

# Class Test (Revision)

• Linear systems  $\rightsquigarrow$  2 different types of questions:

①  $A \in M_{n \times n}(\mathbb{R})$  or  $A \in M_{m \times n}(\mathbb{R})$

1) find (reduced) echelon form  $\rightarrow$  e.r.o.'s

2) find  $A^{-1}$  (for square matrices)  $\rightarrow$  Gauß-Jordan.

3) find  $L, U$  s.t.  $A = LU \rightarrow$  (LU decomp)

$\swarrow$  lower triang  
 $\searrow$  upper triang

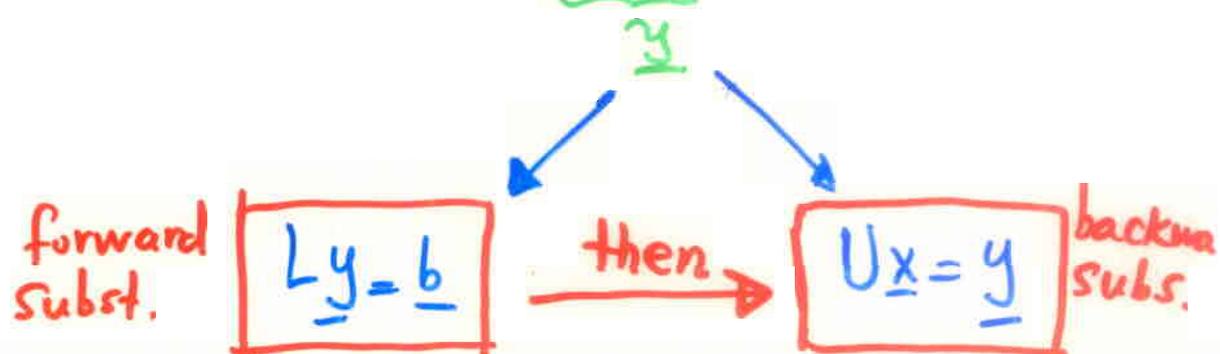
4) find  $P, L, U$  s.t.  $PA = LU$

$\downarrow$  permutation matrix

②  $A\underline{x} = \underline{b}$   $\rightsquigarrow$  solving this system by different methods

$A = LU \rightarrow A\underline{x} = \underline{b}$  can be written as

$$(LU)\underline{x} = \underline{b} \Leftrightarrow L(\underline{U}\underline{x}) = \underline{b}$$



Key concept

→ Gaussian elimination

Method for solving  
SYSTEMS OF LINEAR EQUATIONS  
using e.r.o.'s

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 2 & -1 & 4 \\ 6 & 5 & -3 \end{bmatrix}$$

$$AX = \underline{b}$$

$$\text{where } \underline{b} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\left[ \begin{array}{ccc|c} \text{pivot} \\ \textcircled{2} & 2 & 1 & 1 \\ 2 & -1 & 4 & 0 \\ 6 & 5 & -3 & 2 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 2 & 2 & 1 & 1 \\ 0 & \textcircled{-3} & 3 & -1 \\ 0 & -1 & -6 & -1 \end{array} \right]$$

$R_2 \rightarrow R_2 - R_1$   
 $R_3 \rightarrow R_3 - 3R_1$

Second pivot

$$\sim \left[ \begin{array}{ccc|c} 2 & 2 & 1 & 1 \\ 0 & 3 & -3 & 1 \\ 0 & -1 & -6 & -1 \end{array} \right] R_2 \rightarrow (-1)R_2$$

$$\sim \left[ \begin{array}{ccc|c} 2 & 2 & 1 & 1 \\ 0 & 3 & -3 & 1 \\ 0 & 0 & -7 & -2/3 \end{array} \right] \rightarrow \begin{cases} 2x + 2y + z = 1 \\ 3y - 3z = 1 \\ -7z = -2/3 \end{cases}$$

$R_1 \rightarrow R_1 + \frac{1}{3}R_2$

$-7z = -\frac{2}{3} \rightarrow z = +\frac{2}{21}$

$$y = \frac{1}{3}(1 + 3z) = \frac{3}{7}, \quad x = \frac{1}{2}(1 - 2y - z) = \frac{1}{42}$$

## Possible complications

- zero pivot → curable (swap 2 rows, ...)  
incurable (no solutions, ...)

## Gaussian elimination with pivoting

pivots are chosen to be as large as possible (in modulus)

$$\left[ \begin{array}{ccc|c} 2 & 2 & 1 & 1 \\ 2 & -1 & 4 & 0 \\ 6 & 5 & -3 & 2 \end{array} \right] \sim \left[ \begin{array}{ccc|c} \text{largest pivot} \\ \text{6} & 5 & -3 & 2 \\ 2 & -1 & 4 & 0 \\ 2 & 2 & 1 & 1 \end{array} \right] R_1 \leftrightarrow R_3$$

... (at each step select the largest pivot)

$$\left[ \begin{array}{ccc} 2 & 2 & 1 \\ 2 & -1 & 4 \\ 6 & 5 & -3 \end{array} \right] \sim \dots \sim \left[ \begin{array}{ccc} 2 & 2 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & -7 \end{array} \right]$$

upper triangular

→  $A = LU$   
lower triang → requires more work

DE (2006)

Using Gaussian elimination with partial pivoting, find a permutation matrix  $P$  and an LU decomposition for the matrix

$$A = \begin{bmatrix} 2 & -1 & 2 \\ 8 & -3 & -8 \\ 4 & -4 & -3 \end{bmatrix}$$

such that  $PA = LU$ . Use this to solve

$$\begin{cases} 2x - y + 2z = 3 \\ 8x - 3y - 8z = 29 \\ 4x - 4y - 3z = 11 \end{cases}$$

Solution

$A \rightarrow$  adjoin a "permutation counter column" that is initially set to the natural order: 1, 2, 3

$$\left[ \begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ 8 & -3 & -8 & 2 \\ 4 & -4 & -3 & 3 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 8 & -3 & -8 & 2 \\ 2 & -1 & 2 & 1 \\ 4 & -4 & -3 & 3 \end{array} \right] R_1 \leftrightarrow R_2$$

$$\begin{array}{l} \sim \\ \frac{1}{4} \\ \frac{1}{2} \end{array} \left[ \begin{array}{ccc|c} 8 & -3 & -8 & 2 \\ 0 & -1/4 & 4 & 1 \\ 0 & -5/2 & 1 & 3 \end{array} \right] \begin{array}{l} R_2 \rightarrow R_2 - \frac{1}{4} R_1 \\ R_3 \rightarrow R_3 - \frac{1}{2} R_1 \end{array}$$

$$\sim \left[ \begin{array}{ccc|c} 8 & -3 & -8 & 2 \\ 0 & -5/2 & 1 & 3 \\ 0 & -1/4 & 4 & 1 \end{array} \right] R_2 \leftrightarrow R_3$$

must be eliminated

$$-\frac{1}{4} + \frac{5}{2}\lambda = 0 \Rightarrow \lambda = \frac{1}{10}$$

$R_3 \rightarrow R_3 - \lambda R_2$  gives the result

$$\sim \frac{1}{10} \left[ \begin{array}{ccc|c} 8 & -3 & -8 & 2 \\ 0 & -5/2 & 1 & 3 \\ 0 & 0 & \frac{39}{10} & 1 \end{array} \right] R_3 \rightarrow R_3 - \frac{1}{10} R_2$$

upper triangular

$$L = \begin{bmatrix} 1 & 0 & 0 \\ ? & 1 & 0 \\ ? & ? & 1 \end{bmatrix} \rightsquigarrow L = \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 1/2 & 1/10 & 1 \end{bmatrix}$$

$$P \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightsquigarrow P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} 8 & -3 & -8 \\ 0 & -5/2 & 1 \\ 0 & 0 & 39/10 \end{bmatrix}$$

## Check

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_P \underbrace{\begin{bmatrix} 2 & -1 & 2 \\ 8 & -3 & -8 \\ 4 & -4 & -3 \end{bmatrix}}_A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 1/2 & 1/10 & 1 \end{bmatrix}}_L \underbrace{\begin{bmatrix} 8 & -3 & -8 \\ 0 & -5/2 & 1 \\ 0 & 0 & 39/10 \end{bmatrix}}_U$$

etc...

## Second part:

$$A \underline{x} = \underline{b}$$

$$\underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\underline{b} = \begin{bmatrix} 3 \\ 29 \\ 11 \end{bmatrix}$$

$$\left. \begin{array}{l} PA \underline{x} = P \underline{b} \\ PA = LU \end{array} \right\} \Rightarrow \underbrace{LU \underline{x}}_{\underline{w}} = P \underline{b} \quad \text{(*)}$$

$$\text{Let } \underline{w} = U \underline{x}$$

$$\text{Then (*)} \Rightarrow \underbrace{L \underline{w} = P \underline{b}}_{\text{linear system in } \underline{w} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}} \rightarrow \underline{w}$$

$$\text{Once I have } \underline{w} \rightarrow \text{solve } U \underline{x} = \underbrace{\underline{w}}_{\text{available now}} \rightarrow \text{find } \underline{x}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 1/2 & 1/10 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 29 \\ 29 \\ 3 \end{bmatrix} \rightarrow$$

$$u = 29$$

$$v = 11 - \frac{u}{2} = -\frac{7}{2}$$

$$w = 3 - \frac{1}{4}u - \frac{1}{10}v = -\frac{39}{10}$$

$$\begin{bmatrix} 8 & -3 & -8 \\ 0 & -5/2 & 1 \\ 0 & 0 & 39/10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 29 \\ -7/2 \\ -39/10 \end{bmatrix}$$

$\rightarrow \frac{39}{10} z = -\frac{39}{10} \Rightarrow \boxed{z = -1}$

$y = -\frac{2}{5} \left(-\frac{7}{2} + 1\right) \Rightarrow \boxed{y = 1}$

$x = \frac{1}{8} (29 + 3y + 8z) \Rightarrow \boxed{x = 3}$

$\rightarrow$  final solution

DE (2005) Using Gaussian elimination WITHOUT partial pivoting, find an LU decomposition for the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 3 & -4 \\ 2 & -6 & 1 \end{bmatrix}$$

Use this to solve

$$\begin{cases} x & -z = 5 \\ 3x + 3y & -4z = 15 \\ 2x - 6y & + z = 7 \end{cases}$$

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 3 & -4 \\ 2 & -6 & 1 \end{bmatrix} \sim \begin{matrix} 3 \\ 2 \end{matrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & -1 \\ 0 & -6 & 3 \end{bmatrix} \begin{matrix} R_2 \rightarrow R_2 - 3R_1 \\ R_3 \rightarrow R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{matrix} -2 \end{matrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_3 \rightarrow R_3 + 2R_2 \\ R_3 - (-2)R_2 \end{matrix}$$

↓  
this is going to be U (upper triangular)

$$L = \begin{bmatrix} 1 & 0 & 0 \\ ? & 1 & 0 \\ ? & ? & 1 \end{bmatrix} \rightsquigarrow L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}$$

Check:  $\underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 3 & 3 & -4 \\ 2 & -6 & 1 \end{bmatrix}}_A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}}_L \underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_U$

Second part:  $\underbrace{LU}_A \underline{x} = \underline{b}$  ,  $\underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$  ,  $\underline{b} = \begin{bmatrix} 5 \\ 15 \\ 7 \end{bmatrix}$

$LU\underline{x} = \underline{b} \rightarrow$  Let  $\underline{w} = U\underline{x} \rightarrow \underbrace{L\underline{w} = \underline{b}}_{\text{Solve for } \underline{w}}$

$$L\underline{w} = \underline{b} \Leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \\ 7 \end{bmatrix} \Rightarrow \begin{aligned} u &= 5 \\ v &= 15 - 3u = 0 \\ w &= 7 - 2u + 2v = -3 \end{aligned}$$

$$U\underline{x} = \underbrace{\underline{w}}_{\substack{\downarrow \\ \text{available} \\ \text{now}}} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -3 \end{bmatrix} \Rightarrow \begin{aligned} z &= -3 \\ y &= \frac{0+z}{3} = -1 \\ x &= 5+z = 2 \end{aligned}$$

Final solution  $\rightarrow$   $\boxed{x=2, y=-1, z=-3}$

DE (2004) Use Gaussian elimination with partial pivoting to find a permutation matrix  $P$  and an LU decomposition for the matrix

$$A = \begin{bmatrix} 2 & -1 & 4 \\ 1 & 0 & 2 \\ 3 & 1 & -5 \end{bmatrix}$$

such that  $\boxed{PA = LU}$ . Use this to solve

$$\begin{cases} 2x - y + 4z = 17 \\ x + 2z = 8 \\ 3x + y - 5z = -10 \end{cases}$$

Remember the permutation counter column.

$$\underbrace{\begin{bmatrix} 2 & -1 & 4 \\ 1 & 0 & 2 \\ 3 & 1 & -5 \end{bmatrix}}_A \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 & -5 & | & 3 \\ 1 & 0 & 2 & | & 2 \\ 2 & -1 & 4 & | & 1 \end{bmatrix} R_1 \leftrightarrow R_3$$

$$\sim \begin{matrix} \frac{1}{3} \\ \frac{2}{3} \end{matrix} \begin{bmatrix} 3 & 1 & -5 & | & 3 \\ 0 & -\frac{1}{3} & \frac{11}{3} & | & 2 \\ 0 & -\frac{5}{3} & \frac{22}{3} & | & 1 \end{bmatrix} \begin{matrix} R_2 \rightarrow R_2 - \frac{1}{3} R_1 \\ R_3 \rightarrow R_3 - \frac{2}{3} R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 3 & 1 & -5 & | & 3 \\ 0 & -\frac{5}{3} & \frac{22}{3} & | & 1 \\ 0 & -\frac{1}{3} & \frac{11}{3} & | & 2 \end{bmatrix} R_3 \leftrightarrow R_2$$

$$-\frac{1}{3} + \frac{5}{3} \lambda = 0 \Rightarrow \lambda = \frac{1}{5} \quad R_3 \rightarrow R_3 - \lambda R_2$$

$$\sim \begin{matrix} \frac{1}{5} \end{matrix} \begin{bmatrix} 3 & 1 & -5 & | & 3 \\ 0 & -\frac{5}{3} & \frac{22}{3} & | & 1 \\ 0 & 0 & \frac{11}{5} & | & 2 \end{bmatrix} R_3 \rightarrow R_3 - \frac{1}{5} R_2$$

$\mathbb{U}$   
(upper triang)

↓ gives P from  $I_3 \rightarrow \rightarrow \rightarrow$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$I_3$

3  
1  
2

$$\begin{cases} R_3 \rightarrow R_1 \\ R_1 \rightarrow R_2 \\ R_2 \rightarrow R_3 \end{cases}$$

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ ? & 1 & 0 \\ ? & ? & 1 \end{bmatrix} \rightsquigarrow L = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 2/3 & 1/5 & 1 \end{bmatrix}$$

Check:

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_P \underbrace{\begin{bmatrix} 2 & -1 & 4 \\ 1 & 0 & 2 \\ 3 & 1 & -5 \end{bmatrix}}_A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 2/3 & 1/5 & 1 \end{bmatrix}}_L \underbrace{\begin{bmatrix} 3 & 1 & -5 \\ 0 & -5/3 & 22/3 \\ 0 & 0 & 11/5 \end{bmatrix}}_U$$

$$\underline{A}\underline{x} = \underline{b}$$

↓

$$\underline{b} = \begin{bmatrix} 17 \\ 8 \\ -10 \end{bmatrix}$$

$$\underline{P}\underline{A}\underline{x} = \underline{P}\underline{b} \Rightarrow (\underline{P}\underline{A})\underline{x} = \underline{P}\underline{b} \Rightarrow (\underline{L}\underline{U})\underline{x} = \underline{P}\underline{b}$$

$$\underline{L}(\underline{U}\underline{x}) = \underline{P}\underline{b}$$

W

First  $\rightarrow$  solve  $L\underline{w} = P\underline{b}$  (for  $\underline{w}$ )

Then  $\rightarrow$  solve  $U\underline{x} = \underline{w}$  (for  $\underline{x}$ )

$$\begin{pmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & 1/5 & 1 \end{pmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -10 \\ 17 \\ 8 \end{bmatrix} \Rightarrow \begin{cases} u = -10 \\ v = 71/3 \\ w = 33/6 \end{cases}$$

$$\begin{bmatrix} 3 & 1 & -5 \\ 0 & -5/3 & 22/3 \\ 0 & 0 & 11/3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -10 \\ 71/3 \\ 33/6 \end{bmatrix}$$

$$\frac{11}{3}z = \frac{33}{6} \Rightarrow \boxed{z = 3}$$

then  $-\frac{5}{3}y + \frac{22}{3}z = \frac{71}{3} \Rightarrow -\frac{5}{3}y = \frac{71 - 22z}{3} \Rightarrow \boxed{y = -1}$

then  $3x + y - 5z = -10 \Rightarrow \boxed{x = 2}$

• Finite differences  $\rightsquigarrow$  2 types of problems

(I)	$x$	$x_0$	$x_1$	$x_2$	$\dots$	$x_n$
	$y$	$y_0$	$y_1$	$y_2$	$\dots$	$y_n$

$$y_j = f(x_j) \quad 0 \leq j \leq n$$

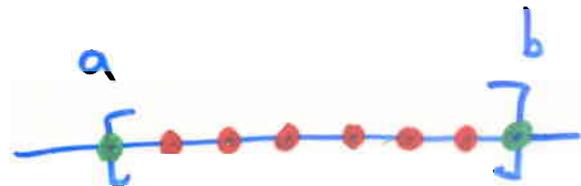
$(x_j)_{j=0, n}$   
in  $[a, b]$

uniform mesh  $\rightsquigarrow$  distance between any 2 consecutive points is the same

$$x_j = a + jh \quad (0 \leq j \leq n)$$

$$h = \frac{b-a}{n}$$

$$\begin{cases} f'(x_j) = \frac{1}{2h} [y_{j+1} - y_j] \\ f''(x_j) = \frac{1}{h^2} [y_{j+1} - 2y_j + y_{j-1}] \end{cases}$$



• internal nodes  
• end points

$\uparrow$   
applicable for internal nodes

forward & back ward  $\rangle$  finite difference formulae  $\rightarrow$  see NOTES

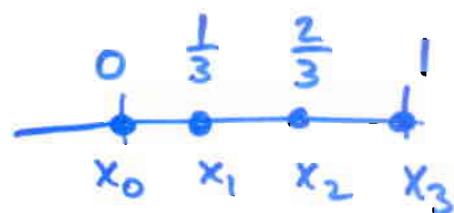
$$\textcircled{\text{II}} \begin{cases} y'' + A(x)y' + B(x)y = C(x) & a \leq x \leq b \\ + \text{ B.C.} \end{cases}$$

Find an approx of  $y$  at the nodes of some (given) uniform mesh.

DE(2004) Use the central difference formulae with  $h = \frac{1}{3}$  to find the approximation of  $y(x)$  at  $x = \frac{1}{3}$  and  $x = \frac{2}{3}$ , when  $y$  satisfies

$$\begin{cases} y'' + xy = x^2 & 0 < x < 1 \\ y(0) = 0 \quad y(1) = \frac{1}{9} \end{cases}$$

Solution:



$$y''(x_j) + x_j y(x_j) = x_j^2 \quad j=1,2 \quad (\text{internal nodes})$$

$\Downarrow$

$$\frac{1}{h^2} [y_{j+1} - 2y_j + y_{j-1}] + x_j y_j = x_j^2$$

$$x_j = \frac{j}{3}$$

$$\begin{cases} 9 [y_2 - 2y_1 + y_0] + \frac{1}{3} y_1 = \frac{1}{9} \\ 9 [y_3 - 2y_2 + y_1] + \frac{2}{3} y_2 = \frac{4}{9} \end{cases}$$

$$\begin{cases} 9 [y_3 - 2y_2 + y_1] + \frac{2}{3} y_2 = \frac{4}{9} \\ y_0 = 0 \\ y_3 = \frac{1}{9} \end{cases}$$

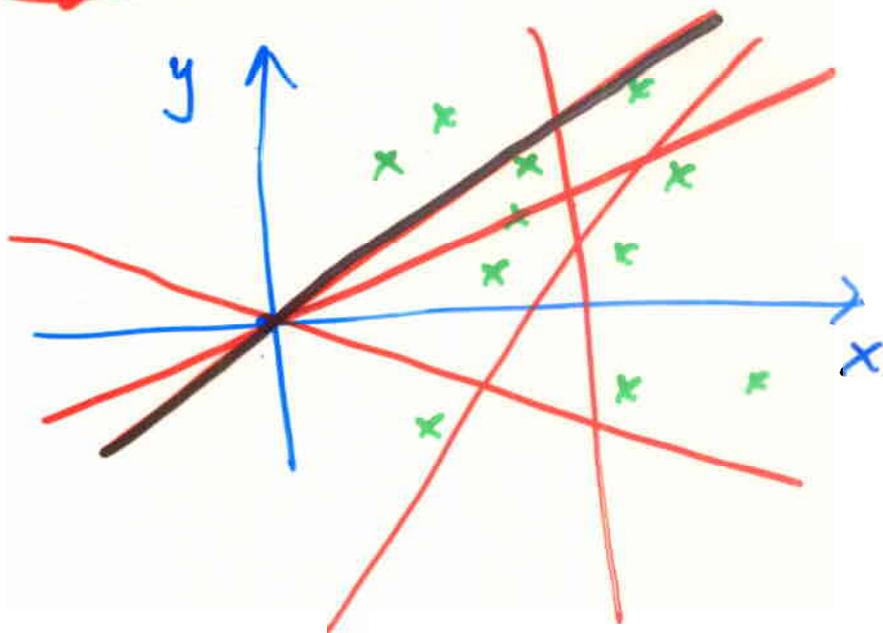
$$\Rightarrow \begin{cases} 81y_0 - 159y_1 + 81y_2 & = 1 \\ 81y_1 - 156y_2 + 81y_3 & = 4 \end{cases}$$

$$y_0 = 0, \quad y_3 = \frac{1}{9}$$

$$\begin{cases} 159y_1 - 81y_2 = -1 \\ 81y_1 - 156y_2 = -5 \end{cases} \rightarrow \dots \begin{cases} y_1 = 0.0136\dots \\ y_2 = 0.0391\dots \end{cases}$$

↑  
Solve by simple substitution

• Least squares fitting  $\rightarrow y = ax + b$  or  $y = ax^2 + bx + c$



$\{(x_j, y_j) \mid 1 \leq j \leq n\}$   $\leftarrow$   $n$  fixed points

?  $y = ax + b$  s.t. some error = minimum (see NOTES)

→ the main question was the determination of  $a$  &  $b$

2 linear equations in 2 unknowns

$$A = \begin{bmatrix} x_1 & | \\ x_2 & | \\ \vdots & | \\ x_n & | \end{bmatrix}$$

$$B = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$Z = \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\text{unknowns}}$$

↓  
normal equations:

$$\underbrace{A^T A} Z = \underbrace{A^T B}$$

→ easy to calculate

DE(2004) Use LSM to find the line that gives the best fit for the data

x	-1	0	0	1	2
y	0	0	1	2	1

and estimate  $y(\frac{1}{2})$



$$y = \frac{6}{13}x + \frac{8}{13}$$

$$y\left(\frac{1}{2}\right) = \frac{6}{13} \times \frac{1}{2} + \frac{8}{13} \Rightarrow$$

$$\boxed{y\left(\frac{1}{2}\right) = \frac{11}{13}}$$