Determinants, eigenvalues & eigenvectors

Last week : $A \in M_{n \times n}(\mathbb{R})$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

- the minor M_{ij} of the entry a_{ij} :

  ↓

  delete row i and column j
  take the determinant

- cofactor A_{ij} of the entry a_{ij}.

  $$A_{ij} = (-1)^{i+j} M_{ij}$$

  # of cofactors $= n^2$

$$\text{cof}(A) = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 1 & -3 & 5 & 6 \\ 2 & 4 & 0 & 3 \\ 1 & 5 & 9 & -2 \\ 4 & 0 & 2 & 7 \end{bmatrix} \longrightarrow ? \ M_{32} \ \& \ M_{24}$$

$M_{32}:$
$$\begin{bmatrix} 1 & -3 & 5 & 6 \\ 2 & 4 & 0 & 3 \\ 1 & 5 & 9 & -2 \\ 4 & 0 & 2 & 7 \end{bmatrix} \longrightarrow M_{32} = \begin{bmatrix} 1 & 5 & 6 \\ 2 & 0 & 3 \\ 4 & 2 & 7 \end{bmatrix}$$

$M_{24}:$
$$\begin{bmatrix} 1 & -3 & 5 & 6 \\ 2 & 4 & 0 & 3 \\ 1 & 5 & 9 & -2 \\ 4 & 0 & 2 & 7 \end{bmatrix} \longrightarrow M_{24} = \begin{bmatrix} 1 & -3 & 5 \\ 1 & 5 & 9 \\ 4 & 0 & 2 \end{bmatrix}$$

$M_{3\times3}(\mathbb{R})$ matrices

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 5 & 6 \\ 2 & 0 & 3 \\ 4 & 2 & 7 \end{vmatrix}$$

$$A_{24} = (-1)^{2+4} \begin{vmatrix} 1 & -3 & 5 \\ 1 & 5 & 9 \\ 4 & 0 & 2 \end{vmatrix}$$

numbers

$$\boxed{\det(A) \equiv |A| = \sum_{j=1}^{n} a_{1j} A_{1j}}$$

Cofactor expansion
by row #1

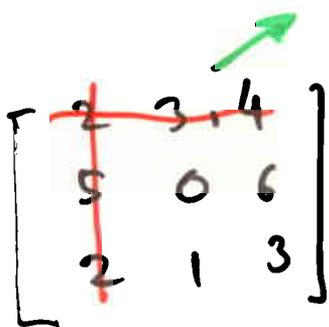$$A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 0 & 6 \\ 2 & 1 & 3 \end{bmatrix}$$

by row 1:

$$\det A = 2(-1)^{1+1}\begin{vmatrix} 0 & 6 \\ 1 & 3 \end{vmatrix} + 3(-1)^{1+2}\begin{vmatrix} 5 & 6 \\ 2 & 3 \end{vmatrix} + 4(-1)^{1+3}\begin{vmatrix} 5 & 0 \\ 2 & 1 \end{vmatrix}$$
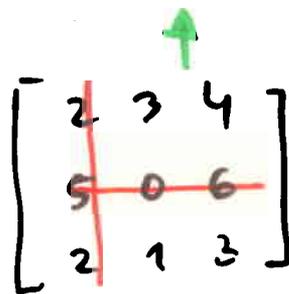
by row 3:

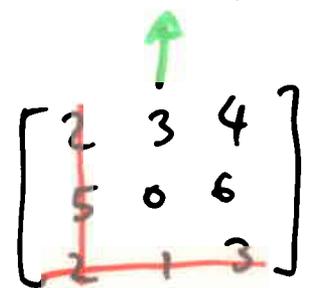$$\det A = 2(-1)^{3+1}\begin{vmatrix} 3 & 4 \\ 0 & 6 \end{vmatrix} + 1(-1)^{3+2}\begin{vmatrix} 2 & 4 \\ 5 & 6 \end{vmatrix} + 3(-1)^{3+3}\begin{vmatrix} 2 & 3 \\ 5 & 0 \end{vmatrix}$$

by column 1:

$$\det A = 2(-1)^{1+1}\begin{vmatrix} 0 & 6 \\ 1 & 3 \end{vmatrix} + 5(-1)^{2+1}\begin{vmatrix} 3 & 4 \\ 1 & 3 \end{vmatrix} + 2(-1)^{3+1}\begin{vmatrix} 3 & 4 \\ 0 & 6 \end{vmatrix}$$

$$\begin{bmatrix} 2 & 3 & 4 \\ 5 & 0 & 6 \\ 2 & 1 & 3 \end{bmatrix} \qquad \begin{bmatrix} 2 & 3 & 4 \\ 5 & 0 & 6 \\ 2 & 1 & 3 \end{bmatrix} \qquad \begin{bmatrix} 2 & 3 & 4 \\ 5 & 0 & 6 \\ 2 & 1 & 3 \end{bmatrix}$$

OBS. → in general, expand by the row/column which contains the largest number of zeros

Practical evaluation → by row & column operations

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

(I)

$$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{12} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$R_2 \leftrightarrow R_1$ ➡ changes the sign of det

$$\left[\; R_1 \leftrightarrow R_3 \quad \text{or} \quad R_3 \leftrightarrow R_2 \; \right]$$

(II)

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+\lambda a_{11} & a_{22}+\lambda a_{12} & a_{23}+\lambda a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$R_2 \to R_2 + \lambda R_1$ ➡ leaves things unchanged

(III)

$$\begin{vmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \lambda \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$R_1 \to \lambda R_1 \quad \rightsquigarrow \quad \lambda \; det$

$$\text{(IV)} \quad \text{If } R_i = R_j \ (i \neq j) \implies \det A = 0$$

$$? \quad \begin{vmatrix} -2 & 1 & 0 & 4 \\ 3 & -1 & 5 & 2 \\ -2 & 7 & 3 & 1 \\ 3 & -7 & 2 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 5 & 6 \\ 12 & 7 & 3 & -27 \\ -11 & -7 & 2 & 33 \end{vmatrix} \quad \begin{array}{l} C_1 \to C_1 + 2C_2 \\ C_4 \to C_4 + 4C_2 \end{array}$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 5 & 6 \\ 7 & 12 & 3 & -27 \\ -7 & -11 & 2 & 33 \end{vmatrix} \quad C_1 \leftrightarrow C_2$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 7 & 12 & -57 & -99 \\ -7 & -11 & 57 & 99 \end{vmatrix} \quad \begin{array}{l} C_3 \to C_3 - 5C_2 \\ C_4 \to C_4 - 6C_2 \end{array}$$

$$= 0 \quad (\text{row } 3 = \text{row } 4)$$

Invertibility of square matrices: $A \in M_{n \times n} (\mathbb{R})$

A is invertible $\Longleftrightarrow$ det $A \neq 0$

(more details in the typed notes)

Adjoints & Cramer's rule:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightsquigarrow \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$\underbrace{\qquad\qquad\qquad}_{\text{matrix of cofactors}}$

cofactor (number) $\rightarrow A_{ij} = (-1)^{i+j} M_{ij} \leftarrow$ minor (matrix)

adjoint of A $\equiv$ transpose of the matrix of cofactors of A

$$\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$\begin{bmatrix} \underline{\text{Aside}} \quad B, C \in M_{n \times n}(\mathbb{R}) \\ BC \qquad\qquad \underbrace{(BC)_{ik}}_{\substack{\text{the } ik \text{ entry} \\ \text{in the product matrix } BC}} = \sum_{j=1}^{n} B_{ij} C_{jk} \end{bmatrix}$$

$\text{adj}(A) = $ adjoint matrix of A

$A \, \text{adj}(A) \rightsquigarrow$ look at $\left(A \, \text{adj}(A) \right)_{ik}$

$$\left(A \, \text{adj}(A) \right)_{ik} = \sum_{j=1}^{n} a_{ij} \left(\text{adj}(A) \right)_{jk} = \sum_{j=1}^{n} a_{ij} A_{kj}$$

if $i = k \implies \left(A \, \text{adj}(A) \right)_{kk} = \underbrace{\sum_{j=1}^{n} a_{kj} A_{kj}}_{\substack{\text{Cofactor expansion of } |A| \\ \text{by row } k.}} = |A|$

if $i \neq k \implies$ cofactor expansion of the matrix obtained
from A by replacing row k with row i $\longrightarrow 0$

$\underbrace{\qquad\qquad\qquad}_{\substack{2 \text{ identical rows} \\ \text{in this matrix}}}$

$A \, \text{adj}(A) = \text{adj}(A) \, A = |A| I_n$

$A \left[\frac{1}{|A|} \text{adj}(A) \right] = \left[\frac{1}{|A|} \text{adj}(A) \right] A = I_n$ $\Bigg\{$

$\underbrace{\qquad\qquad\qquad\qquad}_{\text{inverse is unique}}$

<u>Cramer's Rule</u>

$\implies \boxed{A^{-1} = \frac{1}{|A|} \text{adj}(A)}$

$$\boxed{A^{-1} = \frac{1}{|A|} \, adj(A)}$$

Application: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightsquigarrow adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$A_{11} = d \qquad A_{21} = -b \qquad\qquad det(A) = ad - bc$
$A_{12} = -c \qquad A_{22} = a$

$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \rightsquigarrow$ useful when solving

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

Eigenvalues & eigenvectors

$A \in M_{n \times n}(\mathbb{R})$
λ eigenvalue

$A \underline{x} = \lambda \underline{x}$ for some $\underline{x} \neq \underline{0}$

$\underbrace{}$ eigenvector

$$(A - \lambda I_n) \underline{x} = \underline{0}$$

if $A - \lambda I_n$ invertible $\Rightarrow \underline{x} = \underline{0}$

So $A - \lambda I_n$ must be singular $\iff det(A - \lambda I_n) = 0$

$$\boxed{\det(A - \lambda I_n) = 0} \longrightarrow \text{characteristic eqn. of } A$$

polynomial in of degree n

Examples

$\bullet$ $A = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$

$$A - \lambda I_2 = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 2-\lambda & -1 \\ -4 & 2-\lambda \end{pmatrix}$$

$$\det(A - \lambda I_2) = \begin{vmatrix} 2-\lambda & -1 \\ -4 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 4 = \lambda(\lambda - 4)$$

$$\lambda_1 = 0 \qquad \lambda_2 = 4$$

eigenvalues of A

$\bullet$ $A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$

$$A - \lambda I_3 = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{pmatrix}$$

$$\det(A - \lambda I_3) = \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} = -(\lambda+1)^2(\lambda-8) \begin{cases} \lambda_1 = -1 \\ \lambda_2 = 8 \end{cases}$$

$$\bullet \quad A = \begin{bmatrix} 2 & -1 & 5 \\ 3 & 4 & -1 \\ -2 & -1 & 9 \end{bmatrix}$$

$$A - \lambda I_3 = \begin{bmatrix} 2 & -1 & 5 \\ 3 & 4 & -1 \\ -2 & -1 & 9 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & -1 & 5 \\ 3 & 4-\lambda & -1 \\ -2 & -1 & 9-\lambda \end{bmatrix}$$

$$\det(A - \lambda I_3) = \begin{vmatrix} 2-\lambda & -1 & 5 \\ 3 & 4-\lambda & -1 \\ -2 & -1 & 9-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 2-\lambda & -1 & 5 \\ 3 & 4-\lambda & -1 \\ +(\lambda-4) & 0 & -(\lambda-4) \end{vmatrix} \quad R_3 \to R_3 - R_1$$

$$= -(\lambda-4) \begin{vmatrix} 2-\lambda & -1 & 5 \\ 3 & 4-\lambda & -1 \\ -1 & 0 & 1 \end{vmatrix} \quad C_3 \to C_3 + C_1$$

$$= -(\lambda-4) \begin{vmatrix} 2-\lambda & -1 & 7-\lambda \\ 3 & 4-\lambda & 2 \\ -1 & 0 & 0 \end{vmatrix}$$

$$= +(\lambda-4) \begin{vmatrix} -1 & 7-\lambda \\ 4-\lambda & 2 \end{vmatrix} = (\lambda-4)\left[(\lambda-4)(7-\lambda) - 2\right]$$

$$= (\lambda-4)(-\lambda^2 + 11\lambda - 30) \quad \rightsquigarrow$$

$$= (4-\lambda)(5-\lambda)(6-\lambda)$$

$$\lambda_1 = 4, \quad \lambda_2 = 5, \quad \lambda_3 = 6 \quad \rightsquigarrow \text{ the eigenvalues of } A$$

<u>Aside</u>   Use elementary row & column operations to evaluate the determinant

$$\Delta = \begin{vmatrix} 4 & 1 & 2 & 6 \\ x & x & x^2 & 1 \\ 3 & x & -1 & 2 \\ 2 & x^2 & 1 & 3 \end{vmatrix}$$

(DE 2002)

$$\Delta = \begin{vmatrix} 0 & 1-2x^2 & 0 & 0 \\ x & x & x^2 & 1 \\ 3 & x & -1 & 2 \\ 2 & x^2 & 1 & 3 \end{vmatrix} \quad R_1 \to R_1 - 2R_4$$

$$= (2x^2-1) \begin{vmatrix} x & x^2 & 1 \\ 3 & -1 & 2 \\ 2 & 1 & 3 \end{vmatrix} = (2x^2-1) \begin{vmatrix} x & x^2 & 1 \\ 5 & 0 & 5 \\ 2 & 1 & 3 \end{vmatrix}$$

$$= 5(2x^2-1) \begin{vmatrix} x & x^2 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 5(2x^2-1) \begin{vmatrix} x & x^2 & 1-x \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{vmatrix}$$