

# The Eigenvalue Problem

Consider  $A \in M_{n \times n}(\mathbb{R})$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$$

$\underbrace{\hspace{1.5cm}}_{\underline{c}_1} \quad \underbrace{\hspace{1.5cm}}_{\underline{c}_2} \quad \underbrace{\hspace{1.5cm}}_{\underline{c}_3} \quad \dots \quad \underbrace{\hspace{1.5cm}}_{\underline{c}_n}$

$$\underline{c}_j \in M_{n \times 1}(\mathbb{R}) \quad c_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix}$$

Then  $A = [\underline{c}_1 : \underline{c}_2 : \underline{c}_3 \dots : \underline{c}_n]$

$$[\underline{c}_1 : \underline{c}_2] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \vdots & \vdots \\ a_{n1} & a_{n2} \end{bmatrix}$$

$$[\underline{c}_1 : \underline{c}_2 : \underline{c}_3] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{nn} \end{bmatrix} \quad \text{etc}$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \rightarrow \text{look at the action of } A \text{ on this vector;}$$

$$A \underline{x} = [\underline{c}_1 : \underline{c}_2 : \underline{c}_3 : \dots : \underline{c}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \underline{c}_1 + x_2 \underline{c}_2 + \dots + x_n \underline{c}_n$$

### Particular case

$$A \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \underline{c}_1 + 0 \cdot \underline{c}_2 + 0 \cdot \underline{c}_3 + \dots + 0 \cdot \underline{c}_n = \underline{c}_1$$

$$A \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0 \cdot \underline{c}_1 + \underline{c}_2 + 0 \cdot \underline{c}_3 + \dots + 0 \cdot \underline{c}_n = \underline{c}_2, \text{ etc}$$

$D \in M_{n \times n}(\mathbb{R})$  diagonal matrix

$$D = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) \equiv \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{bmatrix}$$

$$D \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$D \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \lambda_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

proportional  
to the original vector

etc

Question:

In what other circumstances might  $A\underline{x}$  be proportional to  $\underline{x}$ ?

$\underline{x} = \underline{0} \rightarrow$  trivial

$\underline{x} \neq \underline{0}$ , s.t.  $A\underline{x}$  is proportional to  $\underline{x}$ , i.e.  $A\underline{x} = \lambda\underline{x}$  <sup>⊛</sup>  
for some  $\lambda \in \mathbb{R}$

$(\lambda, \underline{x})$  s.t. ⊛ takes place is called an eigenpair

$\underline{x}$  is an eigenvector       $\lambda$  eigenvalue

OBS. Eigenvectors are not unique

$$\underline{x} \neq \underline{0}, \quad A\underline{x} = \lambda\underline{x} \quad | \cdot \alpha \quad (\alpha \in \mathbb{R})$$

$$\alpha(A\underline{x}) = \alpha(\lambda\underline{x})$$

$A(\alpha\underline{x}) = \lambda(\alpha\underline{x})$  i.e.  $\alpha\underline{x}$  is also an eigenvector  $(\forall) \alpha \neq 0$

Eigenvalues  $\rightarrow$  ? unique  
 $\rightarrow$  ? methods for finding them

Example  $B \in M_{2 \times 2}(\mathbb{R})$

$$B = \begin{bmatrix} \frac{11}{8} & \frac{13}{8} \\ \frac{13}{8} & \frac{11}{8} \end{bmatrix}$$

$$B \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{11}{8} & \frac{13}{8} \\ \frac{13}{8} & \frac{11}{8} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow \lambda = 3 \text{ (eigenvalue)} \\ \underline{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ (eigenvector)}$$

$$B \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{11}{8} & \frac{13}{8} \\ \frac{13}{8} & \frac{11}{8} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\frac{1}{4} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightarrow \lambda = -\frac{1}{4} \text{ (eigenvalue)} \\ \underline{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ (eigenvector)}$$

$C \in M_{2 \times 2}(\mathbb{R})$

$$C = \begin{bmatrix} 3 & 0 \\ \frac{13}{4} & -\frac{1}{4} \end{bmatrix}$$

$$C \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ \frac{13}{4} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$C \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ \frac{13}{4} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\frac{1}{4} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

OBS. • Different matrices can have the same eigenvalues

• Can we find any other eigenvalues for B & C?

$$A\underline{x} = \lambda\underline{x} \Rightarrow \underbrace{(A - \lambda I)\underline{x} = \underline{0}}_{\text{must be singular}} \quad \underline{x} \neq \underline{0}$$

$$\underline{x} = (A - \lambda I)^{-1} \underline{0} = \underline{0} \quad \left[ \text{if the inverse of } A - \lambda I \text{ exists} \right]$$

OBS If  $\lambda$  is an eigenvalue of  $A \in M_{n \times n}(\mathbb{R}) \Rightarrow$

$\Rightarrow A - \lambda I$  cannot have an inverse.

best understood  
by using the concept  
of determinant (for a matrix)

## Determinants

$$A \in M_{n \times n}(\mathbb{R}) \rightsquigarrow \underbrace{\det(A)}_{\text{real number}} \quad |A|$$

$$A \in M_{n \times n}(\mathbb{C}) \rightsquigarrow \det(A) \in \mathbb{C}$$

Defined inductively: those of matrices in  $M_{n \times n}$  are defined in terms of determinants of matrices in  $M_{(n-1) \times (n-1)}$

$$A \in M_{n \times n}(\mathbb{R})$$

$$A = (a_{ij})_{1 \leq i, j \leq n}$$

minor  $M_{ij}$  of the entry  $a_{ij}$  ( $1 \leq i, j \leq n$ )

determinant  $\rightarrow$  delete Row  $i$  & Column  $j$

Example:  $A = \begin{bmatrix} 3 & 2 & 1 \\ 5 & 6 & 0 \\ 3 & 1 & 9 \end{bmatrix}$

$$\begin{bmatrix} 3 & 2 & 1 \\ 5 & 6 & 0 \\ 3 & 1 & 9 \end{bmatrix}$$

$$M_{23}(A) = \det \left( \begin{bmatrix} 3 & 2 \\ 3 & 1 \end{bmatrix} \right)$$

$$M_{13}(A) = \det \left( \begin{bmatrix} 5 & 6 \\ 3 & 1 \end{bmatrix} \right) \text{ etc}$$

$\vdots$

cofactor  $A_{ij}$  of the entry  $a_{ij}$  ( $1 \leq i, j \leq n$ )

$$A_{ij} = (-1)^{i+j} M_{ij}$$

Definition of determinants  $\rightarrow \rightarrow \rightarrow \rightarrow$

(D1) For  $A \in M_{1 \times 1}(\mathbb{R})$ , i.e.  $A = [a]$

$$\det(A) = a$$

(D2) For  $A \in M_{n \times n}(\mathbb{R})$

$$\det(A) = \sum_{j=1}^n a_{1j} A_{1j}$$

elements  
in Row 1

their cofactors

(cofactor expansion by the 1st row)

$$\begin{aligned} \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= a A_{11} + b A_{12} \\ &= a (-1)^{1+1} M_{11} + b (-1)^{1+2} M_{12} \\ &= ad - bc \end{aligned}$$

$$\Rightarrow \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

### Important Properties

(P<sub>1</sub>)  $\det(I_n) = 1$  for all  $n \geq 2$

(P<sub>2</sub>)  $\det(A) = \sum_{j=1}^n a_{ij} A_{ij}$  for any  $i$  ( $1 \leq i \leq n$ )  
Cofactor expansion by row  $i$

Example  $A \in M_{3 \times 3}(\mathbb{R})$   $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Cofactor expansion by row 1:

$$\det(A) = \sum_{j=1}^3 a_{1j} A_{1j} = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$$

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = 5 \times 9 - 6 \times 8 = -3$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = -(4 \times 9 - 6 \times 7) = 6$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = 4 \times 8 - 5 \times 7 = -3$$

$$\Rightarrow \det(A) = 1(-3) + 2(6) + 3(-3) = 0$$

Cofactor expansion by row 2:

$$\det(A) = \sum_{j=1}^3 a_{2j} A_{2j} = a_{21} A_{21} + a_{22} A_{22} + a_{23} A_{23}$$

$$= 4(-1)^{2+1} \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} + 5(-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} + (-1)^{2+3} 6 \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix}$$

$$= \dots$$

$$= 0 \quad (\text{same result})$$

$$P_3 \quad \det(A^T) = \det(A) \quad \text{for all } A \in M_{n \times n}(\mathbb{R})$$

consequence

cofactor expansions by columns:

$$\det(A) = \sum_{i=1}^n a_{ij} A_{ij}$$

$P_4$  If any row [or column] of  $A \in M_{n \times n}(\mathbb{R})$  is the zero vector  $\Rightarrow \det(A) = 0$

$P_5$  If  $A$  is upper or lower triangular, then

$$\det(A) = \prod_{i=1}^n a_{ii}$$

$a_{11} a_{22} a_{33} \dots a_{nn}$

$$P_6 \quad \boxed{\det(AB) = \det(A) \det(B)} \quad A, B \in M_{n \times n}(\mathbb{R})$$

$P_7$  If  $A \in M_{n \times n}(\mathbb{R})$  is invertible  $\Rightarrow \det(A) \neq 0$  and

$$\boxed{\det(A^{-1}) = \frac{1}{\det(A)}}$$

$\textcircled{P_8}$  If  $A = \begin{bmatrix} R+S \\ R_2 \\ \vdots \\ R_n \end{bmatrix}$   $R, S = \text{row vectors}$   
 $R_2, \dots, R_n = \text{row vectors}$

then  $\det(A) = \det \left( \begin{bmatrix} R \\ R_2 \\ \vdots \\ R_n \end{bmatrix} \right) + \det \left( \begin{bmatrix} S \\ R_2 \\ \vdots \\ R_n \end{bmatrix} \right)$

Proof  $\det(A) = \sum_{j=1}^n a_{ij} A_{ij}$   $a_{ij} = r_j + s_j$   
 $= \sum_{j=1}^n (r_j + s_j) A_{ij}$   
 $= \underbrace{\sum_{j=1}^n r_j A_{ij}}_{\det \left( \begin{bmatrix} R \\ \vdots \\ R_n \end{bmatrix} \right)} + \underbrace{\sum_{j=1}^n s_j A_{ij}}_{\det \left( \begin{bmatrix} S \\ R_2 \\ \vdots \\ R_n \end{bmatrix} \right)}$

Example

$$\begin{vmatrix} 1+3 & 5+7 & 9+2 \\ 1 & 3 & 0 \\ 6 & 7 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 9 \\ 1 & 3 & 0 \\ 6 & 7 & 4 \end{vmatrix} + \begin{vmatrix} 3 & 7 & 2 \\ 1 & 3 & 0 \\ 6 & 7 & 4 \end{vmatrix}$$