)} + x_2^{(k)} + \dots + x_n^{(k)} = \text{const.} \quad (\forall k \geq 0)$$

In addition, the entries of A are numbers ≥ 0 , and the sum of the elements in each column is 1.

↓ the example discussed involved a Markov process with 2 state variables, G_k and S_k , respectively

Column & row sums

$$A \in M_{m \times n}(\mathbb{R})$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$$A = [\underline{c}_1; \underline{c}_2; \underline{c}_3; \dots; \underline{c}_n] = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix}$$

\underline{c}_i = column i of A

r_j = row j of A

$$(1 \leq i \leq n, 1 \leq j \leq m)$$

Note that:

$$[1: 1: \dots : 1] A = [\sum \underline{c}_1; \sum \underline{c}_2; \dots; \sum \underline{c}_n]$$

$$A \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \sum r_1 \\ \sum r_2 \\ \vdots \\ \sum r_m \end{bmatrix}$$

If $A \in M_{n \times n}(\mathbb{R})$ is a Markov matrix $\Rightarrow \sum \underline{c}_i = 1$

$$\Rightarrow [1: 1: 1: \dots : 1] A = [1: 1: 1: \dots : 1]$$

take T

$$\Rightarrow A^T [1, 1, \dots, 1]^T = [1, 1, \dots, 1]^T$$

i.e. $\lambda = 1$ is an eigenvalue of A^T , eigenvalue of A as well

Positive matrices & Markov processes

$A \in M_{m \times n}(\mathbb{R})$ is said to be non-negative if $a_{ij} \geq 0$
for all i and j

positive if $a_{ij} > 0$
for all i and j

~~Ass~~

$$A \in M_{n \times n}(\mathbb{R}) \rightsquigarrow \mu \stackrel{\text{def}}{=} \max \{ |\lambda| : \lambda \text{ eigenv. } A \}$$

A positive

Then μ is an eigenv. A

(\exists) \underline{p} positive eigenvector associated to μ

$$A \underline{p} = \mu \underline{p}$$

Other facts.

• every other eigenvalue has modulus $< \mu$

•
$$\underbrace{\min_i \left(\sum_j a_{ij} \right)}_{\text{least row sums}} \leq \mu \leq \underbrace{\max_i \left(\sum_j a_{ij} \right)}_{\text{largest row sums}}$$

• $(\mu^{-1} A)^k \underline{x} \longrightarrow (\text{Const.}) \times \underline{p}$ as $k \rightarrow \infty$
for all \underline{x}

• If $A \in M_{n \times n}(\mathbb{R})$ is non-negative s.t. A^q is positive
(for some $q \geq 2$)

Then $(\mu^{-1}A)^k \underline{x} \longrightarrow (\text{const.}) \times \underline{p}$ as $k \rightarrow \infty$
 \downarrow justification:

$$A = SDS^{-1} \Rightarrow A^k = SD^kS^{-1}$$

$$\left. \begin{aligned} (\mu^{-1}A)^k &= \underbrace{(\mu^{-1}A)(\mu^{-1}A)\dots(\mu^{-1}A)}_{k \text{ times}} = \mu^{-k}A^k \end{aligned} \right\} \Rightarrow$$

$$\Rightarrow (\mu^{-1}A)^k \underline{x} = \mu^{-k}SD^kS^{-1}\underline{x}$$

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \Rightarrow D^k = \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{bmatrix}$$

Let $\underline{v} = S^{-1}\underline{x}$ (vector)

Then

$$(\mu^{-1}A)^k \underline{x} = S \begin{bmatrix} \left(\frac{\lambda_1}{\mu}\right)^k & & & \\ & \left(\frac{\lambda_2}{\mu}\right)^k & & \\ & & \ddots & \\ & & & \left(\frac{\lambda_n}{\mu}\right)^k \end{bmatrix} \underline{v} \quad (1)$$

$$\mu = \max \{ |\lambda| : \lambda \text{ eig. } A \} \Rightarrow \left| \frac{\lambda_i}{\mu} \right| < 1 \text{ except for } \lambda_i = \mu$$

