

Linear Algebra

$$U = \{u_1, u_2, \dots, u_m\}$$

$$\alpha_j \in \mathbb{R} \quad (1 \leq j \leq m)$$

$$\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_m u_m$$

linear combination of u_1, u_2, \dots, u_m

F_n = the set of all n -dimensional vectors with entries in F
($F \equiv \mathbb{R}$ or $F \equiv \mathbb{C}$)

$$u \in F_n \Rightarrow u = (x_1, x_2, \dots, x_n)$$

$$x_j \in \mathbb{R} \quad (1 \leq j \leq n)$$

• linear subspace: $V \subset F_n$

↓
if $\alpha_1 u_1 + \alpha_2 u_2 \in V$

for all $u_1, u_2 \in V$
and $\alpha_1, \alpha_2 \in \mathbb{R}$

OBS A linear combination of linear combinations is again a linear combination



V is linear subspace \Rightarrow it contains all linear combinations of its elements

• the span of a set $U = \{u_1, u_2, \dots, u_m\} \subset F_n$:



$$\left\{ \underline{v} \in F_n \mid \underline{v} = \sum_{j=1}^n \alpha_j \underline{u}_j \text{ for some } \alpha_j \in F \right\}$$

(the set of all vectors that can be written as linear comb. of the set of vectors U)

Example: $\underline{u}_1 = (1, 0)$ $\underline{u}_2 = (0, 1)$

What is the span of $U = \{\underline{u}_1, \underline{u}_2\}$?

$$\mathbb{R}_2 = \{(a, b) \mid a, b \in \mathbb{R}\}$$

$$(a, b) = a(1, 0) + b(0, 1) = a\underline{u}_1 + b\underline{u}_2 \in \text{span}\{\underline{u}_1, \underline{u}_2\}$$

• linear independence :

$$\text{let } U = \{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_m\}$$

U is said to be linearly independent in F_n if:

$$\alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2 + \dots + \alpha_m \underline{u}_m = \underline{0} \Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_m = 0$$

linearly dependent \Rightarrow NOT linearly independent, i.e.

$(\exists) \alpha_1, \alpha_2, \dots, \alpha_m$ s.t. $\alpha_1^2 + \alpha_2^2 + \dots + \alpha_m^2 \neq 0$ and not all zero
there exists

$$\alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2 + \dots + \alpha_m \underline{u}_m = \underline{0}$$

Examples **I** $\underline{u}_1 = (1, 0)$ $\underline{u}_2 = (0, 1) \xrightarrow{?}$ l.i.

$$\alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2 = \underline{0} \xrightarrow{?} \alpha_1, \alpha_2 = 0$$

$$\alpha_1 (1, 0) + \alpha_2 (0, 1) = (0, 0)$$

$$(\alpha_1, \alpha_2) = (0, 0) \Rightarrow \alpha_1 = \alpha_2 = 0$$

Thus \underline{u}_1 and \underline{u}_2 are l.i.

II $\underline{u}_1 = (1, 2)$ $\underline{u}_2 = (3, 5)$ $\underline{u}_3 = (1, 1) \xrightarrow{?}$ l.i.

$$\alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2 + \alpha_3 \underline{u}_3 = \underline{0} \xrightarrow{?} \alpha_1, \alpha_2, \alpha_3 = 0$$

$$\alpha_1 (1, 2) + \alpha_2 (3, 5) + \alpha_3 (1, 1) = (0, 0)$$

$$(\alpha_1 + 3\alpha_2 + \alpha_3, 2\alpha_1 + 5\alpha_2 + \alpha_3) = (0, 0)$$

i.e. $\begin{cases} \alpha_1 + 3\alpha_2 + \alpha_3 = 0 \\ 2\alpha_1 + 5\alpha_2 + \alpha_3 = 0 \end{cases} \rightarrow \dots \begin{matrix} \alpha_1 = 2\alpha_3 \\ \alpha_2 = -\alpha_3 \\ \alpha_3 \neq 0 \end{matrix} \rightarrow \underline{\underline{\text{l.i.d.}}}$

Thus $\underline{u}_1, \underline{u}_2,$ and \underline{u}_3 are linearly dependent.

$U \subset F_n$, U linearly dependent

\Downarrow

$(\exists) V \subset U$ s.t. V is l.i. and $\text{span}(V) = U$

A **basis** is a linearly independent spanning set;

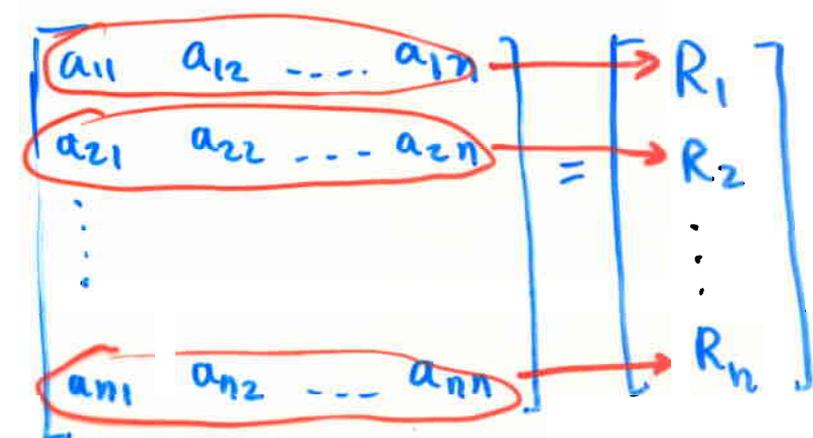
$B \subset F_n$ basis if $\begin{cases} B \text{ is l.i.} \\ \text{span}(B) = F_n \end{cases}$

Example: $u_1 = (1, 0)$, $u_2 = (0, 1)$

$B = \{u_1, u_2\}$ forms a basis of \mathbb{R}^2

The **row space** $RS(A)$ of $A \in M_{n \times n}(F)$

linear space generated by its rows



$$RS(A) = \text{span}\{R_1, \dots, R_n\}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightsquigarrow I_2$$

$$RS(I_2) = \text{Span}\{R_1, R_2\} = \mathbb{R}^2$$

$$R_1 = (1, 0) \quad R_2 = (0, 1)$$

In general $RS(I_n) = F_n$

OBS. If $B = EA$

$E =$ elementary matrix

$$RS(B) = RS(A)$$

Next $\rightarrow U = \{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_m\} \subset F_n$

when is U a basis of F_n ?

$$A_U = \begin{bmatrix} \underline{u}_1 \\ \underline{u}_2 \\ \vdots \\ \underline{u}_m \end{bmatrix} \in M_{m \times n}(F) \xrightarrow{\text{ero's}} B_U$$

its reduced echelon form

$$B_U = P A_U$$

$P =$ product of elementary matrices

$$RS(B_U) = RS(A_U)$$

If $m > n \rightarrow$ Some row of B_U must vanish \rightarrow

Some linear combination of rows of A_U must vanish as well

\rightarrow Some linear combination of U must vanish $\rightarrow U$ is l.d.

Thus $m \leq n$ if U is a basis of F_n 

If $\text{span}(U) = F_n \Rightarrow B_U$ must have at least n non-zero rows

\downarrow
 $m \geq n$  

Theorem A subset $U = \{u_1, u_2, \dots, u_m\}$ of F_n is a basis if it satisfies any two of the following conditions, in which case it also satisfies the third

(i) $m = n$

(ii) $\text{span}(U) = F_n$

(iii) U is l.i.

vectors $\left\{ \begin{array}{l} \text{column vectors} \\ \text{row vectors} \end{array} \right.$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leftarrow x \in \mathbb{R}^3$

$[x_1 \ x_2 \ x_3] \leftarrow x \in \mathbb{R}_3$

F^n = the set of all n -dimensional column vectors

$$A \in M_{n \times n}(F)$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} = [C_1 \ C_2 \ \dots \ C_n]$$

⋮
etc

$A \in M_{n \times n}(F)$ is invertible \Leftrightarrow $SR(A)$ = basis for F^n

$A \in M_{n \times n}(F)$ is invertible \Leftrightarrow $SC(A)$ = basis for F^n

Similar matrices

$$A, B \in M_{n \times n}(F)$$

$$F = \mathbb{R} \text{ or } \mathbb{C}$$

A and B are **similar** if there exists $S \in M_{n \times n}(F)$ s.t.
invertible

$$A = SBS^{-1}$$

⚡
? use

A^k in terms of B^k

$$A = SBS^{-1}$$

$$A^2 = AA = (SBS^{-1})(SBS^{-1}) = SB \underbrace{S^{-1}S}_{I} BS^{-1} = \underbrace{SBBS^{-1}}_{B^2} = SB^2S^{-1}$$

$$A^3 = A^2A = (SB^2S^{-1})(SBS^{-1}) = SB^2 \underbrace{S^{-1}S}_{I} BS^{-1} = \underbrace{SB^2BS^{-1}}_{B^3} = SB^3S^{-1}$$

⋮

$$A^k = SB^kS^{-1}$$

↓ if this is easy to calculate \rightsquigarrow A^k easy to find

$$B = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix}, \dots, B^k = \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}$$

$k \geq 1$

$$B = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} \rightarrow B^k$$

as above,
etc

The diagonal form

$A \in M_{n \times n}(F)$ has a **diagonal form** if it is similar to a diagonal matrix

$[F \equiv \mathbb{R} \text{ or } F \equiv \mathbb{C}]$

i.e. $(\exists) S \in M_{n \times n}(F)$ s.t. $A = SBS^{-1}$ and
invertible

$$B = \underbrace{\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}}_{\text{diagonal matrix}}$$

Theorem 1: $A \in M_{n \times n}(F)$
 $\lambda_1, \lambda_2, \dots, \lambda_k$ distinct eigenvalues of A
 $\underline{u}_1, \underline{u}_2, \dots, \underline{u}_k$ corresponding eigenvec. \Rightarrow

$\Rightarrow \{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_k\}$ are l.i.

OBS. If all eigenvalues of A are distinct \Rightarrow the corresponding eigenvectors form a basis in F^n

$$A\underline{u}_i = \lambda_i \underline{u}_i \quad \& \quad A\underline{u}_j = \lambda_j \underline{u}_j \quad (i \neq j)$$

$\lambda_i \neq \lambda_j \rightsquigarrow ? \rightarrow \{\underline{u}_i, \underline{u}_j\}$ are l.i.

Assume $\alpha_i \underline{u}_i + \alpha_j \underline{u}_j = \underline{0}$ $\textcircled{*}$ for some $\alpha_i, \alpha_j \in \mathbb{R}$

$$\Downarrow \\ A(\alpha_i \underline{u}_i + \alpha_j \underline{u}_j) = \underline{0} \Rightarrow \alpha_i \underbrace{A \underline{u}_i}_{\lambda_i \underline{u}_i} + \alpha_j \underbrace{A \underline{u}_j}_{\lambda_j \underline{u}_j} = \underline{0} \Rightarrow$$

$$\Rightarrow \alpha_i \lambda_i \underline{u}_i + \alpha_j \lambda_j \underline{u}_j = \underline{0}$$

$$\alpha_i \lambda_i \underline{u}_i + \alpha_j \lambda_i \underline{u}_j = \underline{0}$$

subtract

$$\left. \begin{array}{l} \alpha_j \underline{u}_j (\lambda_j - \lambda_i) = \underline{0} \\ \underline{u}_j \neq \underline{0}, \lambda_i \neq \lambda_j \end{array} \right\} \Rightarrow \alpha_j = 0 \textcircled{*} \Rightarrow \alpha_i = 0$$

Summary: $\alpha_i \underline{u}_i + \alpha_j \underline{u}_j = \underline{0} \Rightarrow \alpha_i = \alpha_j = 0$

i.e. $\{\underline{u}_i, \underline{u}_j\}$ are l.i

OBS. This procedure can be repeated for any number of vectors

Theorem 2: If $A \in M_{n \times n}(F)$ is symmetric $\Rightarrow A$ has

$$A^T = A$$

a set of eigenvectors that is a basis for F^n

$$A \in M_{n \times n}(F)$$

$\underline{u}_1, \underline{u}_2, \dots, \underline{u}_n$ l.i. eigenvectors \rightsquigarrow $\lambda_1, \lambda_2, \dots, \lambda_n$
eigenvalues

$$S = [\underline{u}_1 : \underline{u}_2 : \dots : \underline{u}_n]$$

$$\underline{u}_j \in M_{n \times 1}(F)$$

$$AS = A [\underline{u}_1 : \underline{u}_2 : \dots : \underline{u}_n]$$

$$= [A\underline{u}_1 : A\underline{u}_2 : \dots : A\underline{u}_n]$$

$$= [\lambda_1 \underline{u}_1 : \lambda_2 \underline{u}_2 : \dots : \lambda_n \underline{u}_n]$$

$$= \underbrace{[\underline{u}_1 : \underline{u}_2 : \dots : \underline{u}_n]}_S \underbrace{\text{diag}(\lambda_1, \dots, \lambda_n)}_{\text{let's call it } D}$$

$$AS = SD \mid S^{-1} \Rightarrow \boxed{A = SDS^{-1}}$$

$\{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_n\}$ are l.i. $\Rightarrow S$ is invertible

Conversely \rightarrow if $A = SDS^{-1} \mid S \Rightarrow AS = SD \Rightarrow$

$\Rightarrow C_j$ is an eigenvector of A corresponding to the eigenvalue d_{jj}

i.e. $\boxed{A C_j = d_{jj} C_j \quad \text{for } j = 1, 2, \dots, n}$

OBS. NOT all matrices admit a diagonal form

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \rightsquigarrow \det(A - \lambda I) = \lambda^2 \Rightarrow \lambda_1 = \lambda_2 = 0$$

repeated
eigenvalue

$$(A - 0 \cdot I)\underline{x} = \underline{0} \Rightarrow \underline{x} = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} \dots ?$$

Assume by contradiction that A were diagonalisable

Then (∃) $S \in M_{2 \times 2}(\mathbb{R})$ invertible

(∃) $D \in M_{2 \times 2}(\mathbb{R})$ diagonal such that

$$S^{-1}AS = D \Rightarrow A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

" $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

contradiction
etc...

Applications Let $A = \begin{bmatrix} 5 & 4 \\ -2 & -1 \end{bmatrix}$. Calculate A^{60} .

Solution: step 1: calculate the eigenvalues of A

step 2: find its eigenvectors

step 3: get S and D

step 4: $A = SDS^{-1} \Rightarrow A^{60} = SD^{60}S^{-1}$

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 4 \\ -2 & -1-\lambda \end{vmatrix} = (\lambda-5)(\lambda+1) + 8$$

$$= \lambda^2 - 4\lambda + 3$$

$$= (\lambda-1)(\lambda-3) \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 3 \end{cases}$$

$$\begin{bmatrix} 4 & 4 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 = -\alpha \\ x_2 = \alpha \end{cases} \Rightarrow \underline{x} = \alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 = -2\alpha \\ x_2 = \alpha \end{cases} \Rightarrow \underline{x} = \alpha \begin{bmatrix} -2 \\ 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$S = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$A^{60} = S D^{60} S^{-1} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3^{60} \end{bmatrix} \begin{bmatrix} +1 & +2 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \cdot 3^{60} - 1 & 2 \cdot 3^{60} - 2 \\ 1 - 3^{60} & 2 - 3^{60} \end{bmatrix}$$