Mathematics 2L — Linear Modelling

Solutions 5

1. Taking $A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$ we have $\begin{bmatrix} y_{k+1} \\ z_{k+1} \end{bmatrix} = A \begin{bmatrix} y_k \\ z_k \end{bmatrix}$. Then

$$\det(A - \lambda I_2) = \lambda^2 - 1.5\lambda + 0.5 = (\lambda - 1)(\lambda - 0.5),$$

so the eigenvalues of A are 1, 0.5 with corresponding eigenvectors $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Taking

$$S = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}, \quad S^{-1} = \frac{1}{5} \begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix},$$

we have $A = S \operatorname{diag}(1, 0.5) S^{-1}$ and so $A^k = S \operatorname{diag}(1, 0.5^k) S^{-1}$. This gives

$$\begin{bmatrix} y_k \\ z_k \end{bmatrix} = A^k \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 3 + 2(0.5^k) & 3 - 3(0.5^k) \\ 2 - 2(0.5^k) & 2 + 3(0.5^k) \end{bmatrix} \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}.$$

With the given initial values we get

$$\begin{bmatrix} y_k \\ z_k \end{bmatrix} = \begin{bmatrix} 3 - 3(0.5^k) \\ 2 + 3(0.5^k) \end{bmatrix} \to \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \text{as } k \to \infty.$$

2 (i). Let b_k, g_k, m_k be the numbers of vehicles in Birmingham, Glasgow and Manchester in month k. Then

$$\begin{bmatrix} b_{k+1} \\ g_{k+1} \\ m_{k+1} \end{bmatrix} = A \begin{bmatrix} b_k \\ g_k \\ m_k \end{bmatrix}, \quad \text{where} \quad A = \begin{bmatrix} 0.2 & 0.4 & 0.5 \\ 0.4 & 0.5 & 0.1 \\ 0.4 & 0.1 & 0.4 \end{bmatrix}.$$

(ii) To find the eigenvalues A, solve $\det(A - \lambda I_3) = 0$. We know that 1 is an eigenvalue so this polynomial has $(\lambda - 1)$ as a factor and in fact

$$\det(A - \lambda I_3) = -(\lambda - 1)\left(\lambda^2 - \frac{1}{10}\lambda - \frac{9}{100}\right).$$

The quadratic factor has roots $(1 \pm \sqrt{37})/20$. These are approximately $0.35, -0.25$, both smaller than 1 in absolute value .
For eigenvectors corresponding to the eigenvalues 1, $(1+\sqrt{37})/20$, $\left(1-\sqrt{37}\right)/20$, we may take

$$\mathbf{v}_1 = \begin{bmatrix} 29 \\ 28 \\ 24 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -7-\sqrt{37} \\ 5+\sqrt{37} \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -7+\sqrt{37} \\ 5-\sqrt{37} \end{bmatrix}.$$

(iii) Writing $\begin{bmatrix} b_k \\ g_k \\ m_k \end{bmatrix} = p_k\mathbf{v}_1 + q_k\mathbf{v}_2 + r_k\mathbf{v}_3$, then, since $\left(\frac{1 \pm \sqrt{37}}{20}\right)^k \to 0$ as $k \to \infty$,

$$\begin{bmatrix} b_k \\ g_k \\ m_k \end{bmatrix} = p_0A^k\mathbf{v}_1 + q_0A^k\mathbf{v}_2 + r_0A^k\mathbf{v}_3 \to p_0\mathbf{v}_1 \quad \text{as } k \to \infty.$$

(iv) We have $b_0 = g_0 = m_0 = N_0$, hence if we write

$$\begin{bmatrix} N_0 \\ N_0 \\ N_0 \end{bmatrix} = p_0\mathbf{v}_1 + q_0\mathbf{v}_2 + r_0\mathbf{v}_3,$$

then

$$p_0 = \frac{11}{315}N_0, \quad q_0 = -\frac{1}{315}N_0, \quad r_0 = -\frac{1}{315}N_0.$$

Hence we have

$$\begin{aligned} \begin{bmatrix} b_k \\ g_k \\ m_k \end{bmatrix} &= p_0A^k\mathbf{v}_1 + q_0A^k\mathbf{v}_2 + r_0A^k\mathbf{v}_3 \\ &= p_0\mathbf{v}_1 + q_0\left(\frac{1+\sqrt{37}}{20}\right)^k\mathbf{v}_2 + r_0\left(\frac{1-\sqrt{37}}{20}\right)^k\mathbf{v}_3 \\ &\approx p_0\mathbf{v}_1 + q_0(0.35)^k\mathbf{v}_2 + r_0(-0.25)^k\mathbf{v}_3. \end{aligned}$$

3. Let the proportions of the initial population of those who are well, ill or dead after k months be w_k, s_k and d_k respectively. Then

$$\begin{bmatrix} w_{k+1} \\ s_{k+1} \\ d_{k+1} \end{bmatrix} = A \begin{bmatrix} w_k \\ s_k \\ d_k \end{bmatrix} \quad \text{where} \quad A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{3}{4} & 0 \\ 0 & \frac{1}{4} & 1 \end{bmatrix}.$$

Eigenvalues and eigenvectors of A are

$$\lambda_1 = 1, \quad \mathbf{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \quad \lambda_2 = \frac{1}{2}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}; \quad \lambda_3 = \frac{3}{4}, \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix},$$

from which obtain an invertible matrix

$$S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & -1 \\ 1 & 1 & 1 \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ -2 & -1 & 0 \end{bmatrix}$$

for which $A = S\,\mathrm{diag}(1, \frac{1}{2}, \frac{3}{4})S^{-1}$. Then

$$A^k = \begin{bmatrix} \frac{1}{2^k} & 0 & 0 \\ -\frac{1}{2^{k-1}} + \frac{3^k}{2^{2k-1}} & \frac{3^k}{4^k} & 0 \\ 1 + \frac{1}{2^k} - \frac{3^k}{2^{2k-1}} & 1 - \frac{3^k}{4^k} & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{as } k \to \infty.$$

This shows that $w_k \to 0$, $s_k \to 0$ and $d_k \to 1$, so in the long run the whole population dies from the epidemic.

4. Let m_k, ℓ_k and w_k be the number of customers of each dairy in year k. Then

$$\begin{bmatrix} m_{k+1} \\ \ell_{k+1} \\ w_{k+1} \end{bmatrix} = A \begin{bmatrix} m_k \\ \ell_k \\ w_k \end{bmatrix} \quad \text{where} \quad A = \begin{bmatrix} \frac{8}{10} & \frac{2}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{7}{10} & \frac{3}{10} \\ \frac{1}{10} & \frac{1}{10} & \frac{6}{10} \end{bmatrix}.$$

Setting $B = 10A$, we have $\chi_B(\mu) = \mu^3 - 21\mu^2 + 140\mu - 300$
$= (\mu - 10)(\mu - 5)(\mu - 6)$, with eigenvalues and eigenvectors

$$\mu_1 = 10, \quad \mathbf{u}_1 = \begin{bmatrix} 9 \\ 7 \\ 4 \end{bmatrix}; \quad \mu_2 = 5, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}; \quad \mu_3 = 6, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Notice that eigenvalues and eigenvectors for A are then

$$\lambda_1 = 1, \quad \mathbf{u}_1 = \begin{bmatrix} 9 \\ 7 \\ 4 \end{bmatrix}; \quad \lambda_2 = \frac{1}{2}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}; \quad \lambda_3 = \frac{3}{5}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Taking

$$S = \begin{bmatrix} 9 & 1 & 1 \\ 7 & -2 & -1 \\ 4 & 1 & 0 \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} \frac{1}{20} & \frac{1}{20} & \frac{1}{20} \\ -\frac{1}{5} & -\frac{1}{5} & \frac{4}{5} \\ \frac{3}{4} & -\frac{1}{4} & -\frac{5}{4} \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 1 & 1 & 1 \\ -4 & -4 & 16 \\ 15 & -5 & -25 \end{bmatrix},$$

we have $A = S\,\mathrm{diag}(1, \frac{1}{2}, \frac{3}{5})S^{-1}$ and so $A^k = S\,\mathrm{diag}\left(1, \frac{1}{2^k}, \frac{3^k}{5^k}\right) S^{-1}$. Given the initial data $\begin{bmatrix} m_0 \\ \ell_0 \\ w_0 \end{bmatrix} = \begin{bmatrix} 5200 \\ 6320 \\ 8340 \end{bmatrix}$ we obtain

$$\begin{bmatrix} m_k \\ \ell_k \\ w_k \end{bmatrix} = \begin{bmatrix} 8937 + \dfrac{4368}{2^k} - 8105\,\dfrac{3^k}{5^k} \\ 6951 - \dfrac{8736}{2^k} + 8105\,\dfrac{3^k}{5^k} \\ 3972 + \dfrac{4368}{2^k} \end{bmatrix}.$$

When $k = 3$ this gives $m_3 = 193308/25 \approx 7732$, $\ell_3 = 190242/25 \approx 7609$, $w_3 = 4518$. As $k \to \infty$ we obtain $m_k \to 8937$, $\ell_k \to 6951$, $w_k \to 3972$.

5. A has the eigenvalues and eigenvectors

$$\lambda_1 = 0, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}; \quad \lambda_2 = -3, \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

where -3 is a repeated root of the characteristic polynomial of A. The matrices

$$S = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad S^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

give $A = S\,\mathrm{diag}(0, -3, -3)S^{-1}$. Then the general solution of the differential equation is

$$\mathbf{u}(t) = c_1 e^0 \mathbf{u}_1 + c_2 e^{-3t}\mathbf{u}_2 + c_3 e^{-3t}\mathbf{u}_3 = c_1\mathbf{u}_1 + c_2 e^{-3t}\mathbf{u}_2 + c_3 e^{-3t}\mathbf{u}_3.$$

6 (i). Here

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 8 \\ -1 & 7-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 7 + 8 = (\lambda - 3)(\lambda - 5).$$

Eigenvalues and eigenvectors for A are

$$\lambda_1 = 3, \quad \mathbf{u}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}; \quad \lambda_2 = 5, \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Putting

$$S = \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}, \text{ so that } S^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix},$$

we have $A = S\,\mathrm{diag}(3, 5)S^{-1}$.
The general solution is $\mathbf{u} = c_1 e^{3t}\mathbf{u}_1 + c_2 e^{5t}\mathbf{u}_2$.
(ii) We have

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = S^{-1}\begin{bmatrix} 50 \\ 300 \end{bmatrix} = \begin{bmatrix} -275 \\ 575 \end{bmatrix},$$

giving $r(t) = -1100e^{3t} + 1150e^{5t}$, $f(t) = -275e^{3t} + 575e^{5t}$.
(iii) As $t \to \infty$,

$$\frac{r(t)}{f(t)} = \frac{-1100e^{3t} + 1150e^{5t}}{-275e^{3t} + 575e^{5t}} = \frac{-1100e^{-2t} + 1150}{-275e^{-2t} + 575} \to 2.$$