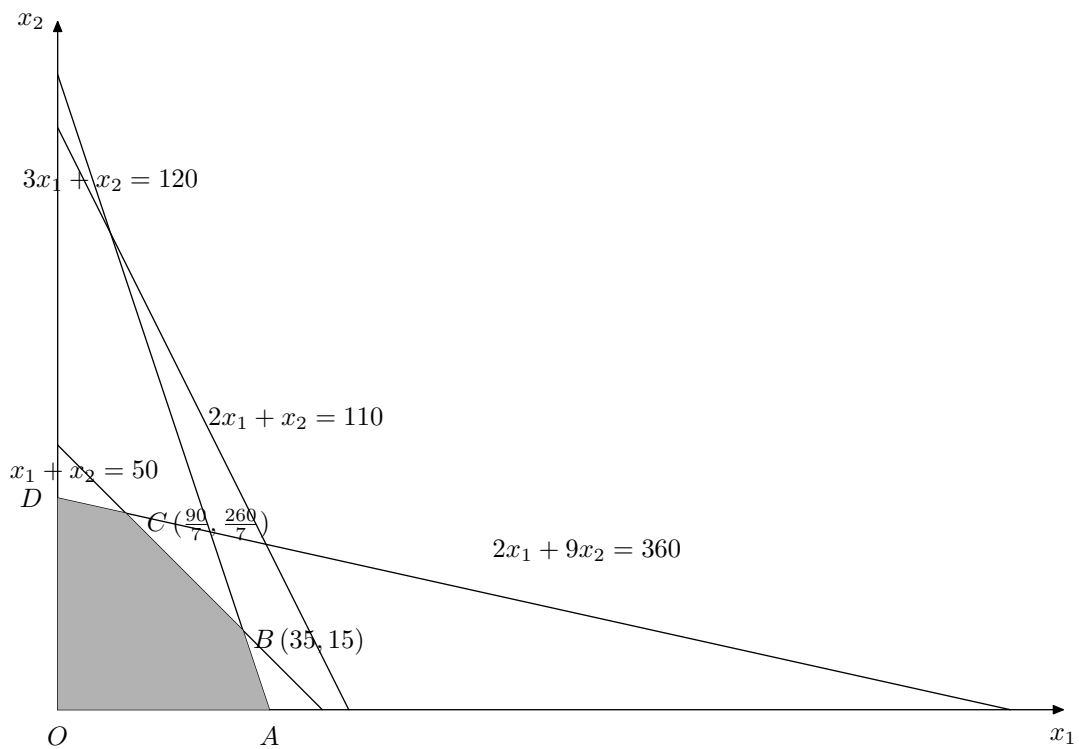


# Mathematics 2L — Linear Modelling

## Solutions 3

1. *Graphically* — The feasible region is as shown



The values of  $P$  at the vertices are

Point	$O$	$A$	$B$	$C$	$D$
$P$	0	120	$\frac{1260}{7}$	$\frac{1570}{7}$	200

from which we see that the maximum of  $P$  is  $\frac{1570}{7}$ , attained at  $(\frac{90}{7}, \frac{260}{7})$ .

*Simplexly* —

Introduce the slack variables

$$\begin{aligned}x_3 &= 50 - x_1 - x_2, \\x_4 &= 110 - 2x_1 - x_2, \\x_5 &= 360 - 2x_1 - 9x_2, \\x_6 &= 120 - 3x_1 - x_2.\end{aligned}$$

The initial tableau is

$$\begin{array}{rcccccccl}P & -3x_1 & -5x_2 & & & & & = & 0 & (P) \\ & x_1 & +x_2 & +x_3 & & & & = & 50 & (1) \\ & 2x_1 & +x_2 & & +x_4 & & & = & 110 & (2) \\ & 2x_1 & \boxed{+9x_2} & & & +x_5 & & = & 360 & (3) \\ & 3x_1 & +x_2 & & & & +x_6 & = & 120 & (4)\end{array}$$

We choose  $x_2$  to enter the basis and, according to the row-ratio criterion,  $x_5$  to leave. This gives

$$\begin{array}{rcccccccl}9P & -17x_1 & & & +5x_5 & & & = & 1800 & (P') \\ & \boxed{+7x_1} & & +7x_3 & & -x_5 & & = & 90 & (1') \\ & 16x_1 & & & +9x_4 & -x_5 & & = & 630 & (2') \\ & 2x_1 & +9x_2 & & & +x_5 & & = & 360 & (3) \\ & 25x_1 & & & & -x_5 & +9x_6 & = & 720 & (4')\end{array}$$

Now we choose  $x_1$  to enter and  $x_3$  to leave. This gives

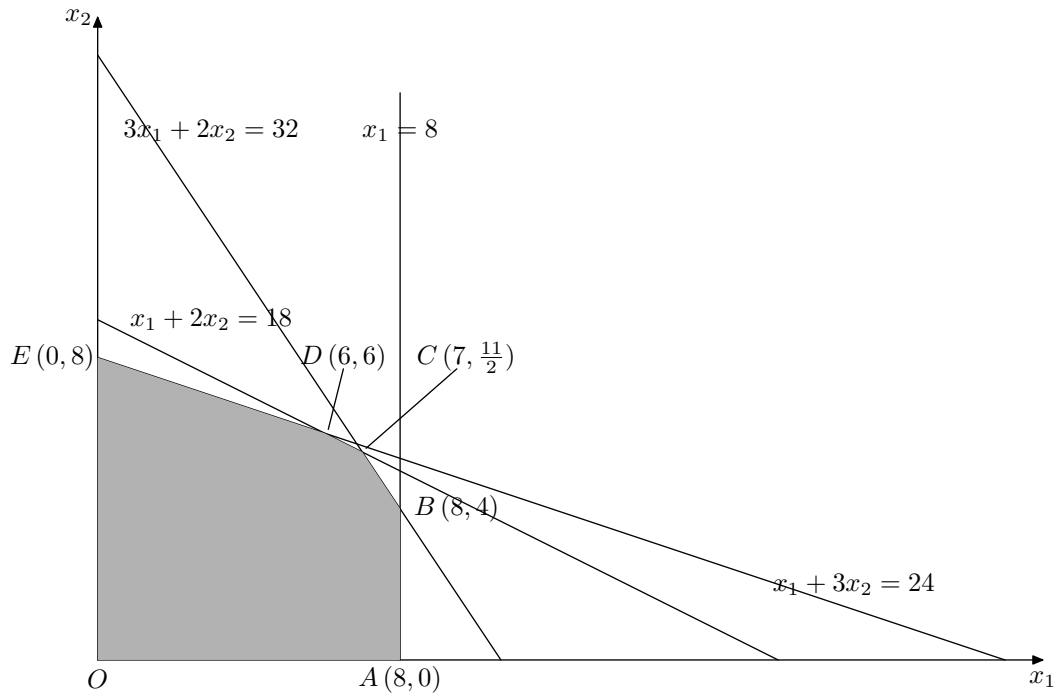
$$63P + 153x_3 + 18x_5 = 90 \times 157 \quad (P'')$$

*ie*

$$7P + 17x_3 + 2x_5 = 1570 \quad (P''')$$

We see that the optimal point is the one given by  $x_3 = x_5 = 0$  *ie*  $\left(\frac{90}{7}, \frac{260}{7}\right)$ , and that the maximum of  $P$  is  $\frac{1570}{7}$ .

2. *Graphically* — The feasible region is as shown



The values of  $P$  at the vertices are

Point	$O$	$A$	$B$	$C$	$D$	$E$
$P$	0	16	36	$\frac{83}{2}$	42	40

from which we see that the maximum of  $P$  is 42, attained at  $D(6, 6)$ .

*Simplexly* —

Introduce the slack variables

$$\begin{aligned} x_3 &= 8 - x_1, \\ x_4 &= 32 - 3x_1 - 2x_2, \\ x_5 &= 24 - x_1 - 3x_2, \\ x_6 &= 18 - x_1 - 2x_2. \end{aligned}$$

The initial tableau is

$$\begin{array}{rcccccccl}
 P & -2x_1 & -5x_2 & & & & & = & 0 & (P) \\
 & x_1 & & +x_3 & & & & = & 8 & (1) \\
 & 3x_1 & +2x_2 & & +x_4 & & & = & 32 & (2) \\
 & x_1 & \boxed{+3x_2} & & & +x_5 & & = & 24 & (3) \\
 & x_1 & +2x_2 & & & & +x_6 & = & 18 & (4)
 \end{array}$$

We choose  $x_2$  to enter the basis and, according to the row-ratio criterion,  $x_5$  to leave. This gives

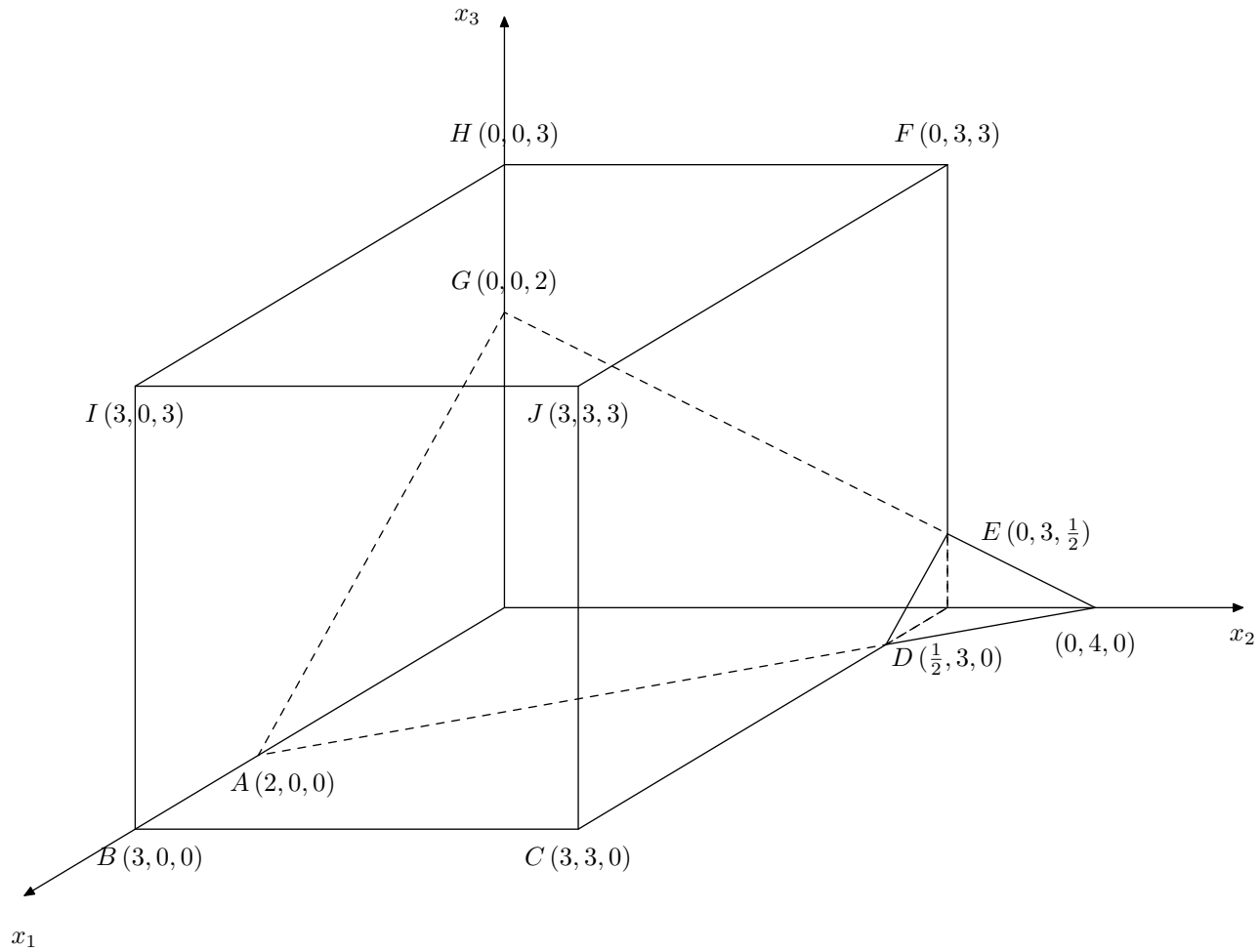
$$\begin{array}{rcccccccl}
 3P & -x_1 & & & +5x_5 & & & = & 120 & (P') = 3(P) + 5(3) \\
 & x_1 & & +x_3 & & & & = & 8 & (1) \\
 & 7x_1 & & +3x_4 & -2x_5 & & & = & 48 & (2') = 3(2) - 2(3) \\
 & x_1 & +3x_2 & & +x_5 & & & = & 24 & (3) \\
 \boxed{x_1} & & & & 2x_5 & +3x_6 & & = & 6 & (4') = 3(4) - 2(3)
 \end{array}$$

We choose  $x_1$  to enter the basis and, according to the row-ratio criterion,  $x_6$  to leave. This gives

$$\begin{array}{rcccccl}
 3P & & +7x_5 & +3x_6 & = & 126 & (P'') = (P') + (4') \\
 & & & & & \vdots &
 \end{array}$$

The optimal point is given by  $x_5 = x_6 = 0$ , so is  $D(6, 6)$ . The optimal value is 42.

3. The region is as shown



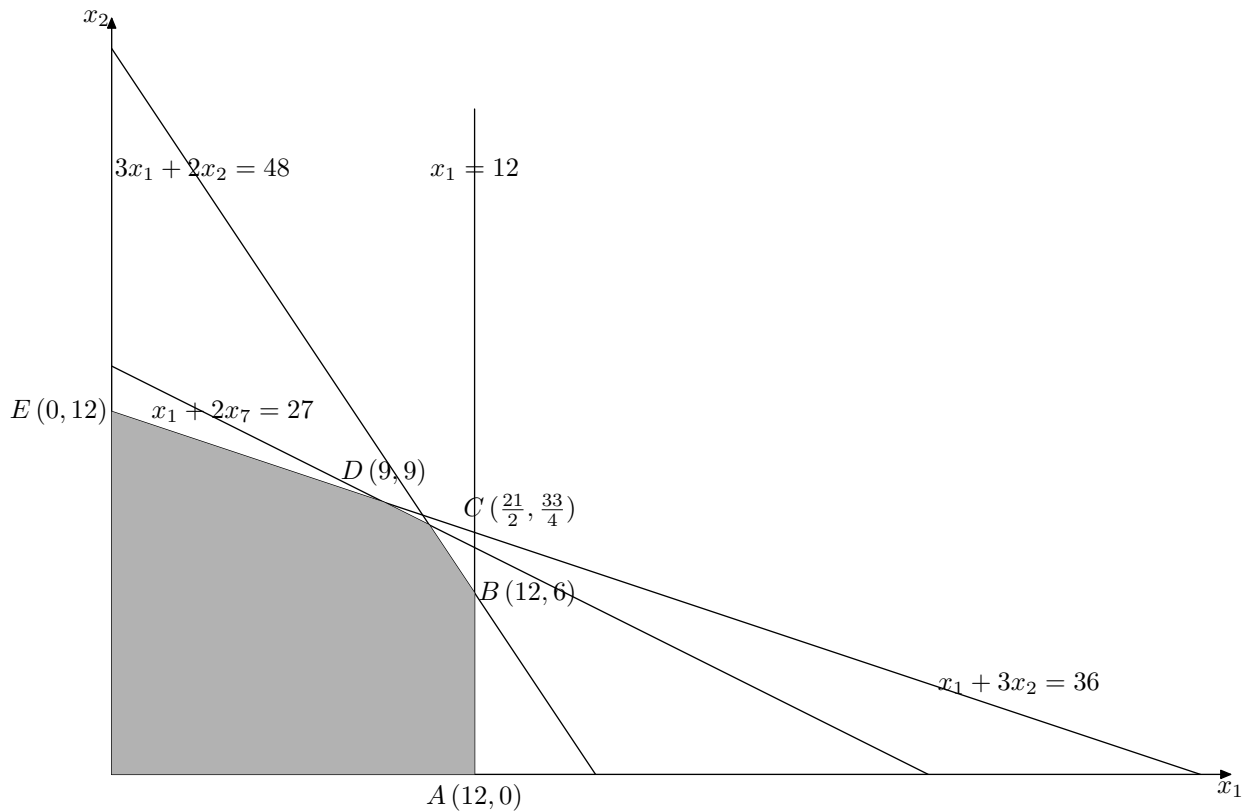
The function values at the extreme vertices are —

	$P(x_1, x_2, x_3)$ = $-4x_1 - 2x_2 - x_3$	$Q(x_1, x_2, x_3)$ = $x_1 + x_2 + x_3$	$R(x_1, x_2, x_3)$ = $x_1 - 4x_2 + x_3$
$A(2, 0, 0)$	-8	$\boxed{2}_{min}$	2
$B(3, 0, 0)$	-12	3	3
$C(3, 3, 0)$	-18	6	-9
$D(\frac{1}{2}, 3, 0)$	-8	$\frac{7}{2}$	$-\frac{11}{2}$
$E(0, 3, \frac{1}{2})$	$-\frac{13}{2}$	$\frac{7}{2}$	$\boxed{-\frac{23}{2}}_{min}$
$F(0, 3, 3)$	-9	6	-9
$G(0, 0, 2)$	$\boxed{-2}_{max}$	$\boxed{2}_{min}$	2
$H(0, 0, 3)$	-3	3	3
$I(3, 0, 3)$	-15	6	$\boxed{6}_{max}$
$J(3, 3, 3)$	$\boxed{-21}_{min}$	$\boxed{9}_{max}$	-6

4. Let  $x_1, x_2$  be the numbers of scrambler and regular models produced; the profit  $P$  produced will be proportional to  $12x_1 + 10x_2$  and we want to maximise this quantity. Using units of 1000 for  $x_1, x_2$ , we can take the constraints to be

$$\begin{aligned}
2x_1 + 0x_2 &\leq 24 \quad \text{ie} \quad x_1 \leq 12, \\
3x_1 + 2x_2 &\leq 48, \\
1x_1 + 3x_2 &\leq 36, \\
2x_1 + 4x_2 &\leq 54, \quad \text{ie} \quad x_1 + 2x_2 \leq 27.
\end{aligned}$$

Graphically — The feasible region is as shown



The values of  $P$  at the vertices are

Point	$O$	$A$	$B$	$C$	$D$	$E$
$P$	0	144	204	$\frac{417}{2}$	198	120

from which we see that the maximum of  $P$  is  $\frac{417}{2}$ , attained where  $x_4 = x_6 = 0$ , i.e. at  $C\left(\frac{21}{2}, \frac{33}{4}\right)$ , corresponding to 10500 scramblers and 8250 regulars.

*Simplexly* — Introduce the slack variables

$$\begin{aligned}x_3 &= 12 - x_1, \\x_4 &= 48 - 3x_1 - 2x_2, \\x_5 &= 36 - 1x_1 - 3x_2, \\x_6 &= 27 - x_1 - 2x_2.\end{aligned}$$

The problem is

$$\begin{array}{rcccccc}P & -12x_1 & -10x_2 & & & & = & 0 & (P) \\ & \boxed{x_1} & & & +x_3 & & & = & 12 & (1) \\ & 3x_1 & +2x_2 & & & +x_4 & & = & 48 & (2) \\ & x_1 & +3x_2 & & & & +x_5 & = & 36 & (3) \\ & x_1 & +2x_2 & & & & & +x_6 & = & 27 & (4)\end{array}$$

We choose  $x_1$  to enter the basis and  $x_3$  to leave, so obtaining:

$$\begin{array}{rcccccc}P & & -10x_2 & +12x_3 & & & = & 144 & (P') \\ x_1 & & & +x_3 & & & & = & 12 & (1) \\ & & 2x_2 & -3x_3 & +x_4 & & & = & 12 & (2') \\ & & 3x_2 & -x_3 & & +x_5 & & = & 24 & (3') \\ & & 2x_2 & -x_3 & & & +x_6 & = & 15 & (4')\end{array}$$

Next we choose  $x_2$  to enter the basis and  $x_4$  to leave, so obtaining:

$$\begin{array}{rcccccc}P & & & -3x_3 & +5x_4 & & = & 204 & (P'') \\ x_1 & & & +x_3 & & & & = & 12 & (1) \\ 2x_2 & & & -3x_3 & +x_4 & & & = & 12 & (2'') \\ & & & 7x_3 & -3x_4 & +2x_5 & & = & 12 & (3'') \\ & & & 2x_3 & -x_4 & & +x_6 & = & 3 & (4'')\end{array}$$

Next we choose  $x_3$  to enter the basis and  $x_6$  to leave, so obtaining:

$$2P + 7x_4 + 3x_6 = 417 \quad (P''')$$

from which we see that  $P$  attains its maximum value,  $\frac{417}{2}$ , where  $x_4 = x_6 = 0$ ,  
ie at  $C \left( \frac{21}{2}, \frac{33}{4} \right)$ .

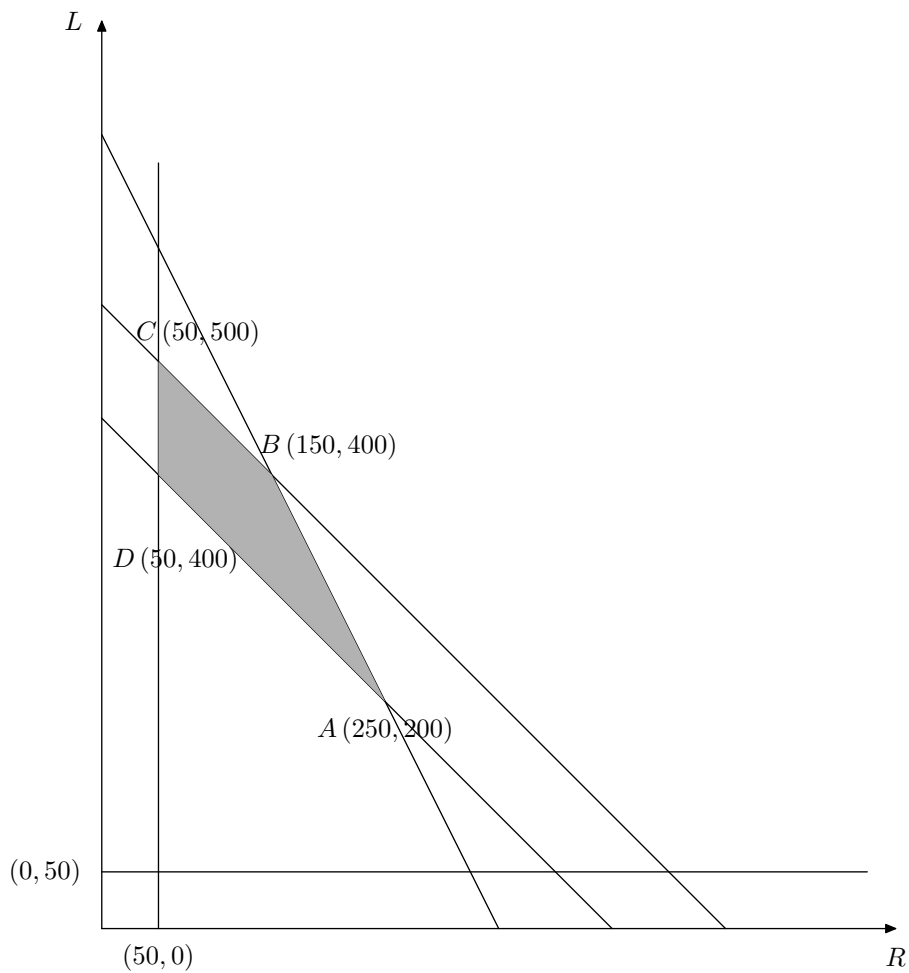
5. Write  $R$  for the number of hectares of root crop,  $L$  for the number of hectares of leaf crop, and  $F$  for the number of hectares left fallow.

The constraints are  $R + L + F = 600$ ;  $R \geq 50$ ,  $L \geq 50$ ,  $F \geq 50$ ;  $R + L \geq 75\% \cdot 600 [= 450]$ ;  $4R + 2L \leq 1400$ . The profit  $P$  is  $25R + 10L + 5F$ .

If we eliminate  $F$  we get, equivalently,

$$R \geq 50, \quad L \geq 50, \quad R + L \leq 550, \quad R + L \geq 450, \quad 2R + L \leq 700.$$

Graphically — The feasible region is as shown





6. Write  $a$  for proportion of *Anthracite*,  $l$  for that of *Lignite* and  $c$  for that of *Coke*.

Then  $a + l + c = 1$ ; the sulphur constraint is  $2a + 4l + 3c \leq 3$ ; the ash content constraint is  $3a + 4l + 2c \leq 3.25$ . The cost is [proportional to]  $C = 9a + 4l + 8c$ .

If we eliminate  $c$  we find, equivalently, that

$$2a + 4l + 3c = 2a + 4l + 3(1 - a - l) = 3 - a - l$$

$$3a + 4l + 2c = 3a + 4l + 2(1 - a - l) = 2 + a + 2l$$

$$C = 9a + 4l + 8c = 9a + 4l + 8(1 - a - l) = 8 + a - 4l$$

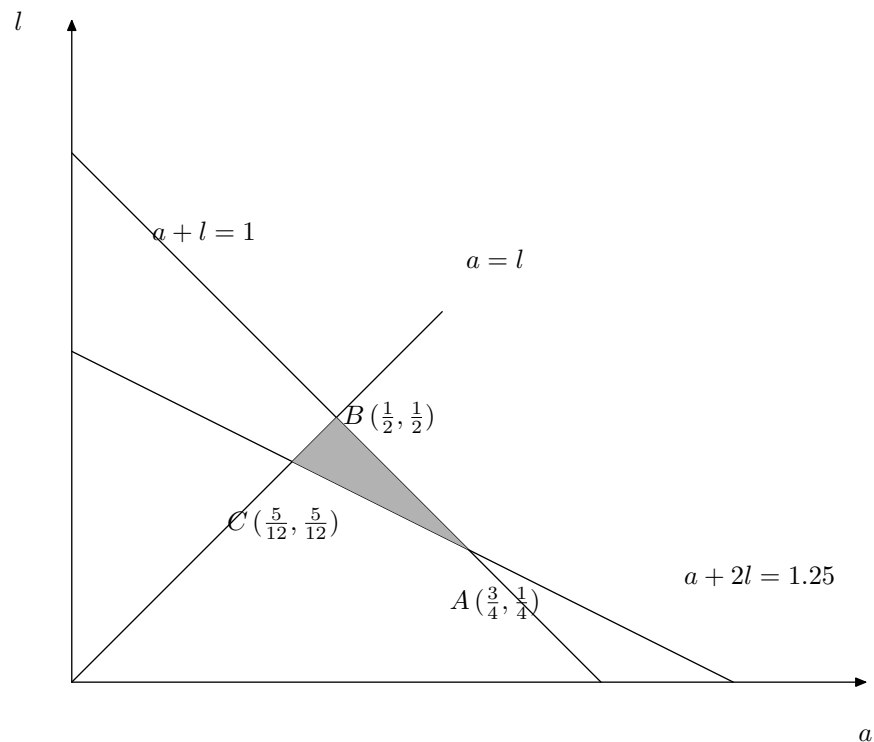
so the constraints are

$$a + l \leq 1$$

$$l \leq a$$

$$a + 2l \leq 1.25$$

Graphically —



The values of the cost  $C$  [£/tonne] at the vertices of the feasible region are

Point	$A$	$B$	$C$
$C$	77.5	75	67.5

The cheapest mixture is  $\frac{5}{12}$  A :  $\frac{5}{12}$  L :  $\frac{1}{6}$  C, at a cost of £ 67.5 per tonne.

*Simplexly* — Introduce slack variables  $d$  and  $e$ . Get system

$$\begin{array}{rclcl}
 C & -a & +4l & & = & 8 \\
 & -a & \boxed{+l} & +d & = & 0 \\
 & a & +2l & +e & = & 1.25
 \end{array}$$

Using pivot shown this gives

$$\begin{array}{rclcl}
 C & +3a & & -4d & = & 8 \\
 & -a & +l & & +d & = & 0 \\
 & \boxed{3a} & & -2d & +e & = & 1.25
 \end{array}$$

and using pivot shown this gives

$$C \quad -2d \quad -e = \frac{27}{4}$$

⋮

which gives the same result.