<u>Last week</u> → linear programming

$\left\{ \begin{array}{l} \text{Maximise} \quad P(x_1, x_2, \ldots x_n) \text{ subject to} \\[2mm] A\underline{x} \leq \underline{b} \quad \longleftarrow n \text{ inequalities in } x_1, x_2, \ldots x_n \\[2mm] \underline{x} \geq 0 \quad \longleftarrow [x_j \geq 0, \; j = 1, 2, \ldots n] \end{array} \right.$

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \qquad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{pmatrix}$$

<u>example</u>

$\left\{ \begin{array}{l} \text{maximise} \quad P = 2x_1 + 3x_2 \\[2mm] x_1 + x_2 \leq 10 \quad \longrightarrow \text{ non-trivial constraints} \\[2mm] x_1 + 3x_2 \leq 12 \quad \nearrow \\[2mm] x_1 \geq 0, \quad x_2 \geq 0 \end{array} \right.$

feasible set
↓
(has corners)
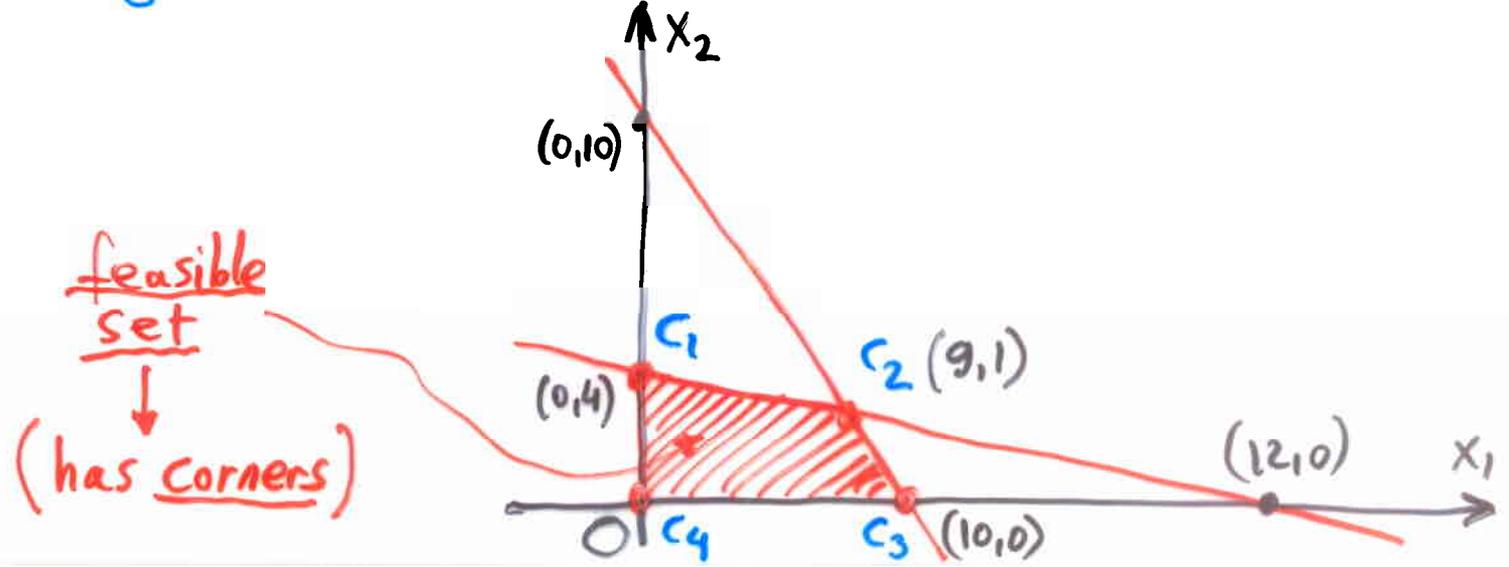
STEP 1 Introduce **slack variables**

$$x_3 = 10 - x_1 - x_2$$
$$x_4 = 12 - x_1 - 3x_2$$

STEP 2 Restate the problem

$$\begin{cases} \text{Maximise} \quad P(x_1, x_2, x_3, x_4) = 2x_1 + 3x_2 \quad \text{subject to} \\ \quad x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0 \end{cases}$$

and $\begin{cases} x_1 + x_2 + x_3 = 10 \\ x_1 + 3x_2 + x_4 = 12 \end{cases}$

STEP 3 Choose **basic** and **free variables**

$$x_1, x_2 = \text{free}$$
$$x_3, x_4 = \text{basic}$$

$\longrightarrow$ feasible corner $\quad x_1 = x_2 = 0 \longrightarrow x_3 = 10$
$$x_4 = 12$$
$$(0, 0, 10, 12)$$

STEP 4 Want to write P as

$$C - \sum_{j=1}^{4} \lambda_j x_j$$

$\oplus$ positive constants $\oplus$

$\longrightarrow$ need to decide on an **entering** & a **leaving** variable

feasible corner

If $x_1 =$ entering $\longrightarrow$? x_3 or x_4 leaves

$x_1 = 10 - x_3 \longrightarrow x_1 = 10, x_2 = 0, x_4 = 12 - 10 = 2$ ✓

$x_1 = 12 - x_4 \longrightarrow x_1 = 12, x_2 = 0, x_3 = 10 - 12 = -2$ ✗

$\Downarrow$

x_3 is the leaving variable

feasible (new) corner $(10, 0, 0, 2)$

$P = 2x_1 + 3x_2 \longrightarrow P = 20 + x_2 - 2x_3$

the value of P
at new corner $= 20$

STEP 5 Re-write the problem withe the new choice
of free & basic variables AND repeat all the
previous steps

x_2, x_3 x_1, x_4

$\begin{cases} \text{Maximise } P(x_1, x_2, x_3, x_4) = 20 + x_2 - 2x_3 \\ x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0, \ x_4 \geq 0 \\ x_1 + x_2 + x_3 = 10 \\ x_1 + 3x_2 + x_4 = 12 \end{cases}$

$x_2 =$ entering $\longrightarrow$? x_1 or x_4 leaves

$x_2 = 10 - x_1 \longrightarrow x_2 = 10, x_1 = 0, x_3 = 0, x_4 = -18$ ✗

$x_2 = \frac{1}{3}(12 - x_4 - x_1) \longrightarrow$ only choice we have

→ x_4 is the leaving variable

feasible (new) corner $(9,1,0,0)$

$P = 20 + x_2 - 2x_3 \longrightarrow P = 20 + 1 - 0 = 21$

the value of P
at new corner $= 21 > 20$

NOW:

free variables: x_3, x_4

basic: x_1, x_2

original equ...

$$P = 20 + \boxed{x_2} - 2x_3$$

has to be expressed in terms of x_3 & x_4

$x_1 = 10 - x_3 - x_2 \longrightarrow (10 - x_3 - x_2) + 3x_2 + x_4 = 12$

$$2x_2 = 2 + x_3 - x_4$$

$$x_2 = 1 + \frac{1}{2}x_3 - \frac{1}{2}x_4$$

$$P = 20 + \left(1 + \frac{1}{2}x_3 - \frac{1}{2}x_4\right) - 2x_3$$

$$P = 21 - \frac{3}{2}x_3 - \frac{1}{2}x_4$$

Since $x_3 \geq 0$, $x_4 \geq 0$ $\Big\} \Rightarrow$

$\Rightarrow$ max $P = 21 - 0 - 0 = 21$
x_3, x_4

A simpler approach to the SM

$$
\begin{cases}
P \quad -2x_1 - 3x_2 \qquad\qquad = 0 \quad (P) \\
\qquad \boxed{x_1} + x_2 + x_3 \quad = 10 \quad (1) \\
\qquad x_1 + 3x_2 \qquad + x_4 = 12 \quad (2)
\end{cases}
$$

$x_1, x_2 = $ basic $\qquad\qquad x_3, x_4 = $ free

- use (1) to eliminate x_1 from the othe 2 eqns.

$$
\begin{cases}
P \qquad\quad -x_2 + 2x_3 \qquad\qquad = 20 \quad (P') \\
\qquad x_1 + x_2 + x_3 \qquad\qquad = 10 \quad (1') = (1) \\
\qquad\quad \boxed{2x_2} - x_3 + x_4 = 2 \quad (2')
\end{cases}
$$

$(P') = (P) + 2 \times (1)$
$(1) \rightarrow (1)$
$(2') = (2) - (1)$

<u>OBS.</u> $x_1 \rightarrow$ has <u>entered</u> the basis

$\dfrac{x_3}{3}$ - has <u>left</u> the basis

- eliminate x_2 from (P') and $(1')$ using $(2')$

$$
\begin{cases}
2P \qquad\qquad\quad +3x_3 + \underline{x_4} = 42 \quad (P'') \rightarrow 2(P') + (2') \\
2x_1 \qquad\qquad +3x_3 \leftarrow x_4 = 18 \quad (1'') \rightarrow 2(1) - (2') \\
\qquad 2x_2 \quad - x_3 + x_4 = 2 \quad (2'') = (2')
\end{cases}
$$

<u>OBS.</u> $x_1 = x_2 = x_3 = 0 \longrightarrow x_4 = 8, \; x_5 = 9 \longrightarrow (0,0,0,8,9)$

feasible corner

$\boxed{C = 16}$

Choose x_3 to enter the basis $\longrightarrow$? x_4 OR x_5 leaves

If x_4 leaves $\longrightarrow$ $\left.\begin{array}{l} x_1, x_2, x_4 \text{ free} \\ x_3, x_5 \text{ basic} \end{array}\right\} \longrightarrow$? feasible corner

$x_1 = x_2 = x_4 = 0$

$x_3 = 4$

$x_5 = 9 - 3x_3 = -3 \; \times$

not feasible

If x_5 leaves $\longrightarrow$ $\left.\begin{array}{l} x_1, x_2, x_5 \text{ free} \\ x_3, x_4 \text{ basic} \end{array}\right\} \longrightarrow$? feasible corner

$x_1 = x_2 = x_5 = 0$

$x_3 = 3, \; x_4 = 2 \quad \checkmark$

$(0,0,3,2,0)$

$\boxed{C = 7}$

So choose x_5 to leave

Use (2) to eliminate x_3 from the other two eqns.

$$\begin{cases} C - 8x_1 + x_2 & -x_5 = 7 \quad (C') = (C) - (2) \\ x_1 + \boxed{18x_2} + 3x_4 - 2x_5 = 6 \quad (1') = 3(1) - 2(2) \\ x_1 - + 3x_3 + x_5 = 9 \quad (2') = (2) \end{cases}$$

$$P = 21 - \frac{3}{2}X_3 - \frac{1}{2}X_4 \left. \begin{array}{l} \\ \\ \end{array} \right\} \Rightarrow \max P = 21$$

$$X_3 \geq 0, \quad X_4 \geq 0 \qquad (\text{when } X_3 = X_4 = 0)$$

$$X_1 = 9, \quad X_2 = 1$$

Example

Minimise $C = 16 + 7X_1 - X_2 - 3X_3$ subject to

$$X_1 + 6X_2 + 2X_3 \leq 8$$

$$X_1 + 3X_3 \leq 9$$

$$X_1, X_2, X_3 \geq 0$$

Introduce slack variables

$$X_5 = 9 - X_1 - 3X_3 \qquad\qquad X_4 = 8 - X_1 - 6X_2 - 2X_3$$

$$\begin{cases} C \quad -7X_1 + X_2 + 3X_3 &= 16 \quad (C) \\ \quad\quad X_1 + 6X_2 + 2X_3 + X_4 &= 8 \quad (1) \\ \quad\quad X_1 \quad\quad + 3X_3 \quad\quad + X_5 &= 9 \quad (2) \end{cases}$$

Aim: want the coefficients of the <u>x</u> variables in the C-equation to be negative.

$X_1, X_2, X_3 =$ free

$X_4, X_5 =$ basic

$X_2 \rightarrow$ entering variable

$$\begin{cases} 18C & -145X_1 & & & -3X_4 -16X_5 = 120 \; (C'') \\ & X_1 & +18X_2 & & +3X_4 -2X_5 = 6 \; (1'')=(1') \\ & X_1 & & +3X_3 & + X_5 = 9 \; (2'')=(2') \end{cases}$$

$$\left[\; (C'') = 18\,(C') - (1') \; \right]$$

$\downarrow$

$$C = \frac{120}{18} + \underbrace{\frac{145}{18}X_1}_{\geqslant 0} + \underbrace{\frac{3}{18}X_4}_{\geqslant 0} + \underbrace{\frac{16}{18}X_5}_{\geqslant 0}$$

$\downarrow$

$$\min C = \frac{120}{18} \quad \text{when} \quad X_1 = X_4 = X_5 = 0 \rightarrow X_2 = \tfrac{1}{3}, X_3 = 3$$

Row ratios

$\Big\{$ constraint equation featuring the entering variable

$$\left| \; \frac{RHS}{\text{Coeff entering var}} \; \right|$$

the variable with the smallest ratio is the one to leave

Simplex Algorithm

(i) introduce slack variables

(ii) write down the 'tableau' of function & constraints

(iii) decide on an entering variable (usually the one with the largest adverse coeff. in the C- eqn.)

(iv) decide on a leaving variable (the one with the lowest row ratio —— if there is a choice)

(v) STOP if

(A) the elimination process gives an eqn. for the function to be optimised with the coeffs of all free variables of the correct sign.

and

(B) if the choice of these free variables to be zero ⟿ feasible corner

ELSE ⟶ return to step(iii)