Mathematics 2L — Linear Modelling

Solutions 4

1.

$$\begin{vmatrix} 3 & 1 & -2 \\ -1 & 4 & -5 \\ 4 & 0 & 1 \end{vmatrix} = 25, \qquad \begin{vmatrix} -2 & 1 & 2 \\ 4 & 0 & 1 \\ -3 & 5 & 2 \end{vmatrix} = 39, \qquad \begin{vmatrix} x & -x & 4x \\ y & 2y & 0 \\ -z & 3z & 2z \end{vmatrix} = 26xyz.$$

2. 110.

3.

(i) False: *eg*, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$.

(ii) True: suppose that $\mathbf{u} \neq \mathbf{0}$ is a column vector such that $B\mathbf{u} = \mathbf{0}$; then $AB\mathbf{u} = A\mathbf{0} = \mathbf{0}$ which means AB cannot be non-singular

or $\det(AB) = \det(A)\ \det(B) = \det(A)\,0 = 0$.

(iii) (a) For an $n \times n$ matrix M, $\det(-M) = (-1)^n \det M$. Hence, if $CD = -DC$ then $\det(C)\det(D) = (-1)^n \det(D)\det(C)$. So: False if n is even, true if n is odd.

(b) Not if n is even: consider $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

If, however, n is odd then $\det(C)\det(D) = (-1)^n\ \det(D)\det(C) = -\det(C)\det(D)$ which can only happen when $\det(C) = 0$ or $\det(D) = 0$.

So C or D must be singular.

(iv) True: $\det(Q^TQ) = \det I = 1$ and, as $\det(Q^TQ) = \det Q \det Q^T = (\det Q)^2$, we have $(\det Q)^2 = 1$ so $\det Q = \pm 1$.

4.

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & x^3 & x \\ x & x & x^3 \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 0 & -x+x^3 & -x^2+x \\ x & x & x^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 0 & -x+x^3 & -x^2+x \\ 0 & -x^2+x & 0 \end{vmatrix}$$

$$= \begin{vmatrix} -x+x^3 & -x^2+x \\ -x^2+x & 0 \end{vmatrix}$$

$$= -(x-x^2)^2 = -x^2(x-1)^2.$$

The solutions are $x = 0, 1$.

5. Have seen [worked example] that the eigenvalues are 0 and 1 (twice). Next, for $\lambda = 0$:

$$A - 0I = \begin{bmatrix} -1 & 2 & -3 \\ 0 & 1 & -1 \\ 1 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

so a 0-eigenvector is $[1, -1, -1]^T$.
For $\lambda = 1$ we have

$$A - 1I = \begin{bmatrix} -2 & 2 & -3 \\ 0 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so any 1-eigenvector is a multiple of $[1, 1, 0]^T$.

$B = A + 3I$ so the eigenvalues of B are 3 and 4 (twice)

6. The eigenvalues of the matrices are

$$A\colon 0, 2, 4, \quad B\colon 1, 1+2i, 1-2i, \quad C\colon -1, 2, 4.$$

7. For A we have eigenvalues $-1, 4$ with corresponding eigenvectors $\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

For B we have eigenvalues $-4, 1, 2$ with corresponding eigenvectors $\begin{bmatrix} 7 \\ 2 \\ -10 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

8 (i). The eigenvalues of B are $-2, 1, 3$.

(ii). The largest eigenvalue is 3 and an eigenvector is $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

(iii). We have

$$B^2 = \begin{bmatrix} 5 & 4 & -4 \\ 8 & 6 & -2 \\ 4 & 1 & 3 \end{bmatrix}$$

and also $C = 3B^2 + 20I$. Hence for each eigenvalue λ of B the number $3\lambda^2 + 20$ is an eigenvalue of C. This shows that $14, 23, 29$ are eigenvalues of C.

9. The matrix $S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ is invertible and $A = S\,\mathrm{diag}\{2, 0\}\,S^{-1}$, $B = S\,\mathrm{diag}\{0, 2\}\,S^{-1}$.

10. If $S = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ then $S^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$ and we may take

$$A = S\,\mathrm{diag}\{1, 4\}\,S^{-1} = \begin{bmatrix} -5 & 18 \\ -3 & 10 \end{bmatrix}.$$

11. $\det(A - \lambda I) = -\lambda^3 + 3\lambda^2 = -\lambda^2(\lambda - 3)$, so the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 3$. The eigenvectors for λ_1 are of the form $u \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, while for λ_2 they are of the form $v \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + w \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ for arbitrary choices of u and v (not both zero).
The invertible matrices

$$S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

satisfy $S\,\mathrm{diag}\{3, 0, 0\}\,S^{-1} = A = T\,\mathrm{diag}\{3, 0, 0\}\,T^{-1}$.

12. $\begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$ has only one eigenvalue, 0, and its eigenvectors are all of the form $t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, hence this matrix cannot be diagonalised.

$\begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix}$ has two distinct eigenvalues $-2, 2$ with corresponding eigenvectors $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, hence this matrix can be diagonalised.

$\begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$ has only one eigenvalue 2, and its eigenvectors are all of the form $t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, hence this matrix cannot be diagonalised.

13. From **Q 12**, we may take $S = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$ with $S^{-1} = \frac{1}{2} \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}$ and $A = S \operatorname{diag}\{-2, 2\}\, S^{-1}$. Then

$$A^{100} = S \operatorname{diag}\{-2^{100},\, 2^{100}\}\, S^{-1} = 2^{100} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}.$$